MR3025559 81S25 (60H10)

Ogundiran, M. O. [Ogundiran, Michael O.] (WAN-IFE); **Ayoola, E. O.** (WAN-IBAD)

Caratheodory solution of quantum stochastic differential inclusions. (English summary)

J. Nigerian Math. Soc. **31** (2012), 81–90.

{A review for this item is in process.}

MR2905708 (Review) 81S25 (34A60 60H99)

Ogundiran, M. O. [Ogundiran, Michael O.] (WAN-IFE); **Ayoola, E. O.** (WAN-IBAD)

An application of Michael selection theorems. (English summary)

Int. J. Funct. Anal. Oper. Theory Appl. **4** (2012), *no. 1,* 65–79.

Michael's selection theorem for lower semicontinuous multifunctions is a famous result used in the solution of differential inclusions. In this paper the authors attempt to extend this idea to lower semicontinuous multivalued stochastic processes. The first of the main results is as follows:

Assume the following hold:

(1) For arbitrary $\eta, \xi \in (D\underline{\otimes}E)$, the map $(t,x) \to G(t,x)(\eta,\xi)$ is lower semicontinuous with respect to a seminorm.

(2) $g\colon I \times \widetilde{A} \to \mathrm{sesq}(D\underline{\otimes}E)$ is continuous.

(3) $\varepsilon\colon \widetilde{A} \to \mathbb{R}_+$ is lower semicontinuous.

Then the map $(t,x) \to \Phi(t,x)(\eta,\xi)$ defined by

$$\Phi(t,x)(\eta,\xi) = B_{\varepsilon(x)}(g(t,x)(\eta,\xi)) \cap G(t,x)(\eta,\xi)$$

is lower semicontinuous with respect to the seminorm. Here $\widetilde{A}$ is the completion of a certain locally convex space.

The above result and another proposition are used to establish the main theorem below, which is a noncommutative extension to Michael's selection theorem.

Theorem: Suppose that $\Psi\colon I \times \widetilde{A} \to 2^{\mathrm{sesq}(D\underline{\otimes}E)^2}$ is a multivalued stochastic process such that for arbitrary elements $\eta, \xi \in (D\underline{\otimes}E)$,

(1) $(t,x) \to \Psi(t,x)(\eta,\xi)$ is lower semicontinuous with respect to a seminorm,

(2) Ψ is closed and convex-valued.

Then there exists $\psi\colon I \times \widetilde{A} \to \mathrm{sesq}(D\underline{\otimes}E)^2$, which is a continuous selection from Ψ.

A corollary provides an application to quantum stochastic evolution inclusions which requires an understanding of hypermaximal monotone multifunctions.

Reviewed by *John S. Spraker*

MR2953241 (Review) 81S25 (34A60 34F05)

Ogundiran, M. O. [Ogundiran, Michael O.] (WAN-IFE); **Ayoola, E. O.** (WAN-IBAD)

Upper semicontinuous quantum stochastic differential inclusions via Kakutani-Fan fixed point theorem. (English summary)

Dynam. Systems Appl. **21** (2012), *no. 1,* 121–132.

The existence and uniqueness of solutions of quantum stochastic differential equations was first obtained, in the case of bounded coefficients, by Hudson and Parthasarathy and later extended, to unbounded coefficients, by Franco Fagnola. Existence was extended to quantum stochastic differential inclusions by Ekhaguere. The authors extend the results to discontinuous quantum stochastic differential inclusions with upper semicontinuous coefficients. In the process, they prove a noncommutative generalization of the Kakutani-Fan fixed point theorem.

Reviewed by *Andreas Boukas*

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Note: This list reflects references listed in the original paper as accurately as possible with no attempt to correct errors.

MR2905434 (Review) 81S25 (60H10)
Ogundiran, M. O. [Ogundiran, Michael O.] (WAN-IFE); **Ayoola, E. O.** (WAN-IBAD)
Directionally continuous quantum stochastic differential equations. (English summary)
Far East J. Appl. Math. **57** (2011), *no. 1,* 33–48.

The authors show that the following Hudson-Parthasarathy type quantum stochastic differential equation with directionally continuous coefficients:

$$dX(t) = E(t, X(t))\, d\Lambda_\pi(t) + F(t, X(t))\, dA_f(t) + G(t, X(t))\, dA_g^+(t) + H(t, X(t))\, dt,$$

with initial condition

$$X(t_0) = x_0,$$

admits a solution in the sense of Carathéodory. They also show that, under certain regularity conditions, the solution of the directionally continuous quantum stochastic differential equation

$$\frac{d}{dt}\langle \eta, X(t)\xi\rangle = P(t, X(t))(\eta, \xi)$$

coincides with the solution set of the upper semicontinuous quantum stochastic differential inclusion

$$\frac{d}{dt}\langle \eta, X(t)\xi\rangle \in \mathbb{P}(t, X(t))(\eta, \xi).$$

Reviewed by *Andreas Boukas*

MR2605072 (2011d:81180) 81S25 (81Q93)
Ogundiran, M. O. [Ogundiran, Michael O.] (WAN-IFE); **Ayoola, E. O.** (WAN-IBAD)
Mayer problem for quantum stochastic control. (English summary)
J. Math. Phys. **51** (2010), *no. 2,* 023521, 8 *pp*.1089-7658

Motivated by Ekhaguere's multi-valued analog of Hudson-Parthasarathy quantum stochastic calculus, the authors study quantum stochastic control via set-valued analysis. In particular, they study the regularity properties of the value function inherited from the multi-valued stochastic processes involved, and they show that, under the assumption of directional differentiability of the value function, the associated Mayer problem has at least one optimal solution. The authors' theory covers earlier work on quantum stochastic control by Belavkin and by the reviewer.

Reviewed by *Andreas Boukas*

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MR2510253 (2010d:60153) 60H35 (60H15 65C30 65M15 65M60)

Njoseh, Ignatius N.; **Ayoola, Ezekiel O.** (WAN-IBAD)

Maximum-norm error estimate for a strongly damped stochastic wave equation. (English summary)

J. Math. Sci. (Dattapukur) **20** (2009), *no. 1,* 21–30.

Summary: “We discuss the finite element analysis of a strongly damped stochastic wave equation driven by space-time noise. Maximum-norm error estimates are proved for solutions $u_h(t)$ and U^n of semidiscrete and fully discrete schemes, respectively.”

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MR2569523 (2011d:81177) 81S25 (60H10)

Ayoola, E. O. (WAN-IBAD)

Further results on the existence of continuous selections of solution sets of quantum stochastic differential inclusions. (English summary)

Dynam. Systems Appl. **17** (2008), *no. 3-4,* 609–624.

Quantum stochastic differential inclusions (QSDI) are adapted operator-valued processes X satisfying

$$X(t) \in X_0 + \int_0^t E(s,X(s))d\Lambda_\pi(s) + F(s,X(s))dA_f(s) \\ + G(s,X(s))dA_g^+(s) + H(s,X(s)), \quad t \in [0,T],$$

where E, F, G, H are maps endowed with some regularity, the integral is in the sense of Hudson-Parthasarathy and $T > 0$. For more details on the domains of linear operators and their adjoints, see [E. O. Ayoola, Internat. J. Theoret. Phys. **43** (2004), no. 10, 2041–2059; MR2107450

(2005i:81076)].

In the paper under review, the author proves the existence of a selection of solution sets of QSDI, which is continuous from a compact set of initial values (included in the locally convex space of stochastic processes) into the locally convex space of adapted and weakly continuous quantum stochastic processes.

The main result is in the spirit of [op. cit.], in which the author established the existence of continuous selections of solution sets of QSDI from the set of the matrix elements of initial points to the set of matrix elements of solutions.

Reviewed by *Vitonofrio Crismale*

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MR2569515 (2011d:81176) 81S25 (60H10 81Q93)

Ayoola, E. O. (WAN-IBAD)

Quantum stochastic differential inclusions satisfying a general Lipschitz condition. (English summary)

Dynam. Systems Appl. **17** (2008), *no. 3-4,* 487–502.

The quantum stochastic differential inclusions (QSDI) studied here are adapted operator-valued processes X satisfying

$$(2)\quad X(t) \in X_0 + \int_0^t E(s,X(s))d\Lambda_\pi(s) + F(s,X(s))dA_f(s) + G(s,X(s))dA_g^+(s) + H(s,X(s)), \quad t \in [0,T],$$

where E, F, G, H are maps endowed with some regularity, the integral is à la Hudson-Parthasarathy and $T > 0$. For more details on the domains of linear operators and their adjoints, see [E. O. Ayoola, Internat. J. Theoret. Phys. **43** (2004), no. 10, 2041–2059; MR2107450 (2005i:81076)].

According to the author's motivations, QSDI are involved in quantum stochastic control theory and quantum dynamical systems. Some results on the existence and non-uniqueness of solutions for a class of QSDI (2) had been given in a previous paper by G. O. S. Ekhaguere [Internat. J. Theoret. Phys. **31** (1992), no. 11, 2003–2027; MR1186301 (93k:81120)]. Here the author extends them by means of a more general Lipschitz condition.

Reviewed by *Vitonofrio Crismale*

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Note: This list reflects references listed in the original paper as accurately as possible with no attempt to correct errors.

MR2438149 (2009e:65016) 65C30 (60H15 60H35)
Njoseh, Ignatius N.; **Ayoola, Ezekiel O.** (WAN-IBAD)

Finite element method for a strongly damped stochastic wave equation driven by space-time noise. (English summary)

J. Math. Sci. (Dattapukur) **19** (2008), *no. 1,* 61–72.

The authors extend the works of B. Li [“Numerical method for a parabolic stochastic partial equation”, Masters thesis, Chalmers Univ. Tech./Univ. Göteborg, Göteborg, 2004, available at http://www.md.chalmers.se/EM-Masters/theses/archive/pdf/2004-03.pdf] and Y. B. Yan [BIT **44** (2004), no. 4, 829–847; MR2211047 (2007c:60065)].

The authors propose a finite element method for the strongly damped stochastic wave equation using low-order approximations (piecewise linear finite elements in space and backward Euler approximations in time).

Under the assumption the domain is bounded with a smooth border, the authors pose the problem, its finite element analysis, and close with error estimates on the L^2-norm.

Reviewed by *Gonçalo Dos Reis*

MR2438143 (2009h:60103) 60H10
Atonuje, A. O.; **Ayoola, E. O.** (WAN-IBAD)
On the complementary roles of noise and delay in the oscillatory behaviour of stochastic delay differential equations. (English summary)
J. Math. Sci. (Dattapukur) **19** (2008), *no. 1,* 11–20.

Summary: "We study a scalar linear stochastic delay differential equation (SDDE) with constant delay term and multiplicative noise. We analyze the effect that noise can have on the oscillatory behaviour of the solution of the SDDE. We prove that in the absence of the noise term, non-oscillatory solutions can occur for the deterministic case, but with the presence of the noise, all solutions of the SDDE oscillate almost certainly whenever the feedback intensity is negative. Hence delay and noise play complementary roles in the oscillatory behaviour of the solution of the SDDE."

MR2434230 (2009h:60114) 60H15 (35K55 35R60 65C30)
Njoseh, Ignatius N.; **Ayoola, Ezekiel O.** (WAN-IBAD)
On the finite element analysis of the stochastic Cahn-Hilliard equation. (English summary)
J. Inst. Math. Comput. Sci. Math. Ser. **21** (2008), *no. 1,* 47–53.

Summary: "We discuss the finite element analysis for the stochastic Cahn-Hilliard equation. Error estimates are established."

MR2366386 (2008k:81174) 81S25 (34A60 34F05)
Ayoola, E. O. (I-ICTP-MS)
Topological properties of solution sets of Lipschitzian quantum stochastic differential inclusions. (English summary)
Acta Appl. Math. **100** (2008), *no. 1,* 15–37.

The paper is devoted to the topological properties of solution sets of quantum stochastic differential inclusions (QSDI) within the framework of the Hudson-Parthasarathy formulation of boson stochastic calculus. A continuous mapping from the space of matrix elements of an arbitrary nonempty set of quasi-solutions into the space of the matrix elements of its solutions is given satisfying certain conditions. This mapping is used to establish that the space of matrix elements of the solutions is an absolute retract, leading further to the conclusion that this space is connected and contractible in some sense. This result generalizes the previous selection result of the authors by removing the requirement of compactness of the domain of the selection map. It will be used by the authors in their ongoing study of optimization problems for QSDI. The paper is rather technical, the main idea being to extend to the present quantum context the recent achievements (tools and results) in the theory of classical differential inclusions.

Reviewed by *Vassili N. Kolokolʹtsov*

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Note: This list reflects references listed in the original paper as accurately as possible with no attempt to correct errors.

MR2387478 34K50 (34K11 60H10)

Atonuje, A. O.; **Ayoola, E. O.** (WAN-IBAD)

On noise contribution to the oscillatory behaviour of solutions of stochastic delay differential equations. (English summary)

J. Inst. Math. Comput. Sci. Comput. Sci. Ser. **18** (2007), *no. 2,* 51–59.

MR2373411 (2008m:65012) 65C30 (60H99 65D05)

Ayoola, E. O. (WAN-IBAD); **Adeyeye, John O.**

Continuous interpolation of solution sets of Lipschitzian quantum stochastic differential inclusions. (English summary)

J. Appl. Math. Stoch. Anal. **2007,** *Art. ID* 80750, 12 *pp*.

Summary: "Given any finite set of trajectories of a Lipschitzian quantum stochastic differential inclusion (QSDI), there exists a continuous selection from the complex-valued multifunction associated with the solution set of the inclusion, interpolating the matrix elements of the given trajectories. Furthermore, the difference of any two of such solutions is bounded in the seminorm of the locally convex space of solutions."

MR2122403 (2005m:81179) 81S25 (60H99)

Ayoola, E. O. (WAN-IBAD); **Gbolagade, A. W.**

Further results on the existence, uniqueness and stability of strong solutions of quantum stochastic differential equations. (English summary)

Appl. Math. Lett. **18** (2005), *no. 2,* 219–227.

The article under review is a continuation of the authors' work on the existence, uniqueness and stability of strong solutions of quantum stochastic differential equations (q.s.d.e.) of the form

$$(*)\quad dX(t)=E(s,X(s))d\Lambda_\pi(s)+F(s,X(s))dA_f(s) \\ +G(s,X(s))dA_g^\dagger(s)+H(s,X(s))ds,\quad t\in[0,T],\ X(0)=X_0,$$

with nonlinear maps E, F, G, H satisfying suitable Lipschitz conditions. The reader may be referred to [K. R. Parthasarathy, *An introduction to quantum stochastic calculus*, Birkhäuser, Basel, 1992; MR1164866 (93g:81062)] for the meaning of the above equation and relevant details.

Let $\mathcal{D}$ be a dense subspace of a Hilbert space h and $\mathcal{E}$ be the linear span of exponential vectors of the form $e(\varphi)$, with $\varphi \in L^2(\mathbb{R}_+, \gamma)$, where γ is some fixed Hilbert space. The authors consider the linear space $\mathcal{B}$ consisting of all linear (possibly unbounded) operators S from $\mathcal{D} \otimes_{\rm alg} \mathcal{E}$ to $h \otimes \Gamma(L^2(\mathbb{R}_+, \gamma))$ such that $\mathcal{D} \otimes_{\rm alg} \mathcal{E}$ is contained in the domain of the adjoint of S. This space is made into a locally convex topological vector space by equipping it with a family of seminorms $\|\cdot\|_\xi$, $\xi \in \mathcal{D} \otimes_{\rm alg} \mathcal{E}$, given by $\|S\|_\xi = \|S\xi\|$. The completion of this locally convex space is denoted by $\tilde{\mathcal{B}}$. The main results obtained by the authors state that under a suitable Lipschitz property and local square integrability of the maps E, F, G, H from $[0, T] \times \tilde{\mathcal{B}}$ to $\tilde{\mathcal{B}}$, the q.s.d.e. $(*)$ admits unique strong solution and is also stable with respect to a change in the initial value X_0.

However, the arguments in the proof seem to be incomplete. For example, on page 223, no arguments have been given to prove that $R_\xi = \sup_{n \in \mathbb{N}} R_{\xi_n}$ is finite. It does not seem to be a trivial fact, and the proof does not go through if this quantity is infinite. It would have been better if the authors had explained this step in more detail.

Reviewed by *Debashish Goswami*

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Note: *This list reflects references listed in the original paper as accurately as possible with no attempt to correct errors.*

MR2283764 (2007i:81131) 81S25
Ayoola, E. O. (WAN-IBAD); **Gbolagade, A. W.**
On the existence of weak solutions of quantum stochastic differential equations. (English summary)
J. Niger. Assoc. Math. Phys. **8** (2004), 5–8.

Summary: "We establish further results concerning the existence, uniqueness and stability of weak solutions of quantum stochastic differential equations (QSDEs). Our results are achieved by considering a more general Lipschitz condition on the coefficients than our previous considerations in [E. O. Ayoola, Acta Appl. Math. **67** (2001), no. 1, 19–58; MR1847883 (2002f:65017)]. We exhibit a class of Lipschitzian QSDEs in the formulation of this paper, whose coefficients are only continuous on the locally convex space of the weak solution."

Reviewed by *B. V. Rajarama Bhat*

MR2107450 (2005i:81076) 81S25 (34A60)
Ayoola, E. O. (I-ICTP)
Continuous selections of solution sets of Lipschitzian quantum stochastic differential inclusions. (English summary)
Internat. J. Theoret. Phys. **43** (2004), *no. 10,* 2041–2059.

This is a continuation of work of the author and of Ekhaguere (upon which it is heavily reliant for notation) concerning solutions to quantum stochastic differential inclusions, i.e., adapted, operator-valued processes X such that

$$(1)\quad X(t) \in a + \int_0^t E(s,X(s))d\Lambda_\pi(s) + F(s,X(s))dA_f(s) \\ + G(s,X(s))dA_g^\dagger(s) + H(s,X(s))ds \quad \text{a.a. } t\in[0,T],$$

where E, F, G and H are multi-valued functions satisfying suitable regularity conditions, $T > 0$ is fixed, and the integral takes its sense from the Hudson-Parthasarathy theory of quantum stochastic calculus. The starting point of this programme is to rewrite (1) in terms of first-order ordinary differential inclusions associated to the matrix elements of a solution; see the previous article by the author [Stochastic Anal. Appl. **21** (2003), no. 3, 515–543; MR1978232 (2004e:81073)] and the references therein.

As is customary, linear operators and their adjoints are assumed to have domains containing $\mathbb{D}\underline{\otimes}\mathbb{E}$, the algebraic tensor product of $\mathbb{D}$, a dense subspace of some initial Hilbert space, and $\mathbb{E}$, the linear span of a suitable collection of exponential vectors in boson Fock space. A family A of such operators is fixed so that $\{\langle\eta, a\xi\rangle\colon\ a \in A\}$ is a compact set of complex numbers for all $\eta, \xi \in \mathbb{D}\underline{\otimes}\mathbb{E}$.

If $S(a)$ denotes the set of adapted, weakly absolutely continuous solutions to (1) and, for all η, $\xi \in \mathbb{D}\underline{\otimes}\mathbb{E}$,

$$S(a)(\eta,\xi) = \{X(\eta,\xi) = t \mapsto \langle\eta, X(t)\xi\rangle\colon\ X \in S(a)\} \subseteq \mathrm{AC}[0,T],$$

the absolutely continuous functions on $[0,T]$, then Theorem 3.1, the paper's main result, is as follows: if $X_0 \in S(a_0)$ for some $a_0 \in A$ then, for all η, $\xi \in \mathbb{D}\underline{\otimes}\mathbb{E}$, there exists a continuous map $W_{\eta,\xi}\colon A \to \mathrm{AC}[0,T]$ such that $W_{\eta,\xi}(a_0) = X_0(\eta,\xi)$ and $W_{\eta,\xi}(a) \in S(a)(\eta,\xi)$ for all $a \in A$.

{Reviewer's remarks: There seems to be a problem with Proposition 2.2: in its statement, one is invited to provide any partition satisfying certain hypotheses; the proof, however, proceeds by constructing a particular one. Happily, it does not appear that this affects the validity of Theorem 3.1, as the proposition is applied in a way which is compatible with the proof given.}

Reviewed by *Alexander C. R. Belton*

MR2014555 (2005a:60109) 60H99 (81S25)

Ayoola, E. O. (WAN-IBAD)

Error estimates for discretized quantum stochastic differential inclusions. (English summary)

Stochastic Anal. Appl. **21** (2003), *no. 6,* 1215–1230.

Summary: "This paper is concerned with the error estimates involved in the solution of a discrete approximation of a quantum stochastic differential inclusion (QSDI). Our main results rely on certain properties of the averaged modulus of continuity for multivalued sesquilinear forms associated with QSDI. We obtain results concerning the estimates of the Hausdorff distance between the set of solutions of the QSDI and the set of solutions of its discrete approximation. This extends the results of A. L. Dontchev and E. M. Farkhi [Computing **41** (1989), no. 4, 349–358; MR0993830 (90f:34023)] concerning classical differential inclusions to the present noncommutative quantum

setting involving inclusions in certain locally convex spaces.”

Reviewed by *Habib Ouerdiane*

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Note: This list reflects references listed in the original paper as accurately as possible with no attempt to correct errors.

MR1978232 (2004e:81073) 81S25 (60H10 65C99)
Ayoola, E. O. (I-ICTP)
Exponential formula for the reachable sets of quantum stochastic differential inclusions. (English summary)
Stochastic Anal. Appl. **21** (2003), *no. 3,* 515–543.

This paper is a continuation of the author's previous article [Stochastic Anal. Appl. **19** (2001), no. 3, 461–471; MR1841541 (2002f:65018)] concerning quantum stochastic differential inclusions (QSDI for short) of the form

$$(*)\quad dX(t)\in E(t,X(t))d\Lambda_\pi(t)+F(t,X(t))dA_f(t)+ G(t,X(t))dA_g^\dagger(t)+H(t,X(t))dt;\quad X(0)=X_0;$$

where E,F,G,H are assumed to have suitable regularity properties. The basic motivation behind considering such QSDI comes from the need to develop a reasonable numerical scheme for solving quantum stochastic differential equations (QDSE) with discontinuous coefficients, since many such interesting QSDE can be reformulated in some sense as QSDI with regular coefficients.

The basic set-up of the paper is that of multivalued functions (multifunctions for short). The author's article mentioned above may serve as a good reference for the notation and terminology used in the present paper. Given an "initial Hilbert space" $\mathfrak{h}$ and a "noise space" $\mathfrak{k}$, and some suitable subspaces $\mathcal{D}$ of $\mathfrak{h}$ and $\mathcal{E}$ of the symmetric Fock space $\Gamma(L^2(\mathbf{R}_+,\mathfrak{k}))$ spanned by exponential vectors $\{e(\alpha)\}$ with α varying over a suitable subset of $L^2(\mathbf{R}_+,\mathfrak{k})$ let $\tilde{\mathcal{A}}$ denote the locally convex "state space" of noncommutative stochastic processes having $\mathcal{D}\otimes\mathcal{E}$ in the domain, as described in [op. cit.] for example. Fix $T>0$, and let us assume that the coefficients E,F,G,H lie in $L^2([0,T],\tilde{\mathcal{A}})_{\rm mvs}$ in the notation of [op. cit.]. Given α,β in a suitable subset of $L^2(\mathbf{R}_+,\mathfrak{k})$, the author defines a canonical multifunction $P_{\alpha,\beta}\colon [0,T]\times\tilde{\mathcal{A}}\to 2^{\tilde{\mathcal{A}}}$ (see also [op. cit.]), and using it, he defines the multifunction $P\colon[0,T]\times\tilde{\mathcal{A}}\to 2^{{\rm sesq}(\mathcal{D}\otimes\mathcal{E})}$ by

$$P(t,x)(\eta,\xi)=\{\langle\eta,Z(t,x)\xi\rangle\colon Z(t,x)\in P_{\alpha,\beta}(t,x)\},$$

where $\eta=u\otimes e(\alpha)$, $\xi=v\otimes e(\beta)$ for some $u,v\in\mathcal{D}$. Note that ${\rm sesq}(V)$ for a vector space V denotes the space of sesquilinear maps on $V\times V$, and for any set A, 2^A denotes the power set of A.

In the above notation, let us now state the main result of the paper. The set $R^{(T)}(X_0)$ consisting of all $X(T)$ such that $X(\cdot)$ is a solution of $(*)$ is called the "reachable set" for the QSDI $(*)$, and is of special importance. The main result states that

$$R^{(T)}(X_0)=\lim_{N\to\infty}(I+\tfrac{T}{N}P)^N(X_0),$$

where I is the identity multifunction $x\mapsto\{x\}$, repeated composition of multifunctions is understood in a suitable way described in the paper, and the limit in the above formula has to be interpreted as the Kuratowski limit of sets. Using the above formula, some results concerning convergence of the discrete approximations to the reachable sets are obtained, generalising similar results for classical differential inclusions.

Reviewed by *Debashish Goswami*

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165–182. MR1042664 (91c:49009)
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Note: This list reflects references listed in the original paper as accurately as possible with no attempt to correct errors.

MR1900360 (2003b:60081) 60H10 (60H20 81S25)
Ayoola, E. O. (WAN-IBAD)
Existence and stability results for strong solutions of Lipschitzian quantum stochastic differential equations. (English summary)
Stochastic Anal. Appl. **20** (2002), *no. 2,* 263–281.

Some results on existence, uniqueness and stability of strong solutions of Lipschitzian quantum stochastic differential equations in a locally convex space are obtained by a method of successive approximations. It is shown that these results generalize the analogous results on classical stochastic differential equations driven by a Brownian motion. The construction of the quantum stochastic integral of Hudson and Parthasarathy is briefly reviewed at the beginning of the paper.

Reviewed by *Vassili N. Kolokol′tsov*

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MR1897940 (2003e:60121) 60H10 (60H20 65C30 81S25)

Ayoola, E. O. (WAN-IBAD)

Lagrangian quadrature schemes for computing weak solutions of quantum stochastic differential equations. (English summary)

SIAM J. Numer. Anal. **39** (2002), *no. 6,* 1835–1864 (*electronic*).

The paper is devoted to the analysis of the Lagrangian quadrature schemes for computing weak solutions of Lipschitzian quantum stochastic differential equations with matrix elements that are smooth enough. Results on the convergence of such schemes to a solution are obtained. Precise estimates for an error term are given in the case when the nodes of approximations are chosen to be the roots of the Chebyshev polynomials. Numerical examples are presented.

Reviewed by *Vassili N. Kolokol′tsov*

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MR1891476 (2002m:81124) 81S25 (60H20)

Ayoola, E. O. (WAN-IBAD)

On computational procedures for weak solutions of quantum stochastic differential equations. (English summary)

Stochastic Anal. Appl. **20** (2002), *no. 1,* 1–20.

Summary: "A continuous time Euler approximation scheme and a computational theorem for weak solutions of Lipschitzian quantum stochastic differential equations (QSDEs) are established. The work is accomplished within the framework of the Hudson-Parthasarathy formulation of quantum stochastic calculus and subject to the equivalent forms of the equations satisfying the Carathéodory conditions. Our results generalize analogous results concerning classical Itô stochastic differential equations and those on differential equations in a Banach space context to the noncommutative quantum setting involving unbounded linear operators on a Hilbert space. Numerical examples are given."

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MR1847883 (2002f:65017) 65C30 (60H10 60H20 60H35)

Ayoola, E. O. (WAN-IBAD)

On convergence of one-step schemes for weak solutions of quantum stochastic differential equations. (English summary)

Acta Appl. Math. **67** (2001), *no. 1,* 19–58.

Summary: "Several one-step schemes for computing weak solutions of Lipschitzian quantum stochastic differential equations driven by certain operator-valued stochastic processes associated with creation, annihilation and gauge operators of quantum field theory are introduced and studied. This is accomplished within the framework of the Hudson-Parthasarathy formulation of quantum stochastic calculus and subject to the matrix elements of the solution being sufficiently differen-

tiable. Results concerning convergence of these schemes in the topology of the locally convex space of solution are presented. It is shown that the Euler-Maruyama scheme, with respect to weak convergence criteria for Itô stochastic differential equation, is a special case of Euler schemes in this framework. Numerical examples are given."

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MR1841945 (2002g:81078) 81S25 (60H10 65C30)
Ayoola, E. O. (WAN-IBAD)
Lipschitzian quantum stochastic differential equations and the associated Kurzweil equations. (English summary)
Stochastic Anal. Appl. **19** (2001), *no. 4,* 581–603.

In the article under review, Kurzweil equations [cf. J. Kurzweil, Czechoslovak Math. J. **7 (82)** (1957), 418–449; MR0111875 (22 #2735)] associated with Lipschitzian quantum stochastic differential equations (QSDE's) are introduced and studied. After a nice discussion of Kurzweil integrals and Kurzweil equations, including some useful technical results, the author proves the interesting equivalence between the QSDE of the form

$$X(t) = X_0 + \int_{t_0}^{t} (E(X(s),s)d\Lambda_\pi(s) + F(X(s),s)dA_f(s) + G(X(s),s)dA_g^\dagger(s) + H(X(s),s)ds), t \in [t_0,T],$$

and the Kurzweil equation of the form

$$\frac{d}{d\tau}\langle \eta, X(\tau)\xi\rangle = D\Phi(X(\tau),t)(\eta,\xi)$$

on $[t_0,T]$ and for $t \in [t_0,T]$, for a suitable map Φ and ξ,η belonging to an appropriate class. In the above, $X\colon [t_0,T] \to \tilde{\mathcal{A}}$, where $\tilde{\mathcal{A}}$ is as in references [E. O. Ayoola, "On numerical procedures for solving Lipschitzian quantum stochastic differential equations", Ph.D. Thesis, Univ.

Ibadan, Nigeria, 1995; per bibl.; Stochastic Anal. Appl. **18** (2000), no. 4, 525–554; MR1763939 (2001e:81065)] of the article under review, and E, F, G, H, π, f, g satisfy suitable properties described carefully in those references. Finally, some numerical examples are given to demonstrate the effectiveness of the methods using Kurzweil equations.

Reviewed by *Debashish Goswami*

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MR1841541 (2002f:65018) 65C30 (60H10 81S25)

Ayoola, E. O. (WAN-IBAD)

Construction of approximate attainability sets for Lipschitzian quantum stochastic differential inclusions. (English summary)

Stochastic Anal. Appl. **19** (2001), *no. 3,* 461–471.

The author considers solving, numerically, quantum stochastic integral inclusions or, equivalently, certain first-order (non-classical) ordinary differential inclusions. The solutions are represented with the help of what are called attainable sets (non-void closed sets in the complex plane). An algorithm is described for numerically approximating the attainable sets within any prescribed accuracy.

Reviewed by *Volker Wihstutz*

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MR1991401 (2004e:81072) 81S25 (60H10 60H20 65C30 65L06)

Ayoola, E. O. (WAN-IBAD)

Convergence and stability of general multistep schemes for weak solutions of quantum stochastic differential equations. (English summary)

Ordinary differential equations (Abuja, 2000), 43–55, *Proc. Natl. Math. Cent. Abuja Niger.*, 1.1, *Natl. Math. Cent., Abuja*, 2000.

Consider the weak integral formulation of the initial value problem related to quantum stochastic differential equations (QSDEs) of the form

$$dX(t) = E(t, X(t))d\Lambda_\pi(t) + F(t, X(t))dA_f^\dagger(t) + G(t, X(t))dA_g(t) + H(t, X(t))dt$$

with $t \in [t_0, T]$ and initial condition $X(t_0) = X_0$, where $\Lambda_\pi, A_f^\dagger, A_g$ are the stochastic integrators in the boson Fock quantum stochastic calculus. The author presents convergence results for some equidistant, implicit, multi-step numerical approximations to weak solutions to such QSDEs in integral form. The main convergence result as the nonrandom step size h tends to 0 is established with respect to some appropriate seminorms within the framework of the Hudson-Parthasarathy formulation of QSDEs and under Lipschitz-type conditions on the equivalent initial value problem of a related nonclassical ordinary differential equation. The effect of round-off errors is also taken into account. However, a considerable effort is needed on the part of any non-quantum-calculus specialist reader to understand the author's notation.

{For the entire collection see MR1991396 (2004b:34005)}

Reviewed by *Henri Schurz*

MR1763939 (2001e:81065) 81S25 (60H35)

Ayoola, E. O. (WAN-IBAD)

Converging multistep schemes for weak solutions of quantum stochastic differential equations. (English summary)

Stochastic Anal. Appl. **18** (2000), *no. 4,* 525–554.

The author proposes an algorithm for solving numerically a quantum stochastic differential equation of the form $dX(t) = E(t, X(t))d\Lambda_\pi(t) + F(t, X(t))dA_f^\dagger(t) + G(t, X(t))dA_g(t) + H(t, X(t))dt$ in an interval $[t_0, T]$ with initial condition $X(t_0) = X_0$, where $\Lambda_\pi, A_f^\dagger, A_g$ are the stochastic integrators in the boson Fock quantum stochastic calculus. The exact formulation and the solution of the problem depend heavily on the author's Ph.D. thesis. Considerable effort would be needed on the part of the reader to understand the notation and the statement of the main result.

Reviewed by *Kalyanapuram R. Parthasarathy*

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Note: This list reflects references listed in the original paper as accurately as possible with no attempt to correct errors.

AMERICAN MATHEMATICAL SOCIETY
MathSciNet
Mathematical Reviews

MR3758107 81S25 60H99 81P20
Dikko, D. A. [**Dikko Dauda, A.**] (WAN-IBAD);
Ayoola, E. O. [**Ayoola, Ezekiel O.**] (WAN-IBAD)

Arcwise connectedness of solution sets of Lipschitzian quantum stochastic differential inclusions. (English summary)

J. Phys. Conf. Ser. **819** (2017), 012005, 13 *pp.*

Given η, ξ in the space generated by the exponential vectors in the Fock space, the authors consider the set of functions $[0, T] \ni t \mapsto \langle \eta, \Phi(\cdot)\xi \rangle$, where Φ runs through the adapted noncommutative solutions of a quantum stochastic differential inclusion. Based on earlier selection results for multivalued stochastic processes obtained by the second author, they show that this set is arcwise connected in $C([0, T]; \mathbb{C})$. *Nicolas Privault*

AMERICAN MATHEMATICAL SOCIETY
MathSciNet
Mathematical Reviews

MR3683008 81S25
Bishop, S. A. [**Bishop, Sheila Amina**] (WAN-COV-M);
Ayoola, E. O. [**Ayoola, Ezekiel O.**] (WAN-IBAD);
Oghonyon, G. J. (WAN-COV-M)

Existence of mild solution of impulsive quantum stochastic differential equation with nonlocal conditions. (English summary)

Anal. Math. Phys. **7** (2017), *no. 3,* 255–265.

Summary: "New results on existence and uniqueness of solution of impulsive quantum stochastic differential equation with nonlocal conditions are established. The nonlocal conditions are completely continuous. The methods applied here are simple extension of the methods applied in the classical case to this noncummutative quantum setting."

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MR3644986 81S25 60H15

Ayoola, E. O. [**Ayoola, Ezekiel O.**] (WAN-IBAD);
Ajibola, S. A. [**Ajibola, S. Adetona**] (WAN-POLYI-MS)

★**Existence and uniqueness of solutions of a class of quantum stochastic partial differential equations. (English summary)**

Dynamic systems and applications. Vol. 7, 45–50, *Dynamic*, *Atlanta*, *GA*, 2016.

Summary: "By employing the theory of iterated stochastic integration with respect to quantum martingale measures taking values in a linear space $\mathcal{A}$ of unbounded linear operators on a Hilbert space, we present a rigorous formulation of quantum stochastic partial differential equations (QSPDE). The solutions of certain classes of these equations are closable operators and they are known to provide examples of irreversible quantum dynamics which have found applications as models of open quantum systems and models of electric currents in neutrons among many other applications. Existence and uniqueness of a class of semi-linear quantum stochastic partial differential equations are studied."

{For the collection containing this paper see MR3560307}

MR3459309 60H10 60H20 65C30 81S25

Bishop, S. A. [**Bishop, Sheila Amina**] (WAN-COV-M);
Ayoola, E. O. [**Ayoola, Ezekiel O.**] (WAN-IBAD)

On topological properties of solution sets of non Lipschitzian quantum stochastic differential inclusions. (English summary)

Anal. Math. Phys. **6** (2016), *no. 1,* 85–94.

Summary: "In this paper, we establish results on continuous mappings of the space of the matrix elements of an arbitrary nonempty set of pseudo solutions of non Lipschitz quantum Stochastic differential inclusion (QSDI) into the space of the matrix elements of its solutions. We show that under the non Lipschitz condition, the space of the matrix elements of solutions is still an absolute retract, contractible, locally and integrally connected in an arbitrary dimension. The results here generalize existing results in the literature."

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chastic differential inclusions. Acta. Appl. Math. **100**, 15–37 (2006) MR2366386
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MR3025559 81S25 60H10
Ogundiran, M. O. **[Ogundiran, Michael Oluniyi]** (WAN-IFE);
Ayoola, E. O. **[Ayoola, Ezekiel O.]** (WAN-IBAD)
Caratheodory solution of quantum stochastic differential inclusions. (English summary)
J. Nigerian Math. Soc. **31** (2012), 81–90.

The research conducted in the paper considers a class of quantum stochastic differential inclusions, i.e., inclusions having as their coefficients multivalued stochastic processes that are non-convex and Scorza-Dragoni lower semi-continuous. Furthermore, the inclusions are driven by annihilation, creation, and gauge operators. The major purpose of the paper is to establish the existence of solutions of the inclusions in the sense of Carathéodory. *Wan-yang Dai*

MR2953241 81S25 34A60 34F05
Ogundiran, M. O. **[Ogundiran, Michael Oluniyi]** (WAN-IFE);
Ayoola, E. O. **[Ayoola, Ezekiel O.]** (WAN-IBAD)
Upper semicontinuous quantum stochastic differential inclusions via Kakutani-Fan fixed point theorem. (English summary)
Dynam. Systems Appl. **21** (2012), *no. 1,* 121–132.

The existence and uniqueness of solutions of quantum stochastic differential equations was first obtained, in the case of bounded coefficients, by Hudson and Parthasarathy and later extended, to unbounded coefficients, by Franco Fagnola. Existence was extended to quantum stochastic differential inclusions by Ekhaguere. The authors extend the results to discontinuous quantum stochastic differential inclusions with upper semicontinuous coefficients. In the process, they prove a noncommutative generalization of the Kakutani-Fan fixed point theorem. *Andreas Boukas*

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MR2905708 81S25 34A60 60H99
Ogundiran, M. O. [**Ogundiran, Michael Oluniyi**] (WAN-IFE);
Ayoola, E. O. [**Ayoola, Ezekiel O.**] (WAN-IBAD)
An application of Michael selection theorems. (English summary)
Int. J. Funct. Anal. Oper. Theory Appl. **4** (2012), *no. 1,* 65–79.

Michael's selection theorem for lower semicontinuous multifunctions is a famous result used in the solution of differential inclusions. In this paper the authors attempt to extend this idea to lower semicontinuous multivalued stochastic processes. The first of the main results is as follows:

Assume the following hold:

(1) For arbitrary $\eta, \xi \in (D\underline{\otimes}E)$, the map $(t,x) \to G(t,x)(\eta,\xi)$ is lower semicontinuous with respect to a seminorm.

(2) $g\colon I \times \widetilde{A} \to \mathrm{sesq}(D\underline{\otimes}E)$ is continuous.

(3) $\varepsilon\colon \widetilde{A} \to \mathbb{R}_+$ is lower semicontinuous.

Then the map $(t,x) \to \Phi(t,x)(\eta,\xi)$ defined by

$$\Phi(t,x)(\eta,\xi) = B_{\varepsilon(x)}(g(t,x)(\eta,\xi)) \cap G(t,x)(\eta,\xi)$$

is lower semicontinuous with respect to the seminorm. Here $\widetilde{A}$ is the completion of a certain locally convex space.

The above result and another proposition are used to establish the main theorem below, which is a noncommutative extension to Michael's selection theorem.

Theorem: Suppose that $\Psi\colon I \times \widetilde{A} \to 2^{\mathrm{sesq}(D\underline{\otimes}E)^2}$ is a multivalued stochastic process such that for arbitrary elements $\eta, \xi \in (D\underline{\otimes}E)$,

(1) $(t,x) \to \Psi(t,x)(\eta,\xi)$ is lower semicontinuous with respect to a seminorm,

(2) Ψ is closed and convex-valued.

Then there exists $\psi\colon I \times \widetilde{A} \to \mathrm{sesq}(D\underline{\otimes}E)^2$, which is a continuous selection from Ψ.

A corollary provides an application to quantum stochastic evolution inclusions which requires an understanding of hypermaximal monotone multifunctions.

John S. Spraker

MR2905434 81S25 60H10

Ogundiran, M. O. **[Ogundiran, Michael Oluniyi]** (WAN-IFE);
Ayoola, E. O. **[Ayoola, Ezekiel O.]** (WAN-IBAD)

Directionally continuous quantum stochastic differential equations. (English summary)

Far East J. Appl. Math. **57** (2011), *no. 1,* 33–48.

The authors show that the following Hudson-Parthasarathy type quantum stochastic differential equation with directionally continuous coefficients:

$$dX(t) = E(t, X(t))\, d\Lambda_\pi(t) + F(t, X(t))\, dA_f(t) + G(t, X(t))\, dA_g^+(t) + H(t, X(t))\, dt,$$

with initial condition

$$X(t_0) = x_0,$$

admits a solution in the sense of Carathéodory. They also show that, under certain regularity conditions, the solution of the directionally continuous quantum stochastic differential equation

$$\frac{d}{dt}\langle \eta, X(t)\xi\rangle = P(t, X(t))(\eta, \xi)$$

coincides with the solution set of the upper semicontinuous quantum stochastic differential inclusion

$$\frac{d}{dt}\langle \eta, X(t)\xi\rangle \in \mathbb{P}(t, X(t))(\eta, \xi).$$

Andreas Boukas

AMERICAN MATHEMATICAL SOCIETY
MathSciNet
Mathematical Reviews

Citations	From References: 1	From Reviews: 0

MR2605072 (2011d:81180) 81S25 81Q93

Ogundiran, M. O. **[Ogundiran, Michael Oluniyi]** (WAN-IFE);
Ayoola, E. O. **[Ayoola, Ezekiel O.]** (WAN-IBAD)

Mayer problem for quantum stochastic control. (English summary)

J. Math. Phys. **51** (2010), *no. 2,* 023521, 8 *pp.*

Motivated by Ekhaguere's multi-valued analog of Hudson-Parthasarathy quantum stochastic calculus, the authors study quantum stochastic control via set-valued analysis. In particular, they study the regularity properties of the value function inherited from the multi-valued stochastic processes involved, and they show that, under the assumption of directional differentiability of the value function, the associated Mayer problem has at least one optimal solution. The authors' theory covers earlier work on quantum stochastic control by Belavkin and by the reviewer. *Andreas Boukas*

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ential inclusions," Acta Appl. Math. **100**. 15 (2008). MR2366386
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MR2569523 (2011d:81177) 81S25 60H10
Ayoola, E. O. **[Ayoola, Ezekiel O.]** (WAN-IBAD)
Further results on the existence of continuous selections of solution sets of quantum stochastic differential inclusions. (English summary)
Dynam. Systems Appl. **17** (2008), *no. 3-4,* 609–624.

Quantum stochastic differential inclusions (QSDI) are adapted operator-valued processes X satisfying

$$X(t) \in X_0 + \int_0^t E(s,X(s))d\Lambda_\pi(s) + F(s,X(s))dA_f(s) \\ + G(s,X(s))dA_g^+(s) + H(s,X(s)), \quad t \in [0,T],$$

where E, F, G, H are maps endowed with some regularity, the integral is in the sense of Hudson-Parthasarathy and $T > 0$. For more details on the domains of linear operators and their adjoints, see [E. O. Ayoola, Internat. J. Theoret. Phys. **43** (2004), no. 10, 2041–2059; MR2107450].

In the paper under review, the author proves the existence of a selection of solution sets of QSDI, which is continuous from a compact set of initial values (included in the locally convex space of stochastic processes) into the locally convex space of adapted and weakly continuous quantum stochastic processes.

The main result is in the spirit of [op. cit.], in which the author established the existence of continuous selections of solution sets of QSDI from the set of the matrix elements of initial points to the set of matrix elements of solutions.

Vitonofrio Crismale

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MR2569515 (2011d:81176) 81S25 60H10 81Q93
Ayoola, E. O. **[Ayoola, Ezekiel O.]** (WAN-IBAD)
Quantum stochastic differential inclusions satisfying a general Lipschitz condition. (English summary)
Dynam. Systems Appl. **17** (2008), *no. 3-4,* 487–502.

The quantum stochastic differential inclusions (QSDI) studied here are adapted operator-valued processes X satisfying

$$(2)\quad X(t)\in X_0+\int_0^t E(s,X(s))d\Lambda_\pi(s)+F(s,X(s))dA_f(s) \\ +G(s,X(s))dA_g^+(s)+H(s,X(s)),\quad t\in[0,T],$$

where E, F, G, H are maps endowed with some regularity, the integral is àla Hudson-Parthasarathy and $T>0$. For more details on the domains of linear operators and their adjoints, see [E. O. Ayoola, Internat. J. Theoret. Phys. **43** (2004), no. 10, 2041–2059; MR2107450].

According to the author's motivations, QSDI are involved in quantum stochastic control theory and quantum dynamical systems. Some results on the existence and non-uniqueness of solutions for a class of QSDI (2) had been given in a previous paper by G. O. S. Ekhaguere [Internat. J. Theoret. Phys. **31** (1992), no. 11, 2003–2027; MR1186301]. Here the author extends them by means of a more general Lipschitz condition.

Vitonofrio Crismale

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Note: This list reflects references listed in the original paper as accurately as possible with no attempt to correct errors.

Citations	From References: 0	From Reviews: 0

MR2438143 (2009h:60103) 60H10

Atonuje, A. O. [Atonuje, Augustine Omoghaghare];
Ayoola, E. O. [Ayoola, Ezekiel O.] (WAN-IBAD)

On the complementary roles of noise and delay in the oscillatory behaviour of stochastic delay differential equations. (English summary)

J. Math. Sci. (Dattapukur) **19** (2008), *no. 1,* 11–20.

Summary: "We study a scalar linear stochastic delay differential equation (SDDE) with constant delay term and multiplicative noise. We analyze the effect that noise can have on the oscillatory behaviour of the solution of the SDDE. We prove that in the absence of the noise term, non-oscillatory solutions can occur for the deterministic case, but with the presence of the noise, all solutions of the SDDE oscillate almost certainly whenever

the feedback intensity is negative. Hence delay and noise play complementary roles in the oscillatory behaviour of the solution of the SDDE."

MR2366386 (2008k:81174) 81S25 34A60 34F05
Ayoola, E. O. **[Ayoola, Ezekiel O.]** (I-ICTP-MS)
Topological properties of solution sets of Lipschitzian quantum stochastic differential inclusions. (English summary)
Acta Appl. Math. **100** (2008), *no. 1,* 15–37.

The paper is devoted to the topological properties of solution sets of quantum stochastic differential inclusions (QSDI) within the framework of the Hudson-Parthasarathy formulation of boson stochastic calculus. A continuous mapping from the space of matrix elements of an arbitrary nonempty set of quasi-solutions into the space of the matrix elements of its solutions is given satisfying certain conditions. This mapping is used to establish that the space of matrix elements of the solutions is an absolute retract, leading further to the conclusion that this space is connected and contractible in some sense. This result generalizes the previous selection result of the authors by removing the requirement of compactness of the domain of the selection map. It will be used by the authors in their ongoing study of optimization problems for QSDI. The paper is rather technical, the main idea being to extend to the present quantum context the recent achievements (tools and results) in the theory of classical differential inclusions.
Vassili N. Kolokol′tsov

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Note: This list reflects references listed in the original paper as accurately as possible with no attempt to correct errors.

MR2387478 34K50 34K11 60H10
Atonuje, A. O. [Atonuje, Augustine Omoghaghare];
Ayoola, E. O. [Ayoola, Ezekiel O.] (WAN-IBAD)
On noise contribution to the oscillatory behaviour of solutions of stochastic delay differential equations. (English summary)
J. Inst. Math. Comput. Sci. Comput. Sci. Ser. **18** (2007), *no. 2,* 51–59.

MR2373411 (2008m:65012) 65C30 60H99 65D05
Ayoola, E. O. [Ayoola, Ezekiel O.] (WAN-IBAD); **Adeyeye, John O.** (1-WSSU-M)
Continuous interpolation of solution sets of Lipschitzian quantum stochastic differential inclusions. (English summary)
J. Appl. Math. Stoch. Anal. **2007,** *Art. ID* 80750, 12 *pp.*

Summary: "Given any finite set of trajectories of a Lipschitzian quantum stochastic differential inclusion (QSDI), there exists a continuous selection from the complex-valued multifunction associated with the solution set of the inclusion, interpolating the matrix elements of the given trajectories. Furthermore, the difference of any two of such solutions is bounded in the seminorm of the locally convex space of solutions."

MR2122403 (2005m:81179) 81S25 60H99
Ayoola, E. O. [Ayoola, Ezekiel O.] (WAN-IBAD); **Gbolagade, A. W.**
Further results on the existence, uniqueness and stability of strong solutions of quantum stochastic differential equations. (English summary)
Appl. Math. Lett. **18** (2005), *no. 2,* 219–227.

The article under review is a continuation of the authors' work on the existence, uniqueness and stability of strong solutions of quantum stochastic differential equations (q.s.d.e.) of the form

$$(*)\quad dX(t)=E(s,X(s))d\Lambda_\pi(s)+F(s,X(s))dA_f(s) \\ +G(s,X(s))dA_g^\dagger(s)+H(s,X(s))ds,\quad t\in[0,T],\ X(0)=X_0,$$

with nonlinear maps E,F,G,H satisfying suitable Lipschitz conditions. The reader may be referred to [K. R. Parthasarathy, *An introduction to quantum stochastic calculus*, Birkhäuser, Basel, 1992; MR1164866] for the meaning of the above equation and relevant details.

Let $\mathcal{D}$ be a dense subspace of a Hilbert space h and $\mathcal{E}$ be the linear span of exponential vectors of the form $e(\phi)$, with $\phi\in L^2(\mathbb{R}_+,\gamma)$, where γ is some fixed Hilbert space. The authors consider the linear space $\mathcal{B}$ consisting of all linear (possibly unbounded) operators S from $\mathcal{D} \otimes_{\text{alg}} \mathcal{E}$ to $h \otimes \Gamma(L^2(\mathbb{R}_+, \gamma))$ such that $\mathcal{D} \otimes_{\text{alg}} \mathcal{E}$ is contained in the domain of the adjoint of S. This space is made into a locally convex topological vector space by equipping it with a family of seminorms $\|\cdot\|_\xi$, $\xi \in \mathcal{D} \otimes_{\text{alg}} \mathcal{E}$, given by $\|S\|_\xi = \|S\xi\|$. The completion of this locally convex space is denoted by $\tilde{\mathcal{B}}$. The main results obtained by the authors state that under a suitable Lipschitz property and local square integrability of the maps E, F, G, H from $[0,T] \times \tilde{\mathcal{B}}$ to $\tilde{\mathcal{B}}$, the q.s.d.e. $(*)$ admits unique strong solution and is also stable with respect to a change in the initial value X_0.

However, the arguments in the proof seem to be incomplete. For example, on page 223, no arguments have been given to prove that $R_\xi = \sup_{n\in\mathbb{N}} R_{\xi_n}$ is finite. It does not seem to be a trivial fact, and the proof does not go through if this quantity is infinite. It would have been better if the authors had explained this step in more detail.

Debashish Goswami

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Note: This list reflects references listed in the original paper as accurately as possible with no attempt to correct errors.

MR2283764 (2007i:81131) 81S25

Ayoola, E. O. [**Ayoola, Ezekiel O.**] (WAN-IBAD); **Gbolagade, A. W.**

On the existence of weak solutions of quantum stochastic differential equations. (English summary)

J. Niger. Assoc. Math. Phys. **8** (2004), 5–8.

Summary: "We establish further results concerning the existence, uniqueness and stability of weak solutions of quantum stochastic differential equations (QSDEs). Our results are achieved by considering a more general Lipschitz condition on the coefficients than our previous considerations in [E. O. Ayoola, Acta Appl. Math. **67** (2001), no. 1, 19–58; MR1847883]. We exhibit a class of Lipschitzian QSDEs in the formulation of this paper, whose coefficients are only continuous on the locally convex space of the weak solution."

B. V. Rajarama Bhat

MR2107450 (2005i:81076) 81S25 34A60

Ayoola, E. O. [**Ayoola, Ezekiel O.**] (I-ICTP)

Continuous selections of solution sets of Lipschitzian quantum stochastic differential inclusions. (English summary)

Internat. J. Theoret. Phys. **43** (2004), *no. 10,* 2041–2059.

This is a continuation of work of the author and of Ekhaguere (upon which it is heavily reliant for notation) concerning solutions to quantum stochastic differential inclusions, i.e., adapted, operator-valued processes X such that

$$(1)\quad X(t)\in a+\int_0^t E(s,X(s))d\Lambda_\pi(s)+F(s,X(s))dA_f(s) \\ +G(s,X(s))dA_g^\dagger(s)+H(s,X(s))ds \quad \text{a.a. } t\in[0,T],$$

where E, F, G and H are multi-valued functions satisfying suitable regularity conditions, $T>0$ is fixed, and the integral takes its sense from the Hudson-Parthasarathy theory of quantum stochastic calculus. The starting point of this programme is to rewrite (1) in terms of first-order ordinary differential inclusions associated to the matrix elements of a solution; see the previous article by the author [Stochastic Anal. Appl. **21** (2003), no. 3, 515–543; MR1978232] and the references therein.

As is customary, linear operators and their adjoints are assumed to have domains containing $\mathbb{D}\underline{\otimes}\mathbb{E}$, the algebraic tensor product of $\mathbb{D}$, a dense subspace of some initial Hilbert space, and $\mathbb{E}$, the linear span of a suitable collection of exponential vectors in boson Fock space. A family A of such operators is fixed so that $\{\langle\eta,a\xi\rangle\colon a\in A\}$ is a compact set of complex numbers for all η, $\xi\in\mathbb{D}\underline{\otimes}\mathbb{E}$.

If $S(a)$ denotes the set of adapted, weakly absolutely continuous solutions to (1) and, for all η, $\xi\in\mathbb{D}\underline{\otimes}\mathbb{E}$,

$$S(a)(\eta,\xi)=\{X(\eta,\xi)=t\mapsto\langle\eta,X(t)\xi\rangle\colon\ X\in S(a)\}\subseteq \mathrm{AC}[0,T],$$

the absolutely continuous functions on $[0,T]$, then Theorem 3.1, the paper's main result, is as follows: if $X_0\in S(a_0)$ for some $a_0\in A$ then, for all η, $\xi\in\mathbb{D}\underline{\otimes}\mathbb{E}$, there exists a continuous map $W_{\eta,\xi}\colon A\to\mathrm{AC}[0,T]$ such that $W_{\eta,\xi}(a_0)=X_0(\eta,\xi)$ and $W_{\eta,\xi}(a)\in$

$S(a)(\eta, \xi)$ for all $a \in A$.
{Reviewer's remarks: There seems to be a problem with Proposition 2.2: in its statement, one is invited to provide any partition satisfying certain hypotheses; the proof, however, proceeds by constructing a particular one. Happily, it does not appear that this affects the validity of Theorem 3.1, as the proposition is applied in a way which is compatible with the proof given.} *Alexander C. R. Belton*

MR2014555 (2005a:60109) 60H99 81S25
Ayoola, E. O. **[Ayoola, Ezekiel O.]** (WAN-IBAD)
Error estimates for discretized quantum stochastic differential inclusions. (English summary)
Stochastic Anal. Appl. **21** (2003), *no. 6,* 1215–1230.

Summary: "This paper is concerned with the error estimates involved in the solution of a discrete approximation of a quantum stochastic differential inclusion (QSDI). Our main results rely on certain properties of the averaged modulus of continuity for multivalued sesquilinear forms associated with QSDI. We obtain results concerning the estimates of the Hausdorff distance between the set of solutions of the QSDI and the set of solutions of its discrete approximation. This extends the results of A. L. Dontchev and E. M. Farkhi [Computing **41** (1989), no. 4, 349–358; MR0993830] concerning classical differential inclusions to the present noncommutative quantum setting involving inclusions in certain locally convex spaces." *Habib Ouerdiane*

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tions. Commun. Math. Phys. **1984**, *93*, 301–324. MR0745686
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Note: This list reflects references listed in the original paper as accurately as possible with no attempt to correct errors.

MR1978232 (2004e:81073) 81S25 60H10 65C99
Ayoola, E. O. **[Ayoola, Ezekiel O.]** (I-ICTP)
Exponential formula for the reachable sets of quantum stochastic differential inclusions. (English summary)
Stochastic Anal. Appl. **21** (2003), *no. 3,* 515–543.

This paper is a continuation of the author's previous article [Stochastic Anal. Appl. **19** (2001), no. 3, 461–471; MR1841541] concerning quantum stochastic differential inclusions (QSDI for short) of the form

$$(*)\quad dX(t)\in E(t,X(t))d\Lambda_\pi(t)+F(t,X(t))dA_f(t)+ G(t,X(t))dA_g^\dagger(t)+H(t,X(t))dt;\quad X(0)=X_0;$$

where E,F,G,H are assumed to have suitable regularity properties. The basic motivation behind considering such QSDI comes from the need to develop a reasonable numerical scheme for solving quantum stochastic differential equations (QDSE) with discontinuous coefficients, since many such interesting QSDE can be reformulated in some sense as QSDI with regular coefficients.

The basic set-up of the paper is that of multivalued functions (multifunctions for short). The author's article mentioned above may serve as a good reference for the notation and terminology used in the present paper. Given an "initial Hilbert space" $\mathfrak{h}$ and a "noise space" $\mathfrak{k}$, and some suitable subspaces $\mathcal{D}$ of $\mathfrak{h}$ and $\mathcal{E}$ of the symmetric Fock space $\Gamma(L^2(\mathbf{R}_+,\mathfrak{k}))$ spanned by exponential vectors $\{e(\alpha)\}$ with α varying over a suitable subset of $L^2(\mathbf{R}_+,\mathfrak{k})$ let $\tilde{\mathcal{A}}$ denote the locally convex "state space" of noncommutative stochastic processes having $\mathcal{D}\otimes\mathcal{E}$ in the domain, as described in [op. cit.] for example. Fix $T>0$, and let us assume that the coefficients E,F,G,H lie in $L^2([0,T],\tilde{\mathcal{A}})_{\rm mvs}$ in the notation of [op. cit.]. Given α,β in a suitable subset of $L^2(\mathbf{R}_+,\mathfrak{k})$, the author defines a canonical multifunction $P_{\alpha,\beta}\colon[0,T]\times\tilde{\mathcal{A}}\to 2^{\tilde{\mathcal{A}}}$ (see also [op. cit.]), and using it, he defines the multifunction $P\colon[0,T]\times\tilde{\mathcal{A}}\to 2^{\mathrm{sesq}(\mathcal{D}\otimes\mathcal{E})}$ by

$$P(t,x)(\eta,\xi)=\{\langle\eta,Z(t,x)\xi\rangle\colon Z(t,x)\in P_{\alpha,\beta}(t,x)\},$$

where $\eta=u\otimes e(\alpha)$, $\xi=v\otimes e(\beta)$ for some $u,v\in\mathcal{D}$. Note that $\mathrm{sesq}(V)$ for a vector space

V denotes the space of sesquilinear maps on $V \times V$, and for any set A, 2^A denotes the power set of A.

In the above notation, let us now state the main result of the paper. The set $R^{(T)}(X_0)$ consisting of all $X(T)$ such that $X(\cdot)$ is a solution of $(*)$ is called the "reachable set" for the QSDI $(*)$, and is of special importance. The main result states that

$$R^{(T)}(X_0) = \lim_{N\to\infty} (I + \tfrac{T}{N}P)^N(X_0),$$

where I is the identity multifunction $x \mapsto \{x\}$, repeated composition of multifunctions is understood in a suitable way described in the paper, and the limit in the above formula has to be interpreted as the Kuratowski limit of sets. Using the above formula, some results concerning convergence of the discrete approximations to the reachable sets are obtained, generalising similar results for classical differential inclusions.

Debashish Goswami

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MR1900360 (2003b:60081) 60H10 60H20 81S25
Ayoola, E. O. [**Ayoola, Ezekiel O.**] (WAN-IBAD)
Existence and stability results for strong solutions of Lipschitzian quantum stochastic differential equations. (English summary)
Stochastic Anal. Appl. **20** (2002), *no. 2,* 263–281.

Some results on existence, uniqueness and stability of strong solutions of Lipschitzian quantum stochastic differential equations in a locally convex space are obtained by a method of successive approximations. It is shown that these results generalize the analogous results on classical stochastic differential equations driven by a Brownian motion. The construction of the quantum stochastic integral of Hudson and Parthasarathy is briefly reviewed at the beginning of the paper. *Vassili N. Kolokolʹtsov*

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Quantum Martingales in Fock Space. J. Funct. Anal. **1986**, *67*, 126–151. MR0842607
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Citations	From References: 3	From Reviews: 0

MR1897940 (2003e:60121) 60H10 60H20 65C30 81S25
Ayoola, E. O. **[Ayoola, Ezekiel O.]** (WAN-IBAD)
Lagrangian quadrature schemes for computing weak solutions of quantum stochastic differential equations. (English summary)
SIAM J. Numer. Anal. **39** (2002), *no. 6,* 1835–1864.

The paper is devoted to the analysis of the Lagrangian quadrature schemes for computing weak solutions of Lipschitzian quantum stochastic differential equations with matrix elements that are smooth enough. Results on the convergence of such schemes to a solution are obtained. Precise estimates for an error term are given in the case when the nodes of approximations are chosen to be the roots of the Chebyshev polynomials. Numerical examples are presented. *Vassili N. Kolokolʹtsov*

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MR1891476 (2002m:81124) 81S25 60H20

Ayoola, E. O. **[Ayoola, Ezekiel O.]** (WAN-IBAD)

On computational procedures for weak solutions of quantum stochastic differential equations. (English summary)

Stochastic Anal. Appl. **20** (2002), *no. 1,* 1–20.

Summary: "A continuous time Euler approximation scheme and a computational theorem for weak solutions of Lipschitzian quantum stochastic differential equations (QSDEs) are established. The work is accomplished within the framework of the Hudson-Parthasarathy formulation of quantum stochastic calculus and subject to the equivalent forms of the equations satisfying the Carathéodory conditions. Our results generalize analogous results concerning classical Itô stochastic differential equations and those on differential equations in a Banach space context to the noncommutative quantum setting involving unbounded linear operators on a Hilbert space. Numerical examples are given."

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MR1847883 (2002f:65017) 65C30 60H10 60H20 60H35
Ayoola, E. O. [**Ayoola, Ezekiel O.**] (WAN-IBAD)

On convergence of one-step schemes for weak solutions of quantum stochastic differential equations. (English summary)

Acta Appl. Math. **67** (2001), *no. 1,* 19–58.

Summary: "Several one-step schemes for computing weak solutions of Lipschitzian quantum stochastic differential equations driven by certain operator-valued stochastic processes associated with creation, annihilation and gauge operators of quantum field theory are introduced and studied. This is accomplished within the framework of the Hudson-Parthasarathy formulation of quantum stochastic calculus and subject to the matrix elements of the solution being sufficiently differentiable. Results concerning convergence of these schemes in the topology of the locally convex space of solution

are presented. It is shown that the Euler-Maruyama scheme, with respect to weak convergence criteria for Itô stochastic differential equation, is a special case of Euler schemes in this framework. Numerical examples are given."

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Numer. Anal. **19**(3) (1982). MR0656474
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Note: This list reflects references listed in the original paper as accurately as possible with no attempt to correct errors.

AMERICAN MATHEMATICAL SOCIETY
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Mathematical Reviews

Citations	From References: 4	From Reviews: 0

MR1841945 (2002g:81078) 81S25 60H10 65C30
Ayoola, E. O. **[Ayoola, Ezekiel O.]** (WAN-IBAD)
Lipschitzian quantum stochastic differential equations and the associated Kurzweil equations. (English summary)
Stochastic Anal. Appl. **19** (2001), *no. 4,* 581–603.

In the article under review, Kurzweil equations [cf. J. Kurzweil, Czechoslovak Math. J. **7 (82)** (1957), 418–449; MR0111875] associated with Lipschitzian quantum stochastic differential equations (QSDE's) are introduced and studied. After a nice discussion of Kurzweil integrals and Kurzweil equations, including some useful technical results, the author proves the interesting equivalence between the QSDE of the form

$$X(t) = X_0 + \int_{t_0}^{t} (E(X(s), s)d\Lambda_\pi(s) + F(X(s), s)dA_f(s) + G(X(s), s)dA_g^\dagger(s) + H(X(s), s)ds), t \in [t_0, T],$$

and the Kurzweil equation of the form

$$\frac{d}{d\tau}\langle \eta, X(\tau)\xi\rangle = D\Phi(X(\tau), t)(\eta, \xi)$$

on $[t_0, T]$ and for $t \in [t_0, T]$, for a suitable map Φ and ξ, η belonging to an appropriate class. In the above, $X: [t_0, T] \to \tilde{\mathcal{A}}$, where $\tilde{\mathcal{A}}$ is as in references [E. O. Ayoola, "On numerical procedures for solving Lipschitzian quantum stochastic differential equations", Ph.D. Thesis, Univ. Ibadan, Nigeria, 1995; per bibl.; Stochastic Anal. Appl. **18** (2000), no. 4, 525–554; MR1763939] of the article under review, and E, F, G, H, π, f, g satisfy suitable properties described carefully in those references. Finally, some numerical examples are given to demonstrate the effectiveness of the methods using Kurzweil equations. *Debashish Goswami*

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MR1841541 (2002f:65018) 65C30 60H10 81S25
Ayoola, E. O. **[Ayoola, Ezekiel O.]** (WAN-IBAD)
Construction of approximate attainability sets for Lipschitzian quantum stochastic differential inclusions. (English summary)
Stochastic Anal. Appl. **19** (2001), *no. 3,* 461–471.

The author considers solving, numerically, quantum stochastic integral inclusions or, equivalently, certain first-order (non-classical) ordinary differential inclusions. The solutions are represented with the help of what are called attainable sets (non-void closed sets in the complex plane). An algorithm is described for numerically approximating the attainable sets within any prescribed accuracy. *Volker Wihstutz*

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Note: This list reflects references listed in the original paper as accurately as possible with no attempt to correct errors.

MR1991401 (2004e:81072) 81S25 60H10 60H20 65C30 65L06

Ayoola, E. O. [**Ayoola, Ezekiel O.**] (WAN-IBAD)

★**Convergence and stability of general multistep schemes for weak solutions of quantum stochastic differential equations. (English summary)**

Ordinary differential equations (Abuja, 2000), 43–55, *Proc. Natl. Math. Cent. Abuja Niger.*, 1.1, *Natl. Math. Cent., Abuja*, 2000.

Consider the weak integral formulation of the initial value problem related to quantum stochastic differential equations (QSDEs) of the form

$$dX(t) = E(t, X(t))d\Lambda_\pi(t) + F(t, X(t))dA_f^\dagger(t) \\ + G(t, X(t))dA_g(t) + H(t, X(t))dt$$

with $t \in [t_0, T]$ and initial condition $X(t_0) = X_0$, where $\Lambda_\pi, A_f^\dagger, A_g$ are the stochastic integrators in the boson Fock quantum stochastic calculus. The author presents convergence results for some equidistant, implicit, multi-step numerical approximations to weak solutions to such QSDEs in integral form. The main convergence result as the nonrandom step size h tends to 0 is established with respect to some appropriate seminorms within the framework of the Hudson-Parthasarathy formulation of QSDEs and under Lipschitz-type conditions on the equivalent initial value problem of a related nonclassical ordinary differential equation. The effect of round-off errors is also taken into account. However, a considerable effort is needed on the part of any non-quantum-calculus specialist reader to understand the author's notation.

{For the collection containing this paper see MR1991396}

Henri Schurz

MR1763939 (2001e:81065) 81S25 60H35

Ayoola, E. O. [**Ayoola, Ezekiel O.**] (WAN-IBAD)

Converging multistep schemes for weak solutions of quantum stochastic differential equations. (English summary)

Stochastic Anal. Appl. **18** (2000), *no. 4,* 525–554.

The author proposes an algorithm for solving numerically a quantum stochastic differential equation of the form $dX(t) = E(t, X(t))d\Lambda_\pi(t) + F(t, X(t))dA_f^\dagger(t) + G(t, X(t))dA_g(t) + H(t, X(t))dt$ in an interval $[t_0, T]$ with initial condition $X(t_0) = X_0$, where $\Lambda_\pi, A_f^\dagger, A_g$ are the stochastic integrators in the boson Fock quantum stochastic calculus. The exact formulation and the solution of the problem depend heavily on the author's Ph.D. thesis. Considerable effort would be needed on the part of the reader to understand the notation and the statement of the main result. *Kalyanapuram R. Parthasarathy*

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24. G.F. Vincent-Smith, Unitary quantum stochastic evolutions. *Proc. Lond. Math. Soc.*, (3) 63 (1991), 401–425. MR1114515

Note: This list reflects references listed in the original paper as accurately as possible with no attempt to correct errors.

MR1900360 (2003b:60081) 60H10 60H20 81S25
Ayoola, E. O. [**Ayoola, Ezekiel O.**] (WAN-IBAD)
Existence and stability results for strong solutions of Lipschitzian quantum stochastic differential equations. (English summary)
Stochastic Anal. Appl. **20** (2002), *no. 2,* 263–281.

Some results on existence, uniqueness and stability of strong solutions of Lipschitzian quantum stochastic differential equations in a locally convex space are obtained by a method of successive approximations. It is shown that these results generalize the analogous results on classical stochastic differential equations driven by a Brownian motion. The construction of the quantum stochastic integral of Hudson and Parthasarathy is briefly reviewed at the beginning of the paper. *Vassili N. Kolokol′tsov*

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Quantum Martingales in Fock Space. J. Funct. Anal. **1986**, *67*, 126–151. MR0842607
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MR1897940 (2003e:60121) 60H10 60H20 65C30 81S25
Ayoola, E. O. **[Ayoola, Ezekiel O.]** (WAN-IBAD)
Lagrangian quadrature schemes for computing weak solutions of quantum stochastic differential equations. (English summary)
SIAM J. Numer. Anal. **39** (2002), *no. 6,* 1835–1864.

The paper is devoted to the analysis of the Lagrangian quadrature schemes for computing weak solutions of Lipschitzian quantum stochastic differential equations with matrix elements that are smooth enough. Results on the convergence of such schemes to a solution are obtained. Precise estimates for an error term are given in the case when the nodes of approximations are chosen to be the roots of the Chebyshev polynomials. Numerical examples are presented. *Vassili N. Kolokol′tsov*

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MR1891476 (2002m:81124) 81S25 60H20

Ayoola, E. O. **[Ayoola, Ezekiel O.]** (WAN-IBAD)

On computational procedures for weak solutions of quantum stochastic differential equations. (English summary)

Stochastic Anal. Appl. **20** (2002), *no. 1,* 1–20.

Summary: "A continuous time Euler approximation scheme and a computational theorem for weak solutions of Lipschitzian quantum stochastic differential equations (QSDEs) are established. The work is accomplished within the framework of the Hudson-Parthasarathy formulation of quantum stochastic calculus and subject to the equivalent forms of the equations satisfying the Carathéodory conditions. Our results generalize analogous results concerning classical Itô stochastic differential equations and those on differential equations in a Banach space context to the noncommutative quantum setting involving unbounded linear operators on a Hilbert space. Numerical examples are given."

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Theoret. Phys. **1992**, *31* (11), 2003–2203. MR1186301
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18. Vincent-Smith, G.F. Unitary Quantum Stochastic Evolutions. Proc. Lond. Math. Soc. **1991**, *63* (3), 401–425. MR1114515

Note: This list reflects references listed in the original paper as accurately as possible with no attempt to correct errors.

MR1847883 (2002f:65017) 65C30 60H10 60H20 60H35
Ayoola, E. O. [**Ayoola, Ezekiel O.**] (WAN-IBAD)

On convergence of one-step schemes for weak solutions of quantum stochastic differential equations. (English summary)

Acta Appl. Math. **67** (2001), *no. 1,* 19–58.

Summary: "Several one-step schemes for computing weak solutions of Lipschitzian quantum stochastic differential equations driven by certain operator-valued stochastic processes associated with creation, annihilation and gauge operators of quantum field theory are introduced and studied. This is accomplished within the framework of the Hudson-Parthasarathy formulation of quantum stochastic calculus and subject to the matrix elements of the solution being sufficiently differentiable. Results concerning convergence of these schemes in the topology of the locally convex space of solution

are presented. It is shown that the Euler-Maruyama scheme, with respect to weak convergence criteria for Itô stochastic differential equation, is a special case of Euler schemes in this framework. Numerical examples are given."

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3. Ekhaguere, G. O. S.: Lipschitzian quantum stochastic differential inclusions, *Internat. J. Theoret. Phys.* **31**(11) (1992), 2003–2034. MR1186301
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Numer. Anal. **19**(3) (1982). MR0656474
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Citations	From References: 4	From Reviews: 0

MR1841945 (2002g:81078) 81S25 60H10 65C30
Ayoola, E. O. **[Ayoola, Ezekiel O.]** (WAN-IBAD)
Lipschitzian quantum stochastic differential equations and the associated Kurzweil equations. (English summary)
Stochastic Anal. Appl. **19** (2001), *no. 4,* 581–603.

In the article under review, Kurzweil equations [cf. J. Kurzweil, Czechoslovak Math. J. **7 (82)** (1957), 418–449; MR0111875] associated with Lipschitzian quantum stochastic differential equations (QSDE's) are introduced and studied. After a nice discussion of Kurzweil integrals and Kurzweil equations, including some useful technical results, the author proves the interesting equivalence between the QSDE of the form

$$X(t) = X_0 + \int_{t_0}^{t} (E(X(s),s)d\Lambda_\pi(s) + F(X(s),s)dA_f(s) + G(X(s),s)dA_g^\dagger(s) + H(X(s),s)ds), t \in [t_0,T],$$

and the Kurzweil equation of the form

$$\frac{d}{d\tau}\langle \eta, X(\tau)\xi\rangle = D\Phi(X(\tau),t)(\eta,\xi)$$

on $[t_0,T]$ and for $t\in[t_0,T]$, for a suitable map Φ and ξ,η belonging to an appropriate class. In the above, $X\colon [t_0,T]\to\tilde{\mathcal{A}}$, where $\tilde{\mathcal{A}}$ is as in references [E. O. Ayoola, "On numerical procedures for solving Lipschitzian quantum stochastic differential equations", Ph.D. Thesis, Univ. Ibadan, Nigeria, 1995; per bibl.; Stochastic Anal. Appl. **18** (2000), no. 4, 525–554; MR1763939] of the article under review, and E,F,G,H,π,f,g satisfy suitable properties described carefully in those references. Finally, some numerical examples are given to demonstrate the effectiveness of the methods using Kurzweil equations. *Debashish Goswami*

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MR1841541 (2002f:65018) 65C30 60H10 81S25
Ayoola, E. O. **[Ayoola, Ezekiel O.]** (WAN-IBAD)
Construction of approximate attainability sets for Lipschitzian quantum stochastic differential inclusions. (English summary)
Stochastic Anal. Appl. **19** (2001), *no. 3,* 461–471.

The author considers solving, numerically, quantum stochastic integral inclusions or, equivalently, certain first-order (non-classical) ordinary differential inclusions. The solutions are represented with the help of what are called attainable sets (non-void closed sets in the complex plane). An algorithm is described for numerically approximating the attainable sets within any prescribed accuracy. *Volker Wihstutz*

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MR1991401 (2004e:81072) 81S25 60H10 60H20 65C30 65L06
Ayoola, E. O. [**Ayoola, Ezekiel O.**] (WAN-IBAD)

★**Convergence and stability of general multistep schemes for weak solutions of quantum stochastic differential equations. (English summary)**

Ordinary differential equations (Abuja, 2000), 43–55, *Proc. Natl. Math. Cent. Abuja Niger.*, 1.1, *Natl. Math. Cent.*, *Abuja*, 2000.

Consider the weak integral formulation of the initial value problem related to quantum stochastic differential equations (QSDEs) of the form

$$dX(t) = E(t, X(t))d\Lambda_\pi(t) + F(t, X(t))dA_f^\dagger(t) + G(t, X(t))dA_g(t) + H(t, X(t))dt$$

with $t \in [t_0, T]$ and initial condition $X(t_0) = X_0$, where $\Lambda_\pi, A_f^\dagger, A_g$ are the stochastic integrators in the boson Fock quantum stochastic calculus. The author presents convergence results for some equidistant, implicit, multi-step numerical approximations to weak solutions to such QSDEs in integral form. The main convergence result as the nonrandom step size h tends to 0 is established with respect to some appropriate seminorms within the framework of the Hudson-Parthasarathy formulation of QSDEs and under Lipschitz-type conditions on the equivalent initial value problem of a related nonclassical ordinary differential equation. The effect of round-off errors is also taken into account. However, a considerable effort is needed on the part of any non-quantum-calculus specialist reader to understand the author's notation.

{For the collection containing this paper see MR1991396}

Henri Schurz

MR1763939 (2001e:81065) 81S25 60H35
Ayoola, E. O. [**Ayoola, Ezekiel O.**] (WAN-IBAD)

Converging multistep schemes for weak solutions of quantum stochastic differential equations. (English summary)

Stochastic Anal. Appl. **18** (2000), *no. 4,* 525–554.

The author proposes an algorithm for solving numerically a quantum stochastic differential equation of the form $dX(t) = E(t, X(t))d\Lambda_\pi(t) + F(t, X(t))dA_f^\dagger(t) + G(t, X(t))dA_g(t) + H(t, X(t))dt$ in an interval $[t_0, T]$ with initial condition $X(t_0) = X_0$, where $\Lambda_\pi, A_f^\dagger, A_g$ are the stochastic integrators in the boson Fock quantum stochastic calculus. The exact formulation and the solution of the problem depend heavily on the author's Ph.D. thesis. Considerable effort would be needed on the part of the reader to understand the notation and the statement of the main result.

Kalyanapuram R. Parthasarathy

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