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On Convergence of One-Step Schemes for Weak Solutions of Quantum Stochastic Differential Equations

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Abstract. Several one-step schemes for computing weak solutions of Lipschitzian quantum stochastic differential equations (QSDE) driven by certain operator-valued stochastic processes associated with creation, annihilation and gauge operators of quantum field theory are introduced and studied. This is accomplished within the framework of the Hudson–Parthasarathy formulation of quantum stochastic calculus and subject to the matrix elements of solution being sufficiently differentiable. Results concerning convergence of these schemes in the topology of the locally convex space of solution are presented. It is shown that the Euler–Maruyama scheme,with respect to weak convergence criteria for Itô stochastic differential equation is a special case of Euler schemes in this framework. Numerical examples are given.

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Key words: QSDE, Fock spaces, exponential vectors, Euler, Runge–Kutta, noncommutative stochastic processes.

1. Introduction

This paper is concerned with the development, analysis, and applications of several types of one step schemes for solving the following quantum stochastic differential equation introduced by Hudson and Parthasarathy in [10]

$$\begin{aligned} \mathrm{d}X(t) &= E(t, X(t))\,\mathrm{d}\wedge_\pi(t) + F(t, X(t))\,\mathrm{d}A_f^+(t) + \\ &\qquad + G(t, X(t))\,\mathrm{d}A_g(t) + H(t, X(t))\,\mathrm{d}t, \\ X(t_0) &= X_0, \quad \text{almost all } t \in [t_0, T]. \end{aligned} \tag{1.1}$$

In Equation (1.1), the coefficients E, F, G, H lie in a certain class of stochastic processes for which quantum stochastic integrals against the gauge, creation, and annihilation processes $\wedge_\pi$, A_f^+, A_g and the Lebesgue measure are defined. Equation (1.1) involves unbounded linear operators on a Hilbert space and it is a noncommutative quantum generalization of the classical stochastic differential equations of the form

$$\begin{aligned} &\mathrm{d}X(t, w) = H(t, X)\,\mathrm{d}t + F(t, X)\,\mathrm{d}Q(t), \\ &X(t_0) = X_0, \quad t \in [t_0, T], \end{aligned} \tag{1.2}$$

where the driving process $Q(t)$ is a martingale and H, F are sufficiently smooth ordinary functions. Unlike Equation (1.1), numerical schemes for solving Equation (1.2) are fairly well developed. Each of the schemes exhibits specific features depending on the driving process and the solution space of Equation (1.2) (see [11, 16, 19–23]).

However in [3], Equation (1.1) has been reformulated in the following equivalent form:

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}t}\langle \eta, X(t)\xi \rangle = P(t, X(t))(\eta, \xi), \\ &X(t_0) = X_0, \quad t \in [t_0, T], \end{aligned} \tag{1.3}$$

which is an ordinary differential equation of nonclassical type.

The solution stochastic process $X(t)$ is a densely defined linear operator on some tensor product of two Hilbert spaces, one of which is the Boson Fock space; η, ξ lie in a dense subset of the tensor product Hilbert space and the map $(\eta, \xi) \rightarrow P(t, X)(\eta, \xi)$ is a sesquilinear form for fixed (t, X). The explicit form of this map is given by Equation (2.3) below.

Although the general theory of quantum stochastic differential equations and inclusions has recently undergone rapid developments [2–5, 7–10, 14, 24], there have not been corresponding developments in their numerical solutions. Unique and unitary analytical solutions of some of these equations are known to exist but are difficult to come by and generally are not in a readily usable forms.

The results of this paper are accomplished subject to some smoothness conditions on the map $t \rightarrow \langle \eta, X(t)\xi \rangle$, Lipschitz and continuity conditions on the map $(t, x) \rightarrow P(t, x)(\eta, \xi)$. Linear multistep and quadrature schemes will be addressed elsewhere. We examine the questions of convergence and consistency in respect of discrete schemes that approximate matrix elements of solutions of QSDE. We are able to introduce these schemes since the matrix elements $\langle \eta, X(t)\xi \rangle$ of solution $X(t)$ of problem (1.1) have the advantage of being differentiable and their derivatives are sesquilinear form-valued maps given by Equation (1.3). Moreover the schemes here are independent of any feature of the integrator processes and do not depend on some approximation procedures based on stochastic Taylor expansions as in the classical case for real valued processes. They involve less complicated analysis and their order of convergence are independent of such approximation procedures.

Another important feature of the schemes concerns implementations. Computations of the discrete values of the matrix elements of solution are carried out directly as obtained in the implementations of discrete schemes for solving

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Exponential Formula for the Reachable Sets of Quantum Stochastic Differential Inclusions

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ABSTRACT

We establish an exponential formula for the reachable sets of quantum stochastic differential inclusions (QSDI) which are locally Lipschitzian with convex values. Our main results partially rely on an auxilliary result concerning the density, in the topology of the locally convex space of solutions, of the set of trajectories whose matrix elements are continuously differentiable. By applying the exponential formula, we obtain results concerning convergence of the discrete approximations of the reachable set of the QSDI. This extends similar results of Wolenski[20] for classical differential inclusions to the present noncommutative quantum setting.

Key Words: QSDI; Fock spaces; Exponential vectors and formula; Noncommutative multivalued stochastic processes.

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1. INTRODUCTION

We continue our studies in Ref.[5] concerning the reachable sets (or attainability sets) of quantum stochastic differential inclusions given by

$$dX(t) \in E(t, X(t))d\wedge_{\pi}(t) + F(t, X(t))dA_f(t) + G(t, X(t))dA_g^+(t)$$

$$+ H(t, X(t))dt, \quad \text{almost all} \quad t \in [0, T]$$

$$X(0) = X_0. \tag{1.1}$$

In Eq. (1.1), E, F, G, H lie in $L^2_{loc}([0,T] \times \tilde{\mathcal{A}})_{mvs}$, $X : [0,T] \to \tilde{\mathcal{A}}$ belongs to $L^2_{loc}(\tilde{\mathcal{A}})$ and $\wedge_{\pi}, A_f, A_g^+ : [0,T] \to \tilde{\mathcal{A}}$ are the driving gauge, annihilation and creation processes. As usual $\mathcal{A}$ is the locally convex space of noncommutative stochastic processes whose topology is generated by the family of seminorms $\{\|x\|_{\eta\xi} = |\langle \eta, x\xi\rangle| : x \in \tilde{\mathcal{A}},\ \eta, \xi \in \mathbb{D}\underline{\otimes}\mathbb{E}\}$ (see Refs.[2–5,8–10] for details).

For arbitrary $\eta, \xi \in \mathbb{D}\underline{\otimes}\mathbb{E}$, it is well known (see Refs.[3,4,8]) that Eq. (1.1) is equivalent to the first order initial value nonclassical inclusion given by

$$\frac{d}{dt}\langle \eta, X(t)\xi\rangle \in P(t, X(t)(\eta, \xi)$$

$$X(0) = X_0, \quad \text{almost all} \quad t \in [0, T] \tag{1.2}$$

where $(\eta, \xi) \to P(t, X(t))(\eta, \xi)$ is a multivalued sesquilinear form on $\mathbb{D}\underline{\otimes}\mathbb{E}$ with values in $\mathcal{C}$, the field of complex numbers. The explicit form of the map P is presented in Section 2 below.

By adopting similar notations as in Wolenski,[20] involving reachable sets and sets of trajectories of classical differential inclusions, our considerations in this paper mainly focus on the reachable set $R^{(T)}(X_0)$, which is defined by

$$R^{(T)}(X_0) = \{\Phi(T) : \Phi(\cdot) \text{ solves Eq. (1.2)}\}. \tag{1.3}$$

For arbitrary $\eta, \xi \in \mathbb{D}\underline{\otimes}\mathbb{E}$, we associate with Eq. (1.3) the set

$$R^{(T)}(X_0)(\eta, \xi) := \{\langle \eta, \Phi(T)\xi, \rangle : \Phi(T) \in R^{(T)}(X_0)\}. \tag{1.4}$$

Similarly, the set of trajectories of Eq. (1.2) is defined by

$$S^{(T)}(X_0) := \{\Phi(\cdot) : \Phi(\cdot) \quad \text{solves} \quad \text{Eq. (1.2)}\}. \tag{1.5}$$

Again, associated with Eq. (1.5), we define $S^{(T)}(X_0)(\eta,\xi)$ by:

$$S^{(T)}(X_0)(\eta,\xi) := \{\langle \eta, \Phi(\cdot)\xi, \rangle : \Phi(\cdot) \in S^{(T)}(X_0)\}. \tag{1.6}$$

The main result of this paper is that the exponential formula

$$R^{(T)}(X_0)(\eta,\xi) = \lim_{N\to\infty}\left(I + \frac{T}{N}P\right)^N (X_0)(\eta,\xi) \tag{1.7}$$

for the autonomous version of Eq. (1.2) holds subject to the map $x \to P(x)(\eta,\xi)$ being locally Lipschitzian with convex values and the stochastic processes are defined only on a simple Fock space. For the nonautonomous case, the corresponding formula is given by Eq. (4.13) below. The power of $\left(I + \frac{T}{N}P\right)$ in Eq. (1.7) is that of composition of multivalued sesquilinear forms, defined in Section 4 and the limit is a set limit in the sense of Kuratowski. The identity multifunction $I : \tilde{\mathcal{A}} \to 2^{\tilde{\mathcal{A}}}$ takes $x \to \{x\}$.

An important consequence of formula (1.7) is that solutions of the inclusion (1.1) or (1.2) need not be invoked in order to determine the points in $R^{(T)}(X_0)(\eta,\xi)$. This situation is similar to what obtained in the case of reachable sets for classical differential inclusions as explained in Ref.[20].

Another important feature of Eq. (1.7) concerns discretizations of quantum stochastic differential inclusion (1.2). Equation (1.7) implies that the set of all endpoints of the matrix elements of discrete trajectories of Eq. (1.2) converge to the entire reachable set $R^{(T)}(X_0)(\eta,\xi)$. Consequently, we obtain convergence results concerning discrete approximate reachable sets of Eq. (1.2).

This work is partly motivated by the need to develop numerical analysis of quantum stochastic differential inclusions. As highlighted in Ref.[5], emphasis so far has been on numerical procedures for continuous quantum stochastic differential equations with high degree of differentiability of the matrix elements of solutions (see Refs.[2–4]). The numerical analysis of the discontinuous equations needs to be developed as well since a large number of quantum stochastic differential equations arising from applications are discontinuous but may be reformulated as regularized inclusions. Questions concerning estimations of the Hausdorff distance between the sets of solutions of Eq. (1.2) and the set of solutions of its discrete approximation will be considered in a forthcoming paper.

The plan for the rest of the paper is as follows: Section 2 contains preliminary notations and basic prerequisite results. In Section 3, we establish a result concerning approximation of trajectories of Eq. (1.2) by trajectories whose matrix elements are continuously differentiable. This extends the result of Wolenski[20] concerning approximations of solutions of classical differential inclusions by C^1 trajectories. The main results, concerning the exponential formula, are established in section 4. We formulate the discrete Euler approximations of the reachable set of Eq. (1.2). Finally, we show that the discrete reachable sets converge to the entire reachable set of Eq. (1.2).

2. PRELIMINARY RESULTS AND ASSUMPTIONS

As in Refs.[2–5,8–10], we associate with the locally convex state space $\tilde{\mathcal{A}}$ of non-commutative stochastic processes the spaces $Ad(\tilde{\mathcal{A}})$, $Ad(\tilde{\mathcal{A}})_{wac}$, $L^p_{loc}(\tilde{\mathcal{A}})$, $L^\infty_{\gamma,loc}(\mathbb{R}_+)$ for a fixed Hilbert space γ and for $0 < p < \infty$.

If A is a topological space, then clos(A) (resp. comp(A)) denotes the collection of nonvoid closed (resp. compact) subsets of A.

We employ the Hausdorff topology τ_H on clos($\tilde{\mathcal{A}}$) determined by a family of pseudo-metrics $\{\rho_{\eta\xi}(\cdot),\ \eta, \xi \in \mathbb{D}\underline{\otimes}\mathbb{E}\}$ on clos($\tilde{\mathcal{A}}$) as follows:

For $x \in \tilde{\mathcal{A}}$, $\mathcal{M}, \mathcal{N} \in \text{clos}(\tilde{\mathcal{A}})$,

$$\mathbf{d}_{\eta\xi}(x, \mathcal{N}) \equiv \inf_{y\in\mathcal{N}} \|x - y\|_{\eta\xi},$$

$$\delta_{\eta\xi}(\mathcal{M}, \mathcal{N}) \equiv \sup_{x\in\mathcal{M}} \mathbf{d}_{\eta\xi}(x, \mathcal{N}),$$

and

$$\rho_{\eta\xi}(\mathcal{M}, \mathcal{N}) \equiv \max(\delta_{\eta\xi}(\mathcal{M}, \mathcal{N}),\ \ \delta_{\eta\xi}(\mathcal{N}, \mathcal{M})).$$

For $\mathcal{M} \in \text{clos}(\tilde{\mathcal{A}})$, $\|\mathcal{M}\|_{\eta\xi} := \mathbf{d}_{\eta\xi}(\mathcal{M}, \{0\})$.

Similarly, for $A, B \in \text{clos}(\mathcal{C})$ and $x \in \mathcal{C}$, the complex numbers, let

$$\mathbf{d}(x, A) = \inf_{y\in A} |x - y|$$

$$\delta(A, B) = \sup_{x\in A} \mathbf{d}(x, B)$$

and

$$\rho(A, B) = \max(\delta(A, B), \delta(B, A))$$

Then we employ the metric topology on clos($\mathcal{C}$) induced by ρ. The set - theoretic operations are adopted as usual (see Refs.[8–10] for details).

In the followings, Kuratowski limits of set will be frequently employed. If $\{\mathcal{M}_j\}_{j=1}^\infty$ is a sequence of subsets of $\mathcal{A}$, we define the limsup and liminf of

$\{\mathcal{M}_{j=1}^{\infty}\}$ by

$$\lim_{j\to\infty} \sup \mathcal{M}_j = \{a : \lim_{j\to\infty} \inf \mathbf{d}_{\eta\xi}(a, \mathcal{M}_j) = 0\} \tag{2.1}$$

$$\lim_{j\to\infty} \inf \mathcal{M}_j = \{a : \lim_{j\to\infty} \sup \mathbf{d}_{\eta\xi}(a, \mathcal{M}_j) = 0\}. \tag{2.2}$$

If $\lim\sup \mathcal{M}_j = \lim\inf \mathcal{M}_j$, we say that the limit exists and write $\lim_{j\to\infty}\mathcal{M}_j$ for the common value. We observe that if each $\mathcal{M}_j$ and A are compact in $\mathcal{A}$ and contained in a bounded set, then from Eqs. (2.1) and (2.2), $A = \lim_{j\to\infty}\mathcal{M}_j$ if and only if $\rho_{\eta\xi}(\mathcal{M}_j, A) \to 0$ as $j \to \infty$.

Similar definitions hold for the Kuratowski limit of a sequence of subsets of $\mathcal{C}$, the field of complex numbers. However, the Hausdorff metric ρ will now replace the family of pseudo metric above.

Continuous Multivalued Stochastic Processes

A multivalued stochastic process indexed by the set $[0, T] \subseteq \mathbb{R}_+$ is a multifunction on $[0, T]$ with values in $\text{clos}(\tilde{\mathcal{A}})$. As in Ref.[5], the set of all locally p-integrable multivalued stochastic processes will be denoted by $L^P_{loc}(\tilde{\mathcal{A}})_{mvs}$, $p \in (0, \infty)$ while $L^P_{loc}(I \times \tilde{\mathcal{A}})_{mvs}$ is the set of maps $\Phi : I \times \tilde{\mathcal{A}} \mapsto \text{clos}(\tilde{\mathcal{A}})$ such that the map $t \to \Phi(t, X(t))$, $t \in [0, T]$ lies in $L^P_{loc}(\tilde{\mathcal{A}})_{mvs}$ for every $X \in L^P_{loc}(\tilde{\mathcal{A}})$.

For $f, g \in L^\infty_{\gamma,loc}(\mathbb{R}_+)$, $\pi \in L^\infty_{B(\gamma),loc}(\mathbb{R}_+)$, $\mathbf{1}$ is the identity map on $\mathcal{R}\otimes\Gamma(L^2_\gamma(\mathbb{R}_+))$ and M is any of the processes A_f, A^+_g, $\wedge_\pi$ and $s \mapsto s\mathbf{1}$, $s \in \mathbb{R}_+$, then the multivalued stochastic integral $\int_{t_0}^t \Phi(s, X(s))dM(s)$ is adopted as in Ref.[8].

Let $G : [0, T] \to 2^{\tilde{\mathcal{A}}}$ be a given multivalued stochastic process indexed by $[0, T]$. Then we say that G is upper semicontinuous at $t_0 \in [0, T]$ if

$$\lim_{j\to\infty} \sup G(t_j) \subseteq G(t_0)$$

for all sequences $\{t_j\}$ with $t_j \to t_0$.

The map G is lower semicontinuous at t_0 if

$$G(t_0) \subseteq \lim_{j\to\infty} \inf G(t_j)$$

for all $\{t_j\}$ with $t_j \to t_0$.

G is said to be continuous if it is both upper and lower semicontinuous. Similar definitions of continuity hold for a multifunction of the form: $\Phi : \tilde{\mathcal{A}} \to 2^{\tilde{\mathcal{A}}}$.

Lipschitzian Multifunctions

These are defined as follows:

(i) Let $\mathcal{N}$ be an open subset of $\mathcal{A}$. A map $\Phi : \mathcal{N} \to \text{comp}(\tilde{\mathcal{A}})$ will be called Lipschitzian if for all $\eta, \xi \in \mathbb{D}\underline{\otimes}\mathbb{E}$, there exist positive numbers $K_{\eta\xi}$ such that

$$\rho_{\eta\xi}(\Phi(x), \Phi(y)) \leq K_{\eta\xi}\|x - y\|_{\eta\xi}, \quad \forall x, y \in \mathcal{N}.$$

We say that Φ is locally Lipschitzian if it is Lipschitzian on each compact subset of $\mathcal{N}$.

(ii) If $\Phi : \mathcal{N} \to 2^{sesq(\mathbb{D}\underline{\otimes}\mathbb{E})}$, where $sesq(\mathbb{D}\underline{\otimes}\mathbb{E})$ is the linear space of sesquilinear forms on $\mathbb{D}\underline{\otimes}\mathbb{E}$, then Φ is Lipschitzian if

$$\rho(\Phi(x)(\eta, \xi), \Phi(y)(\eta, \xi)) \leq K_{\eta\xi}\|x - y\|_{\eta\xi},$$

where $K_{\eta\xi}$ are positive real numbers.

We remark here that by Proposition (6.2) in Ref.[8], the map $(t, x) \to P(t, x)(\eta, \xi)$ appearing in Eq. (1.2) is Lipschitzian if the coefficients $E, F, G, H : [0, T] \times \tilde{\mathcal{A}} \to 2^{\tilde{\mathcal{A}}}$ in Eq. (1.1) are Lipschitzian.

For E, F, G, H lying in $L^2_{loc}(I \times \tilde{\mathcal{A}})_{mvs}$, Eq. (1.1) is understood as integral inclusion given by

$$\begin{aligned} X(t) \in X_0 + \int_0^t (E(s, X(s))d\wedge_\pi(s) + F(s, X(s))dA_f(s) \\ + G(s, X(s))dA_g^+(s) + H(s, X(s))ds), \quad t \in [0, T], \end{aligned} \tag{2.3}$$

with initial data (t_0, X_0).

The explicit form of the map $P(t, x)(\eta, \xi)$ is given as follows: For $(t, x) \in \mathbb{R}_+ \times \tilde{\mathcal{A}}$, $\eta, \xi \in \mathbb{D}\underline{\otimes}\mathbb{E}$, such that $\eta = c \otimes e(\alpha)$, $\xi = d \otimes e(\beta)$, $c, d \in \mathbb{D}$, $\alpha, \beta \in L^\infty_{\gamma,loc}(\mathbb{R}_+)$, define the multifunction

$$P_{\alpha\beta} : [0, T] \times \tilde{\mathcal{A}} \to 2^{\tilde{\mathcal{A}}}$$

by

$$P_{\alpha\beta}(t, x) = \mu_{\alpha\beta}(t)E(t, x) + \nu_\beta(t)F(t, x) + \sigma_\alpha(t)G(t, x) + H(t, x)$$

where

$$\mu_{\alpha\beta}(t) = \langle \alpha(t), \pi(t)\beta(t) \rangle_\gamma$$

$$\nu_\beta(t) = \langle f(t), \beta(t) \rangle_\gamma$$

and

$$\sigma_{\alpha}(t) = \langle \alpha(t), g(t) \rangle_{\gamma}.$$

This leads to the multifunction

$$P : [0, T] \times \tilde{\mathcal{A}} \to 2^{sesq(\mathbb{D}\underline{\otimes}\mathbb{E})}$$

defined by

$$P(t, x)(\eta, \xi) := \langle \eta, P_{\alpha\beta}(t, x)\xi, \rangle = \{\langle \eta, Z(t, x)\xi, \rangle : Z(t, x) \in P_{\alpha\beta}(t, x)\}.$$

As in Ref.[20], we shall introduce the notion of escape times. In what follows, unless otherwise indicated, we first consider the autonomous version of Eqs. (1.1) and (1.2).

Let $\mathcal{N} \subseteq \tilde{\mathcal{A}}$ be an open subset and $x_0 \in \mathcal{N}$. Assume that $P(x)(\eta, \xi)$ has compact values and is locally Lipschitzian on $\mathcal{N}$. Then we define the escape time $\tilde{T}$ by

$$\tilde{T} := \sup\left\{ T : cl\bigcup_{0 \le t \le T} R^{(t)}(x_0) \quad \text{is compact in } \mathcal{N} \right\},$$

where "cl" denotes the closure of the set.

Next, we present a non-commutative generalization of the Fillipov existence theorem for inclusion (1.2) due to Ref.[8], in a form suitable for our purpose. To this end, for an arbitrary process $Z : [0, T] \to \tilde{\mathcal{A}}$ lying in $Ad(\tilde{\mathcal{A}})_{wac}$, we define

$$\sigma(Z) := \int_0^T \mathbf{d}\left(\frac{d}{dt}\langle \eta, Z(T)\xi \rangle,\ P(Z(t))(\eta, \xi)\right) dt.$$

Theorem 2.1. *Assume that the following conditions hold:*

(a) $Z : I \mapsto \tilde{\mathcal{A}}$ *is an arbitrary process lying in* $Ad(\tilde{\mathcal{A}})_{wac}$ *such that there exists positive functions* $W_{\eta\xi}(t)$ *satisfying*

$$\boldsymbol{d}\left(\frac{d}{dt}\langle \eta, Z(t)\xi \rangle,\ P(Z(t))(\eta, \xi)\right) \le W_{\eta\xi}(t).$$

(b) There exists $\theta > 0$ *and* $\mathcal{N} \subseteq \tilde{\mathcal{A}}$ *such that each of the maps E, F, G, H is Lipschitzian from* $\mathcal{N}$ *to* $(\text{clos}(\tilde{\mathcal{A}}), \tau_H)$ *and that*

$$Q_{Z,\theta} = \{x \in \tilde{\mathcal{A}} : \|x - Z(t)\|_{\eta\xi} \leq \theta, \forall \eta, \xi \in \mathbb{D}\underline{\otimes}\mathbb{E} \quad \textit{for some} \;\; t \in [0,T]\} \subseteq \mathcal{N}.$$

(c) $K_{\eta\xi} > 0$ *are the Lipschitz constants for the map* $P : \mathcal{N} \longrightarrow 2^{sesq(\mathbb{D}\underline{\otimes}\mathbb{E})}$ *on* $\mathcal{N}$.

(d) For arbitrary $\eta, \xi \in \mathbb{D}\underline{\otimes}\mathbb{E}$, $t \in [0,T]$,

$$E_{\eta\xi}(t) = e^{tK_{\eta\xi}} \int_0^t ds W_{\eta\xi}(s).$$

If in addition, E, F, G, H are continuous from $\mathcal{A}$ *to* $(\text{clos}(\tilde{\mathcal{A}}), \tau_H)$ *and*

$$\int_0^T W_{\eta\xi}(t)dt < \theta e^{-K_{\eta\xi}T},$$

then there exists a solution $\Phi \in S^{(T)}(Z(0))$ *of Eq.* (1.2) *satisfying*

$$\|\Phi(t) - Z(t)\|_{\eta\xi} < \sigma(Z)e^{K_{\eta\xi}T}, \quad t \in J$$

and

$$\left|\frac{d}{dt}\langle \eta, \Phi(t)\xi\rangle - \frac{d}{dt}\langle \eta, Z(t)\xi\rangle\right| \leq K_{\eta\xi}E_{\eta\xi}(t) + \mathrm{W}_{\eta\xi}(t)$$

for almost all $t \in J$ *where*

$$J = \{t \in [0,T] : E_{\eta\xi}(t) \leq \sigma(Z)e^{K_{\eta\xi}\mathrm{T}} \leq \theta\}.$$

The next result is a useful lemma due to Wolenski.[20]

Lemma 2.2. *Suppose that R, S,* M_1, M_2, M_N *are real constants satisfying*

$$M_{j+1} = R + SM_j, \quad \text{for} \quad j = 1, 2 \ldots N,$$

then

$$M_N = R\left(\frac{1 - S^N}{1 - S}\right) + S^N M_0 \quad \text{if} \quad S \neq 1,$$

$$= NR + M_0 \quad \text{if} \quad S = 1.$$

3. SOME DENSITY RESULTS

In this section, we show that an arbitrary trajectory of the Inclusion (1.2) may be approximated in $\mathcal{A}$ by a trajectory whose matrix elements are of class $C^1[0,T]$, provided that the coefficients E, F, G, H are locally Lipschitzian and the map $P(x)(\eta,\xi)$ has convex values in $\mathcal{C}$. This extends similar results in Ref.[20] to the present non-commutative quantum setting. In what follows, in this Section and Section 4, we consider the initial space $\mathcal{R} \equiv \mathcal{C}$. Consequently, $\mathbb{D}\underline{\otimes}\mathbb{E} = \mathbb{E}$ and $\mathcal{R}\otimes\Gamma(L^2_\gamma(\mathbb{R}_+)) \equiv \Gamma(L^2_\gamma(\mathbb{R}_+))$.

Theorem 3.1. *Suppose that the following conditions hold.*

(*i*) *$\mathcal{N}$ is an open subset of $\mathcal{A}$ and $\mathrm{P}: \mathcal{N} \to 2^{\mathrm{sesq}(\mathbb{E})}$ is a multivalued sesquilinear form with nonempty, convex and compact values in $\mathcal{C}$.*
(*ii*) *The coefficients E, F, G, H are locally Lipschitzian on $\mathcal{N}$.*
(*iii*) $\Phi(\cdot) \in S^{(T)}(x_0)$ *is a trajectory of Eq.* (1.2) *with matrix element*

$$\langle \eta, \Phi(\cdot)\xi, \rangle := \Phi_{\eta\xi}(\cdot) \in S^{(T)}(x_0)(\eta,\xi).$$

Then for each $\epsilon > 0$, *there exists* $\tilde{\Phi}(\cdot) \in S^{(T)}(x_0)$ *such that*

$$\langle \eta, \tilde{\Phi}(\cdot)\xi, \rangle := \tilde{\Phi}_{\eta\xi}(\cdot) \in S^{(T)}(x_0)(\eta,\xi) \bigcap C^1[0,T]$$

and

$$\|\Phi(t) - \tilde{\Phi}(t)\|_{\eta\xi} < \epsilon.$$

We first establish the following Proposition which will be employed in the proof of Theorem 3.1.

Proposition 3.2. *Let $\mathcal{N}$, the map P and the coefficients E, F, G, H be as in Theorem* (3.1) *and suppose that the following hold.*

(*i*) *$Y : [0,T] \to \tilde{\mathcal{A}}$ is an arbitrary process lying in $Ad(\tilde{\mathcal{A}})_{wac}$ such that its matrix elements $Y_{\eta\xi}(\cdot) := \langle \eta, Y(\cdot)\xi, \rangle$ belong to $C^1[0,T]$ for each pair $\eta, \xi \in \mathbb{E}$.*
(*ii*) *There is a compact set $Q \subseteq \tilde{\mathcal{A}}$ and $\theta > 0$ such that the set*

$$\{x : \|x - Y(t)\|_{\eta\xi} \le \theta, \quad \text{for some} \quad 0 \le t \le T \;\; \forall\, \eta, \xi \in \mathbb{E}\} \subseteq Q \subseteq \mathcal{N}.$$

Let $K_{\eta\xi}$ be Lipschitz constants for the map P on Q. Assume further that

$$\sigma(Y) < \theta e^{-K_{\eta\xi}T}.$$

Then, there exists a trajectory $\tilde{Y}(\cdot) \in S^{(T)}(Y(0))$ with

$$\tilde{Y}_{\eta\xi}(\cdot) \in S^{(T)}(Y(0))(\eta, \xi) \bigcap C^1[0, T]$$

satisfying

$$\|Y(t) - \tilde{Y}(t)\|_{\eta\xi} < \sigma(Y)e^{K_{\eta\xi}T}.$$

We require the following lemma for the proof of Proposion (3.2).

If $A \subseteq \mathcal{C}$ is a closed, convex set and $a \in \mathcal{C}$, we denote by $\text{proj}(a, A)$, the unique element in A closest to the point a.

Lemma 3.3. *Suppose that $G : [0, T] \to 2^{\tilde{\mathcal{A}}}$ is a multivalued stochastic process such that the map $t \to G(t)(\eta, \xi)$ is a continuous multivalued sesquilinear form with nonempty, closed and convex values on* $[0, T]$. *Suppose further that $V : [0, T] \to \tilde{\mathcal{A}}$ is an adapted process such that the map $t \to \langle \eta, V(t)\xi, \rangle$ is continuous for each pair of $\eta, \xi \in \mathbb{E}$. Then the map*

$$t \to \text{proj}(\langle \eta, V(t)\xi, \rangle, G(t)(\eta, \xi))$$

is continuous on $[0, T]$.

Proof. The proof is an adaptation of the arguments in (Ref.[20], Lemma 3.3) as follows: For each pair of η,ξ, set

$$P_{\eta\xi}(t) := \text{proj}(\langle \eta, V(t)\xi\rangle, G(t)(\eta, \xi))$$

Let $t_0 \in [0, T]$ and $\{t_j\}_{j\geq 1} \subseteq [0, T]$ with $t_j \to t_0$ as $j \to \infty$.
Since $t \to \mathrm{G}(t)(\eta, \xi)$ is continuous, the sequence $\{P_{\eta\xi}(t_j)\}$ is bounded and therefore has a convergent subsequence.
We assume that $P_{\eta\xi}(t_j) \to \tilde{p}_{\eta\xi}$ as $j \to \infty$ by passing to a subsequence if necessary but retaining the same notation. To conclude the proof, it is sufficient for us to show that

$$P_{\eta\xi}(t_0) = \tilde{p}_{\eta\xi}.$$

Since $t \to G(t)(\eta, \xi)$ is upper semicontinuous at t_0, we have $\tilde{p}_{\eta\xi} \in G(t_0)(\eta, \xi)$, where $\tilde{p}_{\eta\xi} = \langle \eta, \tilde{p}\xi\rangle$, for some $\tilde{p} \in G(t_0) \subseteq \tilde{\mathcal{A}}$.

Again, since $t \longrightarrow G(t)(\eta, \xi)$ is lower semicontinuous at t_0, there exists $q_{\eta\xi,j} \in G(t_j)(\eta, \xi)$ such that $q_{\eta\xi,j} \longrightarrow \mathrm{P}_{\eta\xi}(t_0)$.
Hence, we have

$$|V_{\eta\xi}(t_0) - \tilde{p}_{\eta\xi}| = \lim_{j\to\infty} |V_{\eta\xi}(t_j) - P_{\eta\xi}(t_j)| \leq \lim_{j\to\infty} |V_{\eta\xi}(t_j) - q_{\eta\xi,j}|$$

$$\text{by definition of } P_{\eta\xi}(t_j)$$

$$= |V_{\eta\xi}(t_0) - \mathrm{P}_{\eta\xi}(t_0)|, \tag{3.1}$$

by continuity of $V_{\eta\xi}(\cdot)$ and the absolute value function $|\cdot|$.
But $P_{\eta\xi}(t_0)$ is the unique element in $G(t_0)(\eta, \xi)$ closest to $V_{\eta\xi}(t_0)$. Therefore, the last inequality implies that

$$P_{\eta\xi}(t_0) = \tilde{p}_{\eta\xi}.$$

Proof of Proposition 3.2. The stochastic process $Y \in Ad(\tilde{\mathcal{A}})_{wac}$ is given such that $\langle \eta, Y(\cdot)\xi \rangle \in C^1[0, T]$ for each $\eta, \xi \in \mathbb{E}$ and satisfies

$$\sigma(Y) < \theta e^{-K_{\eta\xi}T}.$$

By Lemma 3.3,

$$t \longrightarrow V_{\eta\xi,0} := \mathrm{proj}\left(\frac{d}{dt}\langle \eta, Y(t)\xi \rangle, \quad P(Y(t))(\eta, \xi)\right)$$

is continuous on $[0, T]$. Set

$$Y_{\eta\xi,1} = \langle \eta, Y(0)\xi \rangle + \int_0^t V_{\eta\xi,0}(s) ds.$$

Then,

$$Y_{\eta\xi,1}(\cdot) \in C^1[0, T],$$

with

$$\frac{d}{dt} Y_{\eta\xi,1}(t) = V_{\eta\xi,0}(t) \in P(Y(t))(\eta, \xi)$$

by definition.
Since $Y_{\eta\xi,1}(t)$ is a sesquilinear form on $[0, T]$, there exists a stochastic process $Y_1 : [0, T] \longrightarrow \tilde{\mathcal{A}}$ such that

$$Y_{\eta\xi,1}(t) = \langle \eta, Y_1 \xi \rangle.$$

Since

$$\begin{aligned}|Y_{\eta\xi,1}(t) - Y_{\eta\xi}(t)| &= \left| \int_0^t (V_{\eta\xi,0}(s) - \frac{d}{ds}\langle \eta, Y(s)\xi\rangle) ds \right| \\ &\leq \int_0^t \left| V_{\eta\xi,0}(s) - \frac{d}{ds}\langle \eta, Y(s)\xi\rangle \right| ds \\ &= \int_0^t \mathbf{d}\left(\frac{d}{ds}\langle \eta, Y(s)\xi\rangle, P(Y(s))(\eta, \xi)\right) ds \leq \sigma(Y) < \theta.\end{aligned}$$

i.e

$$\|Y_1(t) - Y(t)\|_{\eta\xi} < \theta.$$

Then,

$$Y_1(t) \in Q \subseteq \mathcal{N}, \forall t \in [0, T].$$

Next, we set

$$Y_{\eta\xi,0}(\cdot) = Y_{\eta\xi}(\cdot). \tag{3.2}$$

Inductively, suppose that $n \geq 1$ and that adapted processes $\{Y_j\}_{j=1}^n$ have been chosen such that the sequence $\{\langle \eta, Y_j(\cdot)\xi, \rangle\}_{j=1}^n$ are continuously differentiable on $[0, T]$ for all $\eta, \xi \in \mathbb{E}$ satisfying Eqs. (3.3)–(3.6) below, for all $0 \leq t \leq T$ and $j = 1, 2 \ldots n$

$$\frac{d}{dt}\langle \eta, Y_j(t)\xi\rangle \in P(Y_{j-1}(t))(\eta, \xi), \tag{3.3}$$

$$\left| \frac{d}{dt}\langle \eta, Y_j(t)\xi\rangle - \frac{d}{dt}\langle \eta, Y_{j-1}(t)\xi\rangle \right| \leq \frac{\sigma(Y) K_{\eta\xi}^{j-1} t^{j-2}}{(j-2)!}, \tag{3.4}$$

$$|\langle \eta, Y_j(t)\xi\rangle - \langle \eta, Y_{j-1}(t)\xi\rangle| \leq \sigma(Y) \frac{(K_{\eta\xi} t)^{j-1}}{(j-1)!}, \tag{3.5}$$

$$Y_j(t) \in Q. \tag{3.6}$$

First, we observe that when $j = 1$, Eqs. (3.5) and (3.6) follow directly from Eq. (3.2), Eq. (3.3) is obvious and Eq. (3.4) is vacuous.

Next, we begin by defining

$$V_{\eta\xi,n+1}(t) = \text{proj}\left(\frac{d}{dt}\langle \eta, Y_n(t)\xi\rangle,\ \ P(Y_n(t))(\eta,\xi)\right)$$

and

$$Y_{\eta\xi,n+1}(t) = \langle \eta, Y(0)\xi\rangle + \int_0^t V_{\eta\xi,n+1}(s)ds = \langle \eta, Y_{n+1}(t)\xi\rangle,$$

for some stochastic process $Y_{n+1} : [0,T] \longrightarrow \tilde{\mathcal{A}}$.
By Lemma 3.3, $t \longrightarrow V_{\eta\xi,n+1}(t)$ is continuous and

$$\frac{d}{dt}Y_{\eta\xi,n+1}(t) = V_{\eta\xi,n+1}(t).$$

This implies that

$$Y_{\eta\xi,n+1}(\cdot) \in C^1[0,T].$$

Let $t \in [0,T]$, then

$$\frac{d}{dt}Y_{\eta\xi,n+1}(t) \in P(Y_n(t))(\eta,\xi).$$

Therefore,

$$\begin{aligned}
&\left|\frac{d}{dt}Y_{\eta\xi,n+1}(t) - \frac{d}{dt}\langle \eta, Y_n(t)\xi\rangle\right| \\
&\quad = \mathbf{d}\left(\frac{d}{dt}\langle \eta, Y_n(t)\xi\rangle, P(Y_n(t))(\eta,\xi)\right) \\
&\quad \leq \rho(P(Y_{n-1}(t))(\eta,\xi), P(Y_n(t))(\eta,\xi)) \quad \text{by Eq. (3.3)}, \\
&\quad \leq K_{\eta\xi}\|Y_{n-1}(t) - Y_n(t)\|_{\eta\xi} \quad \text{by Lipschitz property}, \\
&\quad = K_{\eta\xi}|\langle \eta, Y_{n-1}(t)\xi\rangle - \langle \eta, Y_n(t)\xi\rangle| \leq \sigma(Y)\frac{K^n_{\eta\xi}t^{n-1}}{(n-1)!} \quad \text{by Eq. (3.5)}.
\end{aligned} \tag{3.7}$$

The last inequality shows that Eq. (3.4) holds for $j = n+1$.

Next, we have

$$\begin{aligned}
&|\langle \eta, Y_{n+1}(t)\xi\rangle - \langle \eta, Y_n(t)\xi\rangle \\
&\quad \le \int_0^t \left|\frac{d}{ds} Y_{\eta\xi,n+1}(s) - \frac{d}{ds}\langle \eta, Y_n(s)\xi\rangle\right| ds \\
&\quad \le \sigma(Y)\int_0^t \frac{K_{\eta\xi}^n s^{n-1}}{(n-1)!} ds \quad \text{by (3.7)} \\
&\quad = \sigma(Y)\frac{(K_{\eta\xi}t)^n}{n!}. \qquad (3.8)
\end{aligned}$$

The last inequality shows that Eq. (3.5) holds for $j = n+1$. Finally, we have

$$\begin{aligned}
&\|Y_{n+1}(t) - Y(t)\|_{\eta\xi} \\
&\quad = |\langle \eta, Y_{n+1}(t)\xi\rangle - \langle \eta, Y(t)\xi\rangle| \\
&\quad \le \sum_{j=0}^{n} |\langle \eta, Y_{j+1}(t)\xi\rangle - \langle \eta, Y_j(t)\xi\rangle| \\
&\quad \le \sigma(Y)\sum_{j=0}^{n} \frac{(K_{\eta\xi}t)^j}{j!} \quad \text{by Eqs. (3.5) and (3.8)} \\
&\quad \le \sigma(Y)e^{K_{\eta\xi}T} < \theta. \qquad (3.9)
\end{aligned}$$

This shows that $Y_{n+1}(t) \in Q$. The induction proof is complete.
The foregoings imply the existence of a subsequence $\{Y_j\}_{j=1}^{\infty}$ of adapted processes $Y_j : [0,T] \to \tilde{\mathcal{A}}$ lying in Q with the property that the sequence $\{\langle \eta, Y_j(\cdot)\xi\rangle\}_{j=1}^{\infty}$ are continuously differentiable on $[0,T]$.
It follows from Eqs. (3.5) and (3.4) that $\{Y_j(t)\}$ is Cauchy in Q with the property that $\left\{\frac{d}{dt}\langle \eta, Y_j(t)\xi\rangle\right\}$ is also Cauchy in $\mathcal{C}$, the field of complex numbers. Consequently, $\{Y_j(t)\}$ converges uniformly to some $\tilde{Y}(t)$ in Q, i.e

$$\|Y_j(t) - \tilde{Y}(t)\|_{\eta\xi} = |\langle \eta, Y_j(t)\xi, \rangle - \langle \eta, \tilde{Y}(t)\xi\rangle| \to 0 \quad \text{as} \quad j \to \infty.$$

This implies that

$$\langle \eta, Y_j(t)\xi\rangle \to \langle \eta, \tilde{Y}(t)\xi\rangle \quad \text{as} \quad j \to \infty.$$

The function $\langle\eta, \tilde{Y}(\cdot)\xi\rangle$ is continuously differentiable on $[0, T]$ since $\langle\eta, Y_j(\cdot)\xi\rangle$ lies in $C^1[0, T]$ for each j.
Hence

$$\frac{d}{dt}\langle\eta, Y_j(t)\xi\rangle \to \frac{d}{dt}\langle\eta, \tilde{Y}(t)\xi\rangle.$$

By Eq. (3.4), $\frac{d}{dt}\langle\eta, \tilde{Y}(t)\xi\rangle$ is continuous by the continuity of the sequence $\left\{\frac{d}{dt}\langle\eta, Y_j(t)\xi\rangle\right\}$ on $[0, T]$ for each j.
Moreover, for each $t \in [0, T]$ and by Eq. (3.3), we have

$$\frac{d}{dt}\langle\eta, \tilde{Y}(t)\xi\rangle = \lim_{j\to\infty}\frac{d}{dt}\langle\eta, Y_j(t)\xi\rangle \in \lim_{j\to\infty} P(Y_{j-1}(t))(\eta, \xi) = P(\tilde{Y}(t))(\eta, \xi).$$

Hence,

$$\tilde{Y}(\cdot) \in S^{(T)}(Y(0))$$

with

$$\langle\eta, \tilde{Y}(\cdot)\xi\rangle \in S^{(T)}(Y(0))(\eta, \xi).$$

Finally, from Eq. (3.9), we obtain

$$\|\tilde{Y}(t) - Y(t)\|_{\eta\xi} = \lim_{n\to\infty}\|Y_{n+1}(t) - Y(t)\|_{\eta\xi} \leq \sigma(Y)e^{K_{\eta\xi}T}.$$

The next result is a direct application of Lusin's Theorem (see Refs.[17,16]) to elements of $Ad(\tilde{\mathcal{A}})$ with Lebesque measurable matrix elements. The result will be employed in the proof of Theorem (3.1) that follows.

Theorem 3.4. *Assume that the following conditions hold:*
$f : [0, T] \to \tilde{\mathcal{A}}$ *is a stochastic process such that for all* $\eta, \xi \in \mathbb{E}$,
$f_{\eta\xi}(t) := \langle\eta, f(t)\xi\rangle$ *is Lebesque measurable on* $[0, T]$ *and*

$$R_{\eta\xi} = \sup_{[0,T]}|f_{\eta\xi}(t)|.$$

Then, given $\epsilon > 0$, *there exists a borel subset* $J \subseteq [0, T]$ *and a continuous* *function* $Z_{\eta\xi} : [0, T] \to \mathcal{C}$ *such that*

$$Z_{\eta\xi}(t) = f_{\eta\xi}(t), \quad t \in [0, T] - J,$$

$$\sup_{[0,T]} |Z_{\eta\xi}(t)| \leq R_{\eta\xi}$$

and

$$L(J) < \epsilon,$$

where L denotes the Lebesque measure.

Proof. Since $f_{\eta\xi}(t)$ is measurable on $[0, T]$ and $L([0, T]) < \infty$, then, by Lusin's Theorem, (see Ref.[16] page 225) there exists a Borel subset $J \subseteq [0, T]$ such that

$$L(J) < \epsilon$$

and

$$f_{\eta\xi}(\cdot) \in C([0, T] - J).$$

Next we define

$$Z_{\eta\xi}(t) = f_{\eta\xi}(t), \quad t \in [0, T] - J,$$

$$Z_{\eta\xi}(t) = \frac{R_{\eta\xi}}{|\langle \eta, \xi \rangle|} \langle \eta, \xi \rangle, \quad t \in J.$$

We note that $Z_{\eta\xi}(\cdot)$ is well defined since $|\langle \eta, \xi \rangle| \neq 0, \forall \eta, \xi \in \mathbb{E}$, where

$$\langle \eta, \xi \rangle = e^{\langle \alpha, \beta \rangle},$$

$$\eta = e(\alpha), \quad \xi = e(\beta), \quad \alpha, \beta \in L^2_\gamma(\mathbb{R}_+).$$

Consequently,

$$Z_{\eta\xi}(\cdot) \in C[0, T]$$

and the set

$$J = \{t \in [0, T] : Z_{\eta\xi}(t) \neq f_{\eta\xi}(t)\}$$

satisfies

$$L(J) < \epsilon.$$

Again,

$$\sup_{[0,T]} |Z_{\eta\xi}(t)| \leq \sup_{[0,T]} |f_{\eta\xi}(t)| = R_{\eta\xi}.$$

Proof of Theorem 3.1. Given that $\Phi(\cdot) \in S^{(T)}(X_0)$ and $\epsilon > 0$, we have $\Phi_{\eta\xi}(\cdot) \in S^{(T)}(X_0)(\eta, \xi)$ for each pair of $\eta, \xi \in \mathbb{E}$. We show that there exists a trajectory $\tilde{\Phi}(\cdot)$ such that

$$\tilde{\Phi}_{\eta\xi}(\cdot) \in S^{(T)}(X_0)(\eta, \xi) \bigcap C^1[0, T],$$

and $\|\Phi(t) - \tilde{\Phi}(t)\|_{\eta\xi} < \epsilon$, where $\tilde{\Phi}_{\eta\xi}(\cdot) = \langle \eta, \tilde{\Phi}(\cdot)\xi \rangle$.
We assume without loss of generality that ϵ is sufficiently small so that the following hold

$$\{u : \|u - \Phi(t)\|_{\eta\xi} \leq \epsilon, \quad \text{for some } 0 \leq t \leq T, \forall \eta, \xi \in \mathbb{E}\} \subseteq Q \subseteq \mathcal{N},$$

for some compact set Q contained in $\mathcal{N}$.
Let $K_{\eta\xi} \geq 1$ be Lipschitz constants for the map $x \rightarrow P(x)(\eta, \xi)$ on Q and let

$$R_{\eta\xi} = \sup\left\{ |v_{\eta\xi}| : v_{\eta\xi} \in \bigcup_{u \in Q} P(u)(\eta, \xi) \right\}.$$

Since $\Phi(\cdot) \in S^{(T)}(X_0)$, we have

$$\left| \frac{d}{dt} \langle \eta, \Phi(t)\xi \rangle \right| \leq R_{\eta\xi},$$

for almost all t satisfying $0 \leq t \leq T$. Let

$$f_{\eta\xi}(t) = \frac{d}{dt} \langle \eta, \Phi(t)\xi \rangle, \quad t \in [0, T],$$

then by Theorem (3.3), there exists continuous functions $Z_{\eta\xi}(\cdot)$ on $[0, T]$ and a Borel subset $J \subseteq [0, T]$ such that

$$Z_{\eta\xi}(t) = f_{\eta\xi}(t), \quad \text{for} \quad t \in [0, T] - J, \quad \forall \eta, \xi,$$

$$\sup_{[0,T]} |Z_{\eta\xi}(t)| \leq R_{\eta\xi},$$

and

$$L(J) \leq \frac{\epsilon}{4K_{\eta\xi} R_{\eta\xi}(1 + T)e^{K_{\eta\xi}T}},$$

where L is the Lebesque measure on $[0, T]$.
Next we define for each pair of η, ξ

$$Y_{\eta\xi}(t) = \langle \eta, X_0 \xi \rangle + \int_0^t Z_{\eta\xi}(s) ds.$$

Then

$$Y_{\eta\xi}(\cdot) \in C^1[0, T].$$

As $Y_{\eta\xi}(t)$ is a sesquilinear form, there exists a stochastic process $Y : [0, T] \to \tilde{\mathcal{A}}$ such that

$$Y_{\eta\xi}(t) = \langle \eta, Y(t)\xi \rangle, \quad \text{almost all} \quad t \in [0, T].$$

It is immediate that Y lies in $Ad(\tilde{\mathcal{A}})_{wac}$ by definition.
Observe that for all $t \in [0, T] - J$, $\langle \eta, Y(t)\xi \rangle - \langle \eta, \Phi(t)\xi \rangle = 0$.
However, for all $t \in [0, T]$, the following hold:

$$\begin{aligned} |\langle \eta, Y(t)\xi \rangle - \langle \eta, \Phi(t)\xi \rangle| &= \|Y(t) - \Phi(t)\|_{\eta\xi} \\ &\leq \int_J \left| Z_{\eta\xi}(s) - \frac{d}{ds} \langle \eta, \Phi(s)\xi \rangle \right| ds \leq 2R_{\eta\xi} L(J) \\ &\leq \frac{\epsilon}{2K_{\eta\xi} e^{K_{\eta\xi}T}(1 + T)} \leq \frac{\epsilon}{2}. \end{aligned} \tag{3.10}$$

Again, we observe that Eq. (3.10) implies that the set $\left\{u : \|u - Y(t)\|_{\eta\xi} \leq \frac{\epsilon}{2}\right\}$

is contained in Q. Next, we estimate $\sigma(Y)$ as follows:

$$\begin{aligned}\sigma(Y) &:= \int_0^T \mathbf{d}\left(\frac{d}{dt}\langle \eta, Y(t)\xi\rangle, P(Y(t))(\eta,\xi)\right)\\ &\leq \int_{[0,T]-J} \rho(P(\Phi(t))(\eta,\xi), P(Y(t))(\eta,\xi))dt\\ &\quad + \int_J \mathbf{d}\left(\frac{d}{dt}\langle \eta, Y(t)\xi\rangle, P(Y(t))(\eta,\xi\right)dt\\ &\leq K_{\eta\xi}\int_0^T \|\Phi(t) - Y(t)\|_{\eta\xi}dt + 2R_{\eta\xi}L(J).\end{aligned}$$

Consequently, by applying Eq. (3.10), we have

$$\begin{aligned}\sigma(Y) &\leq \frac{K_{\eta\xi}T\epsilon}{2K_{\eta\xi}e^{K_{\eta\xi}T}(1+T)} + \frac{\epsilon}{2K_{\eta\xi}e^{K_{\eta\xi}T}(1+T)},\\ &= \frac{\epsilon}{2}\left[\frac{K_{\eta\xi}T+1}{K_{\eta\xi}(1+T)}\right]e^{-K_{\eta\xi}T} < \frac{\epsilon}{2}e^{-K_{\eta\xi}T}, \quad \text{since} \quad K_{\eta\xi} \geq 1.\end{aligned}$$

Application of Proposition (3.2) to the process $Y : [0,T] \to \tilde{\mathcal{A}}$ with $\theta = \frac{\epsilon}{2}$ and Q, implies that there exists $\tilde{\Phi}(\cdot) \in S^{(T)}(X_0)$ such that

$$\tilde{\Phi}_{\eta\xi}(\cdot) \in S^{(T)}(X_0)(\eta,\xi) \bigcap C^1[0,T] \quad \forall\eta, \quad \xi \in \mathbb{E}$$

and

$$\|Y(t) - \tilde{\Phi}(t)\|_{\eta\xi} < \frac{\epsilon}{2}.$$

Finally, by employing Eq. (3.10) again, we conclude that

$$\|\Phi(t) - \tilde{\Phi}(t)\|_{\eta\xi} \leq \|\Phi(t) - Y(t)\|_{\eta\xi} + \|Y(t) - \tilde{\Phi}(t)\|_{\eta\xi} < \epsilon.$$

i.e

$$\|\Phi(t) - \tilde{\Phi}(t)\|_{\eta\xi} < \epsilon.$$

4. THE EXPONENTIAL FORMULA

We first present the definitions of composition of multifunctions suitable for our purpose. Unless otherwise indicated, $\eta, \xi \in \mathbb{E}$ such that $\eta = e(\alpha)$, $\xi = e(\beta)$, $\alpha, \beta \in L^2_\gamma(\mathbb{R}_+)$. In what follows I is the multifunction that takes $x \longrightarrow \{x\}$.

Definition 4.1. Let G_0 and $G_1 : \tilde{\mathcal{A}} \longrightarrow 2^{\tilde{\mathcal{A}}}$ be multifunctions defined on $\tilde{\mathcal{A}}$. By composition $G_0 \circ G_1 : \tilde{\mathcal{A}} \longrightarrow 2^{\tilde{\mathcal{A}}}$ of G_1 with G_0, we mean the set

$$G_0 \circ G_1(x) = \{z : \quad \text{there exists } y \in G_1(x) \quad \text{with} \quad z \in G_0(y)\}.$$

G_0^N denotes the composition of G_0 with itself N times.

Definition 4.2. The composition of the multivalued sesquilinear form $P : \tilde{\mathcal{A}} \longrightarrow 2^{sesq(\mathbb{E})}$ with itself N times is defined by

$$P^N(x)(\eta, \xi) = \langle \eta, P^N_{\alpha\beta}(x)\xi \rangle,$$

where

$$P^N_{\alpha\beta} : \tilde{\mathcal{A}} \longrightarrow 2^{\tilde{\mathcal{A}}}$$

is the composition of $P_{\alpha\beta}$ with itself N times in the sense of Definition (4.1).

Theorem 4.3. *Suppose $\mathcal{N} \subseteq \tilde{\mathcal{A}}$ is open and $P : \tilde{\mathcal{A}} \longrightarrow 2^{sesq(\mathbb{E})}$ is a locally Lipschitzian multivalued sequilinear form with nonempty, compact values on $\mathcal{N}$. Let $X_0 \in \mathcal{N}$ be fixed.*

(i) For $0 \leq T \leq \tilde{T}$, one has

$$\limsup_{N \to \infty} \left(I + \frac{T}{N} P \right)^N (X_0)(\eta, \xi) \subseteq clR^{(T)}(X_0)(\eta, \xi) \tag{4.1}$$

(ii) If in addition, P is assumed to have convex values, then for all $T \geq 0$, we have

$$R^{(T)}(X_0)(\eta, \xi) \subseteq \liminf_{N \to \infty} \left(I + \frac{T}{N} P \right)^N (X_0)(\eta, \xi). \tag{4.2}$$

Proof. (i) Suppose $0 \leq T \leq \tilde{T}$. Let $Q = cl \bigcup_{0 \leq t \leq T} R^{(t)}(X_0)$.
Then Q is compact by definition of $\tilde{T}$. So there exists $\theta > 0$ such that $Q + \theta B \subseteq \mathcal{N}$, where B is the closed unit ball in $\tilde{\mathcal{A}}$. We put

$$R_{\eta\xi} = \sup\{|V_{\eta\xi}| : V_{\eta\xi} \in P(Q + \theta B)(\eta, \xi)\}$$

$$:= \sup\{|V_{\eta\xi}| : V_{\eta\xi} \in \bigcup_{u \in Q + \theta B} P(u)(\eta, \xi)\}$$

and put $K_{\eta\xi} > 0$ to be Lipschitz constants for P on $Q + \theta B$.
Let $\epsilon > 0$. We show that for all large N satisfying

$$\frac{T}{N} \leq \min\left\{\frac{\epsilon}{R_{\eta\xi} K_{\eta\xi} T e^{K_{\eta\xi} T}}, \frac{\theta}{2R_{\eta\xi}}\right\}, \tag{4.3}$$

the inclusion

$$\left(I + \frac{T}{N} P_{\alpha\beta}\right)^j (X_0) \subseteq R^{(j\frac{T}{N})}(X_0) + \epsilon B \tag{4.4}$$

holds. Consequently, the inclusion

$$\left(I + \frac{T}{N} P\right)^j (X_0)(\eta, \xi) \subseteq R^{(j\frac{T}{N})}(X_0)(\eta, \xi) + \epsilon B(\eta, \xi) \tag{4.5}$$

immediately follows from Eq. (4.4). Here, $B(\eta, \xi) = \{\langle \eta, x\xi \rangle : x \in B\}$. Since ϵ is arbitrarily small, we can then conclude that Eq. (4.1) holds.
In the sequel, we put $h = \frac{T}{N}$, $t_j = jh$ for $j = 0, 1, 2 \ldots N$, where N satisfies Eq. (4.3).
We shall establish Eq. (4.4) by induction on j. The case $j = 0$ is trivial. For the induction hypothesis, suppose Eq. (4.4) holds for all i such that $0 \leq i \leq j < N$. Let $Y_{j+1} \in (I + hP_{\alpha\beta})^{j+1}(X_0)$. Then there exists $Y_0 = X_0, Y_1, Y_2 \ldots Y_j$ and $U_0, U_1 \ldots U_j$ so that for $0 \leq i \leq j$, we have

$$U_i \in P_{\alpha\beta}(Y_i) \quad \text{and} \quad Y_{i+1} = Y_i + hU_i.$$

We remark that when $0 \leq i \leq j$, Eq. (4.4) implies that

$$Y_i \in Q + \frac{\theta}{2} B$$

so that

$$\langle \eta, U_i \xi \rangle \in P(Q + \theta B)(\eta, \xi)$$

and so

$$|\langle \eta, U_i \xi \rangle| = \|\mathrm{U}_i\|_{\eta\xi} \leq \mathrm{R}_{\eta\xi}.$$

Let $\Phi(\cdot)$ be defined on $[0, t_{j+1}]$ as the piecewise linear interpolation of $\{Y_i\}_{i=0}^{j+1}$ equally spaced on $[0, t_{j+1}]$ as follows

$$\Phi(t) = Y_i + (t - t_i)U_i \quad \text{if} \quad t_i \leq t \leq t_{i+1}. \tag{4.6}$$

$\Phi(t)$ is adapted and weakly absolutely continuous on $[0, t_{j+1}]$. The range of $\Phi(\cdot)$ lies within $Q + \theta B$ because

$$Y_i + (t - t_i)U_i \in Q + \frac{\theta}{2}B + hR_{\eta\xi}B \subseteq Q + \theta B.$$

This follows from Eq. (4.3) since $hR_{\eta\xi} \leq \frac{\theta}{2}$. Hence we have

$$\begin{aligned}
\sigma(\Phi) &= \int_0^{t_{j+1}} \mathbf{d}\left(\frac{d}{dt}\langle \eta, \Phi(t)\xi \rangle, P(\Phi(t))(\eta, \xi)\right) dt \\
&\leq \sum_{i=0}^{j} \int_{t_i}^{t_{i+1}} \rho(P(Y_i)(\eta, \xi), P(\Phi(t))(\eta, \xi)) dt \\
&\leq K_{\eta\xi} \sum_{i=0}^{j} \int_{t_i}^{t_{i+1}} \|Y_i - \Phi(t)\|_{\eta\xi} dt \quad \text{by Lipschitz property of P} \\
&\leq K_{\eta\xi} T R_{\eta\xi} h \leq \epsilon e^{-K_{\eta\xi} T} \quad \text{by (4.3).}
\end{aligned}$$

By Theorem (2.1), there exists a solution $\tilde{\Phi}$ of Eq. (1.2) such that

$$\|\tilde{\Phi}(t) - \Phi(t)\|_{\eta\xi} \leq \sigma(\Phi) e^{K_{\eta\xi} T}, \quad 0 \leq t \leq t_{j+1}.$$

In particular, if we put $t = t_{j+1}$, $i = j + 1$, we have from Eq. (4.6) $\Phi(t_{j+1}) = Y_{j+1}$ and

$$\|\tilde{\Phi}(t_{j+1}) - Y_{j+1}\|_{\eta\xi} \leq \sigma(\Phi) e^{K_{\eta\xi} T} < \epsilon.$$

This implies that

$$\mathbf{d}_{\eta\xi}(Y_{j+1}, R^{(t_{j+1})}(X_0)) < \epsilon.$$

Hence, we have

$$Y_{j+1} \in R^{(t_{j+1})}(X_0) + \epsilon B.$$

Thus Eq. (4.4) holds for all $j = 0, 1, 2 \ldots N$. This completes the proof of (i).

(ii) The values of $x \longrightarrow P(x)(\eta, \xi)$ are now assumed to be convex. Thus Theorem (3.1) can be applied.
Let $\Phi(\cdot) \in S^{(T)}(X_0)$ such that its matrix elements $\langle \eta, \Phi(\cdot)\xi\rangle := \Phi_{\eta\xi}(\cdot) \in C^1[0,T]$, for each pair of $\eta, \xi \in \mathbb{E}$.
By Theorem (3.1), any $\tilde{\Phi}(\cdot) \in S^{(T)}(X_0)$ can be approximated to any degree of accuracy by $\Phi(\cdot)$. Consequently, to prove Eq. (4.2), it is sufficient to show that

$$\langle \eta, \Phi(T)\xi\rangle \in \liminf_{N\to\infty} \left(I + \frac{T}{N}P\right)^N (X_0)(\eta, \xi).$$

Denote by Q, the range of $\Phi(\cdot)$, i.e

$$Q = \{x : x = \Phi(t), \quad t \in [0, T]\}$$

and choose $\theta > 0$ so that

$$Q + \theta B \subseteq \mathcal{N}.$$

Let $K_{\eta\xi}$ be Lipschitz constants for the map P on $Q + \theta B$. For each integer N, $h = \frac{T}{N}$, $t_j = jh$, $j = 1, 2 \ldots N$, define

$$\varepsilon_{N,\eta\xi} = \sup_{j=0,1,\ldots N} \left| \frac{\langle \eta, \Phi(t_{j+1})\xi\rangle - \langle \eta, \Phi(t_j)\xi\rangle}{h} - \frac{d}{dt}\langle \eta, \Phi(t)\xi\rangle|_{t=t_j} \right|. \tag{4.7}$$

Since $\langle \eta, \Phi(\cdot)\xi\rangle$ is continuously differentiable on $[0, T]$, $\varepsilon_{N,\eta\xi} \longrightarrow 0$ as $N \longrightarrow \infty$. Assuming that N is large enough so that

$$\varepsilon_{N,\eta\xi} < \frac{\theta K_{\eta\xi}}{e^{K_{\eta\xi}T} - 1},$$

then it can be shown that

$$\langle \eta, \Phi(T)\xi\rangle \in \left(I + \frac{T}{N}P\right)^N (X_0)(\eta, \xi) + \frac{\varepsilon_{N,\eta\xi}}{K_{\eta\xi}}(e^{K_{\eta\xi}T} - 1)B(\eta, \xi). \tag{4.8}$$

To prove Eq. (4.8), we proceed by letting $Y_0 = X_0$ so that

$$Y_{\eta\xi,0} = \langle \eta, Y_0\xi\rangle = \langle \eta, X_0\xi\rangle$$

and

$$U_{\eta\xi,0} = \frac{d}{dt}\langle \eta, \Phi(t)\xi\rangle|_{t=t_0}, \quad M_0 = 0 \in \mathbb{R}.$$

Having chosen

$$Y_{\eta\xi,j} = \langle \eta, Y_j\xi\rangle, \quad U_{\eta\xi,j} = \langle \eta, U_j\xi\rangle$$

for some $U_j, Y_j \in \tilde{\mathcal{A}}$, let

$$Y_{\eta\xi,j+1} = \langle \eta, Y_j\xi \rangle + h\langle \eta, U_j\xi \rangle,$$

where

$$Y_{j+1} = Y_j + hU_j,$$

$$U_{\eta\xi,j+1} = \text{proj}\left(\frac{d}{dt}\langle \eta, \Phi(t)\xi \rangle|_{t=t_{j+1}},\ \ P(Y_{j+1})(\eta, \xi)\right)$$

and

$$M_{j+1} = (1 + K_{\eta\xi}h)M_j + 1.$$

We note that

$$M_j \leq M_{j+1}$$

for each j and therefore by Lemma (2.2) (with $R = 1$, $S = 1 + K_{\eta\xi}h$) we have

$$M_N = \frac{1}{K_{\eta\xi}h}((1 + K_{\eta\xi}h)^N - 1)) \leq \frac{1}{K_{\eta\xi}h}(e^{K_{\eta\xi}T} - 1). \tag{4.9}$$

Inductively, suppose for $0 \leq j < N$, the estimate

$$\|Y_j - \Phi(t_j)\|_{\eta\xi} \leq h\varepsilon_{N,\eta\xi}M_j \tag{4.10}$$

holds. When $j = 0$, Eq. (4.10) is trivial as $Y_0 = \Phi(0)$.
We have from Eq. (4.9) that

$$h\varepsilon_{N,\eta\xi}M_j \leq \frac{\varepsilon_{N,\eta\xi}}{K_{\eta\xi}}(e^{K_{\eta\xi}T} - 1) \leq \theta$$

by the choice of N.
Hence Eq. (4.10) implies that

$$Y_j \in Q + \theta B,$$

since Q consists of elements in the range of $\Phi(\cdot)$ and

$$B = \{x : \|x\|_{\eta\xi} \leq 1\} \subseteq \tilde{\mathcal{A}}.$$

By the Lipschitz property of P on $Q+\theta B$ and the choice of $U_{\eta\xi,j}$, we have

$$\left|\langle\eta, U_j\xi\rangle - \frac{d}{dt}\langle\eta,\Phi(t)\xi\rangle|_{t=t_j}\right| = \rho(P(Y_j)(\eta,\xi), P(\Phi(t_j))(\eta,\xi))$$

$$\leq K_{\eta\xi}\|Y_j-\Phi(t_j)\|_{\eta\xi}$$

$$\leq K_{\eta\xi}h\varepsilon_{\eta\xi,N}M_j \tag{4.11}$$

by Eq. (4.10).
Therefore,

$$|\langle\eta, Y_{j+1}\xi\rangle - \langle\eta,\Phi(t_{j+1})\xi\rangle| \leq |\langle\eta, Y_j\xi\rangle - \langle\eta,\Phi(t_j)\xi\rangle| + h|\langle\eta, U_j\xi\rangle$$

$$-\left|\frac{d}{dt}\langle\eta,\Phi(t)\xi\rangle|_{tj}\right|$$

$$+\left|\langle\eta,\Phi(t_j)\xi\rangle + h\frac{d}{dt}\langle\eta,\Phi(t)\xi\rangle|_{t_j} - \langle\eta,\Phi(t_{j+1})\xi\rangle\right|$$

$$\leq h\varepsilon_{N,\eta\xi}M_j + K_{\eta\xi}h^2\varepsilon_{N,\eta\xi}M_j + h\varepsilon_{N,\eta\xi},$$

by (4.10) (4.11) and (4.7)

$$= h\varepsilon_{N,\eta\xi}[(1+K_{\eta\xi}h)M_j+1] = h\varepsilon_{N,\eta\xi}M_{j+1}.$$

Hence

$$\|Y_{j+1}-\Phi(t_{j+1})\|_{\eta\xi} \leq h\varepsilon_{N,\eta\xi}M_{j+1}.$$

The estimate (4.10) holds for $j+1$.
When $j=N$, Eq. (4.10) combined with Eq. (4.9) leads to

$$|\langle\eta, Y_N\xi\rangle - \langle\eta,\Phi(T)\xi\rangle| = \|Y_N-\Phi(T)\|_{\eta\xi}$$

$$\leq \frac{\varepsilon_{N,\eta\xi}}{K_{\eta\xi}}(e^{K_{\eta\xi}T}-1) \tag{4.12}$$

By the choice of Y_j, $j=0,1,2\ldots N$, $\langle\eta, Y_N\xi,\rangle$ lies in $\left(I+\frac{T}{N}P\right)^N(X_0)(\eta,\xi)$ so that Eq. (4.8) follows directly from Eq. (4.12).
By letting $N\to\infty$ in Eq. (4.8), the conclusion (4.2) follows.

The exponential formula is recorded in the next Corollary as an immediate consequence of Theorem 4.3.

Corollary 4.4. *Suppose that $\mathcal{N} \subseteq \tilde{\mathcal{A}}$ is open and P is locally Lipschitzian on $\mathcal{N}$ with nonempty, compact and convex values. Then for any $X_0 \in \mathcal{N}$, and $0 \le T \le \tilde{T}$, we have*

$$R^{(T)}(X_0)(\eta, \xi) = \lim_{N \to \infty} \left(I + \frac{T}{N} P \right)^N (X_0)(\eta, \xi).$$

The next Corollary indicates that the interval $[0, T]$ may be partitioned in an arbitrary manner provided that the width of the largest subinterval goes to zero. To this end, we need the following definitions.

If $D = \{t_0, t_1, \ldots, t_N\}$ is a patition of $[0, T]$ (that is $t_0 < t_1 < t_2 \cdots < t_N = T$), define

$$|D| = \sup_{0 \le j \le N-1} |t_{j+1} - t_j|.$$

If $\{P_{\alpha\beta,j}\}_{j=1}^N$ is a collection of multifunctions $P_{\alpha\beta,j} : \tilde{\mathcal{A}} \to 2^{\tilde{\mathcal{A}}}$, define the multifunction product by

$$(\Pi_{j=1}^N P_{\alpha\beta,j})(x) = (P_{\alpha\beta,N} \circ P_{\alpha\beta,N-1} \circ \cdots P_{\alpha\beta,1})(x).$$

For η, $\xi \in \mathbb{E}$, this leads to the definition of the sesquilinear form:

$$(\Pi_{j=1}^N P_j)(x)(\eta, \xi) := \langle \eta, (\Pi_{j=1}^N P_{\alpha\beta,j})(x)\xi \rangle.$$

Corollary 4.5. *Suppose that $\mathcal{N}$ and P are as in Corollary 4.4 and let $X_0 \in \mathcal{N}$, $0 \le T < \tilde{T}$ and η, $\xi \in \mathbb{E}$. Then for any sequence of partitions $D_k = \{t_0^k, t_1^k, \ldots t_{N_k}^k\}$ of $[0, T]$ with $|D_k| \to 0$ as $k \to \infty$, we have*

$$R^{(T)}(X_0)(\eta, \xi) = \lim_{k \to \infty} (\Pi_{j=0}^{N_k - 1}(I + (t_{j+1}^k - t_j^k)P)(X_0)(\eta, \xi).$$

Proof. Follows similar steps as in the proof of Theorem 4.3 by replacing h by $h_j^k := t_{j+1}^k - t_j^k$. □

The Nonautonomous Case

Theorem 4.3 has its generalization to the nonautonomous version of Eq. (1.2) as follows. We consider the initial time $t_0 = 0$ as usual. Let $\mathcal{N}$ be open in $\mathcal{A}$. Assume that for each pair $\eta, \xi \in \mathbb{E}$, $(t, x) \in [0, T] \times \mathcal{N}$, the multifunction $P(t, x)(\eta, \xi)$ is compact and convex in $\tilde{\mathcal{C}}$ such that $t \mapsto P(t, x) \times (\eta, \xi)$ is continuous on $[0, \infty)$ and $x \mapsto P(t, x)(\eta, \xi)$ is locally Lipschitzian on $\mathcal{N}$ independent of $t \in [0, T]$.

As in Theorem 4.3, we consider first, an equidistant time discretization of the interval $[0, T]$ and in addition introduce the following notations. For $\eta, \xi \in \mathbb{E}$ such that $\eta = e(\alpha)$, $\xi = e(\beta)$, we set $P_{\alpha\beta}(t_j, x) := P_{\alpha\beta,j}(x)$ where $P_{\alpha\beta,j} : \tilde{\mathcal{A}} \mapsto 2^{\tilde{\mathcal{A}}}$ and $t_j = j\frac{T}{N}$, $j = 1, 2 \ldots N$ being an equidistant partition of the interval $[0, T]$. Consequently, we have the following results.

Theorem 4.6 *Suppose that the multifunction* $(t, x) \mapsto P(t, x)(\eta, \xi)$ *satisfies the above conditions. Then for all* $x_0 \in \mathcal{N}$ *and* $0 \leq T \leq \tilde{T}$, *we have*

$$R^{(T)}(X_0)(\eta, \xi) = \lim_{N \mapsto \infty} \Pi_{j=1}^{N} \left(I + \frac{T}{N} P_j \right) (X_0)(\eta, \xi), \tag{4.13}$$

where

$$\Pi_{j=1}^{N} \left(I + \frac{T}{N} P_j \right) (X_o)(\eta, \xi) = \langle \eta, \left(\Pi_{j=1}^{N} \left(I + \frac{T}{N} P_{\alpha\beta,j} \right) (X_0) \right) \xi \rangle$$

The proof of Theorem 4.6 *is omitted since it involves a routine modification of the proof of Theorem* 4.3. *The nonautonomous version of Corollary* 4.5 *is also straightforward to establish.*

Approximations of Reachable Set for the QSDI

Consider now the autonomous version of QSDI (1.2). Then the Euler approximation to the reachable set in the case of equally spaced partition $\{t_i\}$

can be written as follows

$$R^N_{\eta\xi,1} = (I + hP)(Y_0)(\eta, \xi)$$
$$R^N_{\eta\xi,2} = (I + hP)^2(Y_0)(\eta, \xi)$$
$$\ldots\ldots\ldots\ldots\ldots$$
$$R^N_{\eta\xi,i} = (I + hP)^i(Y_0)(\eta, \xi)$$

where the power of $(I + hP)$ is that of composition of set valued map described in Section 4. Then, by Theorem (4.3) and Corollary (4.4) $R^N_{\eta\xi,N}$, will converge to $R^{(T)}(X_0)(\eta, \xi)$ in the sense that

$$R^{(T)}(X_0)(\eta, \xi) = \lim_{N\to\infty}\left(I + \frac{T}{N}P\right)^N (X_0)(\eta, \xi)$$

provided that the conditions of the Theorem are satisfied.

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REFERENCES

1. Aubin, J.P.; Frankowska, H. *Set-Valued Analysis*; Birkhauser, 1990.
2. Ayoola, E.O. On numerical procedures for solving Lipschitzian quantum stochastic differential equations Ph.D Thesis, University of Ibadan, Nigeria, 1998.
3. Ayoola, E.O. Solutions of Lipschizian quantum stochastic differential equations in a locally convex topological space. J. Sci. Res. **1998**, *4* (1), 15–23.
4. Ayoola, E.O. Converging multistep methods for weak solutions of quantum stochastic differential equations. Stoch. Anal. Appl. **2000**, *18* (4), 525–554.

5. Ayoola, E.O. Construction of approximate attainability sets for Lipschitzian quantum stochastic differential inclusions. Stoch. Anal. Appl. **2001**, *19* (3), 461–471.
6. Dontchev, A.L.; Farkhi, E. Error estimates for discretized differential inclusions. Computing **1989**, *41*, 349–358.
7. Dontchev, A.; Lempio, F. Difference methods for differential inclusions: a survey. SIAM Rev. **1992**, *34* (2), 263–294.
8. Ekhaguere, G.O.S. Lipschitzian quantum stochastic differential inclusions. Int. J. Theor. Phys. **1992**, *31* (11), 2003–2034.
9. Ekhaguere, G.O.S. Quantum stochastic differential inclusions of hypermaximal monotone type. Int. J. Theor. Phys. **1995**, *34* (3), 323–353.
10. Ekhaguere, G.O.S. Quantum stochastic evolutions. Int. J. Theor. Phys. **1996**, *35* (9), 1909–1946.
11. Filipov, A.F. On the existence of solutions of multivalued differential equation. Mat. Zametk **1971**, *10*, 307–313.
12. Filipov, A.F. Classical solutions of differential equations with multi-valued right hand side. SIAM J. Control **1967**, *5*, 609–621.
13. Hudson, R.L.; Parthasarathy, K.R. Quantum Ito's formulae and stochastic evolutions. Commun. Math. Phys. **1984**, *93*, 301–324.
14. Komarov, V.A. Estimates of the attainability sets of differential inclusions. Mat. Zametki **1985**, *37* (6), 916–925.
15. Komarov, V.A.; Pevchikh, K.E. A method of approximating attainability sets for differential inclusions with a specified accuracy. Comput. Math. Math. Phys. **1991**, *31* (1), 109–112.
16. Phillips, E.R. *An Introduction to Analysis and Integration Theory*; Intext Educational Publishers: Scranton, Toronto, London, 1970.
17. Rudin, W. *Real and Complex Analysis*; McGraw-Hill: New York, 1974.
18. Veliov, V. Second order discrete approximation to linear differential inclusions. SIAM J. Num. Anal. **1992**, *2*, 439–451.
19. Wolenski, P.R. A uniqueness theorem for differential inclusions. J. Differential Eq. **1990**, *84*, 165–182.
20. Wolenski, P.R. The exponential formula for the reachable set of a Lipschitz differential inclusion. SIAM J. Control Optim. **1990**, *28* (5), 1148–1161.

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LAGRANGIAN QUADRATURE SCHEMES FOR COMPUTING WEAK SOLUTIONS OF QUANTUM STOCHASTIC DIFFERENTIAL EQUATIONS*

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Abstract. Lagrangian quadrature schemes for computing weak solutions of Lipschitzian quantum stochastic differential equations are introduced and studied. This is accomplished within the framework of the Hudson–Parthasarathy formulation of quantum stochastic calculus and subject to matrix elements of solution being sufficiently differentiable. Results concerning convergence of these schemes in the topology of the locally convex space of solution are presented. Numerical examples are given.

Key words. quantum stochastic differential equations, Boson Fock spaces, exponential vectors, sesquilinear form valued maps, Lagrangian quadrature, error estimates

AMS subject classifications. 60H10, 60H20, 65U05, 81S25

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1. Introduction.

There have been intense research activities in the development of numerical schemes for solving classical stochastic differential equations of the form

$$\begin{aligned} dX(t,w) &= H(t,X)dt + F(t,X)dQ(t), \\ X(t_0) &= X_0, \qquad t \in [t_0,T], \end{aligned} \tag{1.1}$$

where the driving process $Q(t)$ is a martingale and H, F are sufficiently smooth, deterministic, and ordinary functions. Each of the schemes exhibits specific features depending on the driving process and the solution space of (1.1) (see [14, 16, 18, 19, 27, 28, 31, 37]).

A noncommutative generalization of (1.1) is the following quantum stochastic differential equation introduced by Hudson and Parthasarathy [13]:

$$\begin{aligned} dX(t) &= E(t,X(t))d\wedge_\pi(t) + F(t,X(t))dA_f^+(t) \\ &\quad + G(t,X(t))dA_g(t) + H(t,X(t))dt, \\ X(t_0) &= X_0, \quad \text{almost all} \quad t \in [t_0,T]. \end{aligned} \tag{1.2}$$

In (1.2), the coefficients E, F, G, H lie in a certain class of stochastic processes for which quantum stochastic integrals against the gauge, creation, and annihilation processes $\wedge_\pi$, A_f^+, A_g and the Lebesgue measure are defined. These integrator processes are formally defined in the next section. Equation (1.2) involves unbounded linear operators on a Hilbert space.

It is important to emphasize that several benefits have been achieved by interpreting standard probability in a noncommutative quantum setting. Such benefits include a better understanding of classical stochastic flows and of some parts of Wiener space analysis and Wiener chaos expansions, which have recently been renewed through a fundamental discovery of the chaotic representation property of the Azema martingales (see Meyer [17]).

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Quantum stochastic differential equations of the type (1.2) often arise as mathematical models which describe, among other things, quantum dynamical systems and several physical problems in quantum stochastic control theory and quantum stochastic evolutions (see, for example, [5, 6, 7, 10, 11, 13, 34, 35, 36]). Approximate solutions of these models are imperative since analytical solutions are often difficult to obtain in practice. In addition, (1.2) reduces to the classical Ito stochastic differential equation (1.1) by a suitable choice of parameters in a simple Fock space. In this case, the Ito process is regarded as a multiplication operator-valued process. In general, quantum stochastic differential equations have undergone rapid analytical developments without corresponding developments in their numerical analysis (cf. [3, 5, 6, 7, 9, 11, 12, 13, 23, 24, 33]).

In the work of Ekhaguere [5], (1.2) has been reformulated in the following equivalent form:

$$\begin{aligned} \frac{d}{dt}\langle \eta, X(t)\xi\rangle &= P(t, X(t))(\eta,\xi), \\ X(t_0) &= X_0, \quad t \in [t_0, T], \end{aligned} \tag{1.3}$$

which is an ordinary differential equation of nonclassical type. The solution stochastic process $X(t)$ is a densely defined linear operator on some tensor product of two Hilbert spaces, one of which is the Boson Fock space; η, ξ lie in a dense subset of the tensor product Hilbert space and $\langle \eta, X(t)\xi\rangle$ is the inner product of the elements η and $X(t)\xi$ in the space. The map $(\eta,\xi) \to P(t, X)(\eta,\xi)$ is a sesquilinear form for fixed (t, X). For processes which leave their domain invariant and for any positive integer m, the mth power $X^m(t)$ of the unknown process is understood in the sense of composition of the operator $X(t)$ with itself m times. However, our considerations in this work concern (1.2) involving the unknown process of degree $m = 1$.

In this paper we study techniques based on Lagrangian quadrature, which can be applied for computing weak solutions of the quantum stochastic differential equation (1.2). By imposing the conditions that a choice of nodes should force the Lagrangian interpolating projection to yield a best approximation in a suitable norm, the criterion for the quadrature algorithm is derived.

As a special case of (1.2), our present work may be used to obtain discrete approximate solutions, in the weak sense, of a class of the quantum stochastic evolution equation

$$X(t) = X(t_0) + \int_{t_0}^{t} X(s)[L_1 d\Lambda(s) + L_2 dA(s) + L_3 dA^+(s) + L_4 ds], \tag{1.4}$$

considered in [13, 32], where $\Lambda(t),\ A(t),\ A^+(t)$, respectively, the gauge, annihilation, and creation processes of strengths $\pi = g = f = \mathbf{I}$ and $L_j,\ j = 1, 2\cdots 4$, are operators defined on the initial space. As highlighted in [13], the unitary solutions of the evolution equations generate uniformly continuous semigroups of completely positive maps. Consequently, our schemes may be used in the integration of irreversible equation of motion described by such semigroups. These concepts as well as applications to quantum theory of open systems and heat baths will be addressed elsewhere.

The quadrature method may also be used to generate approximations of the expected value of solution of (1.1). We present an example for the linear case. Extension of our work to the case of nonlinear stochastic differential equations will definitely involve a certain class of quantum stochastic processes which leave their domain invariant. This issue will be addressed in our subsequent papers.

An important feature of quadrature methods is that the nodal points in the interval of integration need not be equidistant. The schemes here involve less complicated analysis, and the order of convergence is independent of some approximation procedures similar to those based on stochastic Taylor expansions for real valued processes as obtained in the numerical solution of (1.1) (see, for example, [14, 18, 37]). Another important feature concerns implementations. Computations of the discrete values of the matrix elements of solution are carried out directly as obtained in the implementations of discrete schemes for solving initial value problems for ordinary differential equations. This is an advantage compared with the implementation of discrete Taylor schemes for simulations of sample paths and functionals of the solution of stochastic differential equation (1.1), where the random increments of the driving process have to be computed by a random number generator. We do not have a uniform tightness type of requirement or a correction term in the limit as in the case of the Wong–Zakai approximation of the noise terms in the classical setting (see [14, 18, 31, 37]).

There are many other interesting motivations for studying quadrature solutions of deterministic ordinary and partial differential equations. For an account of these and some limitations of the methods, we refer the reader to [4, 20, 21, 22]. In these references, quadrature methods have been applied to ordinary and partial differential equations. In general, the quadrature method is a fast and efficient method involving substantial savings in computational efforts for a suitable value of the total nodal points N.

Under an appropriate optimality condition, the Lagrangian quadrature transforms the quantum stochastic differential equation to purely algebraic equations in terms of the nodal values only. Since common stochastic processes are not differentiable in the classical sense, quadrature method cannot be applied directly to solve ordinary stochastic differential equations with respect to weak convergence criteria. We are able to apply this method by considering the equivalent form of the quantum analogue of these equations. This yields a benefit of quantization of ordinary stochastic differential equations.

Since the ordinary differential equation (1.3) is of a nonclassical type, stability analysis is much more complicated compared with the classical initial value problems. In subsequent papers, we will examine the stability of the quadrature schemes.

The rest of the paper is organized as follows: In section 2, we outline some fundamental definitions, notations, and structures that are foundations of the Boson Fock space stochastic calculus employed in what follows. These are adopted from [5]. Section 3 is devoted to the derivation of the Lagrangian quadrature appropriate for quantum stochastic differential equations under some optimality conditions. In section 4, we describe the procedures for choosing the Lagrangian interpolants and the computation of quadrature coefficients. The establishment of bounds for the local and global discretization error is done in section 5. Some examples of (2.2) and results of numerical experiments are reported in section 6. In comparison with the accuracy of the Euler and a 2-step scheme employed in [2] to solve the same model problem considered in Example 2, it is discovered that the Lagrangian quadrature scheme produced better results and that numerical values of the matrix elements of solution depend on the chosen exponential vectors.

2. Preliminaries. Let D be an inner product space and H be the completion of D. We denote by $L^+(D,H)$ the set of all linear maps X from D into H such that $\mathrm{Dom}[X^*] \supseteq D$, where X^* is the adjoint of X. We remark that $L^+(D,H)$ is a linear space under the usual notions of addition and scalar multiplication of operators.

If H is a Hilbert space, we denote by $\Gamma(H)$ the Boson Fock space determined by H. For $f \in H$, $e(f)$ denotes the exponential vector in $\Gamma(H)$ corresponding to f. We remark here that the subspace $\mathbb{E}$ of $\Gamma(H)$ generated by the set of exponential vectors in $\Gamma(H)$ is dense in $\Gamma(H)$. Since the exponential vectors are linearly independent, an operator with domain $\mathbb{E}$ is well defined by specifying its action on $e(f)$, $f \in H$. For other properties enjoyed by the exponential vectors and Boson Fock space, we refer the reader to [1, 5, 17, 23, 24].

In what follows, $\mathbb{D}$ is some inner product space with $\mathcal{R}$ as its completion, and Y is some fixed Hilbert space:

(i) For each $t \in \mathbb{R}_+$, we write $L^2_Y(\mathbb{R}_+)$ (resp., $L^2_Y([0,t))$; resp., $L^2_Y([t,\infty))$) for the Hilbert spaces of square integrable, Y-valued maps on $[0,\infty)$ (resp., $[0,t)$; resp., $[t,\infty)$).

(ii) The noncommutative stochastic processes which we shall discuss are densely defined linear operators on $\mathcal{R}\otimes\Gamma(L^2_Y(\mathbb{R}_+))$; the inner product of this complex Hilbert space will be denoted by $\langle\cdot,\cdot\rangle$ and its norm by $\|\cdot\|$.

(iii) Let $\mathbb{E}, \mathbb{E}_t$, and $\mathbb{E}^t, t > 0$, be the linear spaces generated by the exponential vectors in $\Gamma(L^2_Y(\mathbb{R}_+)), \Gamma(L^2_Y([0,t)))$, and $\Gamma(L^2_Y([t,\infty)))$, respectively; then we introduce the following spaces:
   (a) $\mathcal{A} \equiv L^+(\mathbb{D}\underline{\otimes}\mathbb{E}, \mathcal{R}\otimes\Gamma(L^2_Y(\mathbb{R}_+)))$,
   (b) $\mathcal{A}_t \equiv L^+(\mathbb{D}\underline{\otimes}\mathbb{E}_t, \mathcal{R}\otimes\Gamma(L^2_Y([0,t)))) \otimes \mathbf{1}^t$,
   (c) $\mathcal{A}^t \equiv \mathbf{1}_t \otimes L^+(\mathbb{D}\underline{\otimes}\mathbb{E}^t, \mathcal{R}\otimes\Gamma(L^2_Y([t,\infty)))), t > 0$,
   where $\underline{\otimes}$ denotes algebraic tensor product and $\mathbf{1}_t$ (resp., $\mathbf{1}^t$) denotes the identity map on $\mathcal{R}\otimes\Gamma(L^2_Y([0,t)))$ (resp., $\Gamma(L^2_\gamma([t,\infty)))), t > 0$.

We note that $\mathcal{A}^t$ and $\mathcal{A}_t, t > 0$, may be naturally identified with subspaces of $\mathcal{A}$.

For $\eta,\xi \in \mathbb{D}\underline{\otimes}\mathbb{E}$, we define $\|\cdot\|_{\eta\xi}$ on $\mathcal{A}$ by $\|x\|_{\eta\xi} = |\langle\eta, x\xi\rangle|, x \in \mathcal{A}$. Then $\{\|\cdot\|_{\eta\xi}, \eta,\xi \in \mathbb{D}\underline{\otimes}\mathbb{E}\}$ is a family of seminorms on $\mathcal{A}$; we write τ_W for the locally convex Hausdorff topology on $\mathcal{A}$ determined by this family. We denote by $\tilde{\mathcal{A}}$, $\tilde{\mathcal{A}}_t$, and $\tilde{\mathcal{A}}^t$ the completions of the locally convex topological spaces $(\mathcal{A},\tau_W)$, $(\mathcal{A}_t,\tau_W)$, and $(\mathcal{A}^t,\tau_W)$, $t > 0$, respectively. The net $\{\tilde{\mathcal{A}}_t : t \in \mathbb{R}_+\}$ is a filtration of $\tilde{\mathcal{A}}$.

2.1. Boson quantum stochastic integration. Before defining the quantum stochastic integral employed in the subsequent sections, we present a number of important notations and definitions.

Let $I \subseteq \mathbb{R}_+$; then we have the following:

(i) A map $X : I \to \tilde{\mathcal{A}}$ is called a stochastic process indexed by I.

(ii) A stochastic process X is called adapted if $X(t) \in \tilde{\mathcal{A}}_t$ for each $t \in I$. We denote by $Ad(\tilde{\mathcal{A}})$ the set of all adapted stochastic processes indexed by I.

(iii) A member X of $Ad(\tilde{\mathcal{A}})$ is called
   (a) weakly absolutely continuous if the map $t \to \langle\eta, X(t)\xi\rangle$, $t \in I$, is absolutely continuous for arbitrary $\eta,\xi \in \mathbb{D}\underline{\otimes}\mathbb{E}$.—we denote this subset of $Ad(\tilde{\mathcal{A}})$ by $Ad(\tilde{\mathcal{A}})_{wac}$;
   (b) locally absolutely p-integrable if $\|X(\cdot)\|^p_{\eta\xi}$ is Lebesgue measurable and integrable on $[t_0,t) \subseteq I$ for each $t \in I$, $p \in (0,\infty)$ and arbitrary $\eta,\xi \in \mathbb{D}\underline{\otimes}\mathbb{E}$.—we denote this subset of $Ad(\tilde{\mathcal{A}})$ by $L^p_{loc}(\tilde{\mathcal{A}})$.

Stochastic integrators. Let $B(Y)$ denote the Banach space of bounded endomorphisms of Y and let the spaces $L^\infty_{Y,loc}(\mathbb{R}_+)$ (resp., $L^\infty_{B(Y),loc}(\mathbb{R}_+)$) be the linear space of all measurable, locally bounded functions from $\mathbb{R}_+$ to Y (resp., to $B(Y)$). If $f \in L^\infty_{Y,loc}(\mathbb{R}_+)$ and $\pi \in L^\infty_{B(Y),loc}(\mathbb{R}_+)$, then πf is the member of $L^\infty_{Y,loc}(\mathbb{R}_+)$ given by $(\pi f)(t) = \pi(t)f(t),\quad t \in \mathbb{R}_+$. For $f \in L^2_Y(\mathbb{R}_+)$ and $\pi \in L^\infty_{B(Y),loc}(\mathbb{R}_+)$, we define the operators $a(f)$, $a^+(f)$, and $\lambda(\pi)$ in $L^+(\mathbb{D},\Gamma(L^2_Y(\mathbb{R}_+)))$ as follows: $a(f)e(g) = \langle f,g\rangle_{L^2_Y(\mathbb{R}_+)}e(g)$, $a^+(f)e(g) = \frac{d}{d\sigma}e(g+\sigma f)|_{\sigma=0}$, $\lambda(\pi)e(g) = \frac{d}{d\sigma}e(e^{\sigma\pi}f)|_{\sigma=0}$ for $g \in L^2_Y(\mathbb{R}_+)$. These operators give rise to the operator-valued maps A_f, A_f^+, and $\wedge_\pi$ defined by $A_f(t) \equiv a(f\chi_{[0,t)})$, $A_f^+(t) \equiv a^+(f\chi_{[0,t)})$, $\wedge_\pi(t) \equiv \lambda(\pi\chi_{[0,t)})$, $t \in \mathbb{R}_+$, where χ_I denotes the indicator function of the Borel set $I \subseteq \mathbb{R}_+$.

The operators $a(f)$, $a^+(f)$, and $\lambda(\pi)$ are the annihilation, creation, and gauge operators of quantum field theory, respectively. The maps A_f, A_f^+, and $\wedge_\pi$ are stochastic processes, called the annihilation, creation, and gauge processes, respectively, when their values are identified with their ampliations on $\mathcal{R}\underline{\otimes}\Gamma(L^2_Y(\mathbb{R}_+))$; i.e., for any $r \in \{A_f, A_f^+, \wedge_\pi\}$ and $\eta = c \otimes e(\alpha)$, with $\alpha \in L^2_Y(\mathbb{R}_+)$, $c \in \mathcal{R}$, then $r(t)(c \otimes e(\alpha)) = r(t)c \otimes e(\alpha)$.

These are the stochastic integrators in the Hudson and Parthasarathy [13] formulation of Boson quantum stochastic integration which we adopt in the rest of this paper. Next we present the definition of the stochastic integrals, beginning with simple stochastic processes.

A stochastic process $p \in Ad(\tilde{\mathcal{A}})$ is called simple if there exists an increasing sequence $t_n, n = 0,1,2,\ldots$, with $t_0 = 0$ and $t_n \to \infty$ such that for each $n \geq 0$, $p(t) = p(t_n)$ for $t \in [t_n, t_{n+1})$. Let $p,q,u,v \in Ad(\tilde{\mathcal{A}})$ be simple adapted stochastic processes and let $f,g \in L^\infty_{Y,loc}(\mathbb{R}_+)$ and $\pi \in L^\infty_{B(Y),loc}(\mathbb{R}_+)$. Then the family of operators $M = \{M(t) : t \geq 0\}$ in $Ad(\tilde{\mathcal{A}})$ defined by

$$\begin{aligned} M(0) &= 0,\\ M(t) &= M(t_n) + p(t_n)(\wedge_\pi(t) - \wedge_\pi(t_n)) + q(t_n)(A_f(t) - A_f(t_n))\\ &\quad + u(t_n)(A_g^+(t) - A_g^+(t_n)) + v(t_n)(t - t_n), \qquad t_n < t < t_{n+1}, \end{aligned}$$

is called the stochastic integral of p,q,u,v with respect to $\wedge_\pi$, A_f, A_g^+, and the Lebesgue measure. It is understood in integral form by

$$M(t) = \int_0^t (p(s)d\wedge_\pi(s) + q(s)dA_f(s) + u(s)dA_g^+(s) + v(s)ds)$$

and denoted in differential form as

$$\begin{aligned} M(0) &= 0,\\ dM(t) &= p(t)d\wedge_\pi(t) + q(t)dA_f(t) + u(t)dA_g^+(t) + v(t)dt. \end{aligned}$$

Next we state the first fundamental result due to Hudson and Parthasarathy [13] concerning quantum stochastic integrals of simple and adapted processes. In what follows the inner product of the space Y is denoted by $\langle\cdot,\cdot\rangle_Y$.

THEOREM 2.1.

(a) *Let p,q,u,v be simple processes in $Ad(\tilde{\mathcal{A}})$ and let M be their stochastic integral. If $\eta, \xi \in \mathbb{D}\underline{\otimes}\mathbb{E}$ with $\eta = c \otimes e(\alpha)$, $\xi = d \otimes e(\beta), c, d \in \mathbb{D}$, $\alpha, \beta \in L^\infty_{Y,loc}(\mathbb{R}_+)$, and $t \geq 0$, then*

$$\begin{aligned} \langle \eta, M(t)\xi\rangle = \int_0^t &\langle \eta, \{\langle\alpha(s),\pi(s)\beta(s)\rangle_Y p(s) + \langle f(s),\beta(s)\rangle_Y q(s)\\ &+\langle\alpha(s),g(s)\rangle_Y u(s) + v(s)\}\xi\rangle ds. \end{aligned} \tag{2.1}$$

(b) *The result in* (a) *above remains true if for each integrand $F \in \{p,q,u,v\}$ the map $t \to F(t)\xi$ is measurable and satisfies*

$$\int_0^t \|F(s)\xi\|^2 ds < \infty \qquad \forall\, t > 0 \quad \textit{and} \quad \forall\, \xi \in \mathbb{D}\underline{\otimes}\mathbb{E}.$$

Extension of the stochastic integral above to integrands in $L^2_{loc}(\tilde{\mathcal{A}})$ requires estimates of the integral in the family of seminorms $\{\|\cdot\|_{\eta\xi}, \eta, \xi \in \mathbb{D}\underline{\otimes}\mathbb{E}\}$ that generates the topology of $\tilde{\mathcal{A}}$. First, we have the following proposition (Proposition 3.2 in [13]), which is useful in extending Theorem 2.1 to integrands in $L^2_{loc}(\tilde{\mathcal{A}})$.

Proposition 2.2. *Let $p \in L^2_{loc}(\tilde{\mathcal{A}})$. Then there exists a sequence $p^{(n)}$, $n = 1, 2, \ldots,$ of simple adapted processes such that for each $t > 0$ and for arbitrary $\eta, \xi \in \mathbb{D}\underline{\otimes}\mathbb{E}$, $\lim_{n\to\infty} \int_0^t \|p(s) - p^{(n)}(s)\|^2_{\eta\xi} ds = 0$.*

Next we present the following result which can be easily proved by applying Theorem 2.1.

Proposition 2.3. *Assume that the following hold:*

(i) *p, q, u, v are simple processes in $Ad(\tilde{\mathcal{A}})$.*

(ii) *$M(t) = \int_0^t (p(s)d\wedge_\pi(s) + q(s)dA_f(s) + u(s)dA^+_g(s) + v(s)ds)$ for each $t \in [0, T]$, $T > 0$.*

For arbitrary $\eta, \xi \in \mathbb{D}\underline{\otimes}\mathbb{E}$ with $\eta = c \otimes e(\alpha)$, $\xi = d \otimes e(\beta)$, $c, d \in \mathbb{D}$, $\alpha, \beta \in L^\infty_{Y,loc}(\mathbb{R}_+)$, let $K_{\eta\xi,T}$ be given by

$$K_{\eta,\xi,T} = \sup_{0\le s\le T} \max\{|\langle\alpha(s), \pi(s)\beta(s)\rangle|, \ \ |\langle(s), \beta(s)\rangle|, \ \ |\langle\alpha(s), g(s)\rangle|, \ \ 1\};$$

then

$$\|M(t)\|_{\eta\xi} \le K_{\eta,\xi,T} \int_0^T [\|p(s)\|_{\eta\xi} + \|q(s)\|_{\eta\xi} + \|u(s)\|_{\eta\xi} + \|v(s)\|_{\eta\xi}]ds.$$

Extension of the quantum stochastic integral. Let p, q, u, v be elements of $L^2_{loc}(\tilde{\mathcal{A}})$. Then by Proposition 2.2, there exist simple adapted processes p_n, q_n, u_n, v_n which approximate p, q, u, v in $L^2_{loc}(\tilde{\mathcal{A}})$. We now let

$$M_n(t) = \int_0^t (p_n(s)d\wedge_\pi(s) + q_n(s)dA_f(s) + u_n(s)dA^+_g(s) + v_n(s)ds).$$

Applying the estimate of Proposition 2.3 to the difference $M_n(t) - M_m(t)$, $m, n \in \mathbb{N}$, we find that the sequence $\{M_n(t)\}$ is a Cauchy sequence in $\tilde{\mathcal{A}}$ and therefore converges to a limit in $\tilde{\mathcal{A}}$ by the completeness of the locally convex space. The limit $M(t)$ is independent of the choice of approximating sequences and is defined to be the integral

$$M(t) = \int_0^t (p(s)d\wedge_\pi(s) + q(s)dA_f(s) + u(s)dA^+_g(s) + v(s)ds).$$

By employing the uniformity of the convergence on finite intervals, we may pass to the limit of approximations by simple processes, so that Theorem 2.1 for coefficients p, q, u, v belonging to $L^2_{loc}(\tilde{\mathcal{A}})$ remains valid.

Some fundamental notations and definitions.

(a) We denote the space of sesquilinear forms on $\mathbb{D}\underline{\otimes}\mathbb{E}$ by sesq$[\mathbb{D}\underline{\otimes}\mathbb{E}]$. Thus, sesq$[\mathbb{D}\underline{\otimes}\mathbb{E}] = \{a \ : \ \mathbb{D}\underline{\otimes}\mathbb{E} \ \ \times \mathbb{D}\underline{\otimes}\mathbb{E} \longrightarrow \mathbb{C}|$ the map $(\eta, \xi) \to a(\eta, \xi)$ is linear in ξ and conjugate linear in η $\forall\, \eta, \xi \in \mathbb{D}\underline{\otimes}\mathbb{E}\}$.

(b) A stochastic process Φ will be called locally absolutely p-integrable if the map $t \to \|\Phi(t)\|_{\eta\xi}$, $t \in \mathbb{R}_+$, lies in $L^p_{loc}(I)$ for arbitrary $\eta, \xi \in \mathbb{D}\underline{\otimes}\mathbb{E}$ and $p \in (0, \infty)$.

(c) For $p \in (0, \infty)$ and $I \subseteq \mathbb{R}_+$, $L^p_{loc}(I \times \tilde{\mathcal{A}})$ denotes the set of maps $\Phi \ : \ I \times \tilde{\mathcal{A}} \longrightarrow \tilde{\mathcal{A}}$, such that the map $t \longrightarrow \Phi(t, X(t))$ lies in $L^p_{loc}(\tilde{\mathcal{A}})$ for every $X \in L^p_{loc}(\tilde{\mathcal{A}})$.

In what follows, $f, g \in L^{\infty}_{Y,loc}(\mathbb{R}_+)$, $\pi \in L^{\infty}_{B(Y),loc}(\mathbb{R}_+)$, $\mathbf{1}$ is the identity map on $\mathcal{R} \otimes \Gamma(L^2_Y(\mathbb{R}_+))$. We introduce the processes A_f, A^+_g, $\wedge_\pi$, and $s \longrightarrow s\mathbf{1}$, $s \in \mathbb{R}_+$ as integrators.

(d) Let $E, F, G, H \in L^2_{loc}(I \times \tilde{\mathcal{A}})$ and (t_0, X_0) be a fixed point of $I \times \tilde{\mathcal{A}}$. Then a relation of the form

$$\begin{aligned} X(t) = X_0 + \int_{t_0}^{t} (E(s, X(s)) d \wedge_\pi (s) + F(s, X(s)) dA_f(s) \\ + G(s, X(s)) dA^+_g(s) + H(s, X(s)) ds), \qquad t \in I, \end{aligned} \tag{2.2}$$

will be called a stochastic integral equation with coefficients E, F, G, H and initial data (t_0, X_0) if $X(t_0) = X_0$.

As an abbreviation we shall sometimes write the foregoing equation as follows:

$$\begin{aligned} &dX(t) = E(t, X(t)) d \wedge_\pi (t) + F(t, X(t)) dA_f(t) + G(t, X(t)) dA^+_g(t) + H(t, X(t)) dt \\ &X(t_0) = X_0, \quad \text{almost all } \ t \in I. \end{aligned}$$

We refer to this as a stochastic differential equation with coefficients E, F, G, H and initial data (t_0, X_0).

By a solution of the equation, we mean a weakly absolutely continuous stochastic process $\phi \in L^2_{loc}(\tilde{\mathcal{A}})$ such that

$$\begin{aligned} &d\phi(t) = E(t, \phi(t)) d \wedge_\pi (t) + F(t, \phi(t)) dA_f(t) + G(t, \phi(t)) dA^+_g(t) + H(t, \phi(t)) dt, \\ &\phi(t_0) = X_0, \quad \text{almost all } \ t \in I. \end{aligned}$$

(e) If Φ is a map from $I \times \tilde{\mathcal{A}}$ into sesq$[\mathbb{D}\underline{\otimes}\mathbb{E}]$, then for $(t, x) \in I \times \tilde{\mathcal{A}}$, the value of $\Phi(t, x)$ at $\eta, \xi \in \mathbb{D}\underline{\otimes}\mathbb{E}$ will be denoted by $\Phi(t, x)(\eta, \xi)$.

Such a map will be called Lipschitzian if for arbitrary $\eta, \xi \in \mathbb{D}\underline{\otimes}\mathbb{E}$,

$$|\Phi(t, x)(\eta, \xi) - \Phi(t, y)(\eta, \xi)| \leq K^{\Phi}_{\eta\xi}(t) \, \|x - y\|_{\eta\xi}$$

$\forall x, y \in \tilde{\mathcal{A}}$ and almost all t in I for some locally integrable functions $K^{\Phi}_{\eta\xi}(t)$ on I. Φ will be called continuous if the map $(t, x) \longrightarrow \Phi(t, x)(\eta, \xi)$ from $I \times \tilde{\mathcal{A}}$ to $\mathbb{C}$ is continuous.

(f) Unless otherwise stated, E, F, G, H lie in $L^2_{loc}(I \times \tilde{\mathcal{A}})$ and let (t_0, x_0) be a fixed point of $I \times \tilde{\mathcal{A}}$. For $\eta, \xi \in \mathbb{D}\underline{\otimes}\mathbb{E}$, with $\eta = c \otimes e(\alpha)$ and $\xi = d \otimes e(\beta)$, we define $\mu_{\alpha\beta}$, γ_β, σ_α : $I \longrightarrow \mathbb{C}$ by $\mu_{\alpha\beta} = \langle \alpha(t), \pi(t)\beta(t) \rangle_Y$, $\gamma_\beta(t) = \langle f(t), \beta(t) \rangle_Y$, $\sigma_\alpha(t) = \langle \alpha(t), g(t) \rangle_Y$, $t \in I$.

To these functions, we associate the maps μE, γF, σG, P from $I \times \tilde{\mathcal{A}}$ into the set of sesquilinear forms on $\mathbb{D}\underline{\otimes}\mathbb{E}$ defined by

$$\begin{aligned} (\mu E)(t, x)(\eta, \xi) &= \langle \eta, \mu_{\alpha\beta}(t) E(t, x) \xi \rangle, \\ (\gamma F)(t, x)(\eta, \xi) &= \langle \eta, \gamma_\beta(t) F(t, x) \xi \rangle, \\ (\sigma G)(t, x)(\eta, \xi) &= \langle \eta, \sigma_\alpha(t) G(t, x) \xi \rangle, \\ P(t, x)(\eta, \xi) &= (\mu E)(t, x)(\eta, \xi) + (\gamma F)(t, x)(\eta, \xi) + (\sigma G)(t, x)(\eta, \xi) \\ &\quad + H(t, x)(\eta, \xi), \end{aligned} \tag{2.3}$$

$\eta, \xi \in \mathbb{D}\underline{\otimes}\mathbb{E}$, $(t, x) \in I \times \tilde{\mathcal{A}}$, where $H(t, x)(\eta, \xi) := \langle \eta, H(t, x) \xi \rangle$.

The map P is known to have Lipschitz and continuity properties depending on such properties of the coefficients of (2.2). Furthermore, it has been established that if the coefficients E, F, G, H appearing in (2.2) belong to $L^2_{loc}(I \times \tilde{\mathcal{A}})$, then (2.2) is equivalent to (1.3). If, in addition, the coefficients are Lipschitzian, then for any fixed point (t_0, X_0) of $I \times \tilde{\mathcal{A}}$, the existence and uniqueness of a solution $\Phi \in Ad(\tilde{\mathcal{A}})_{wac}$ of (2.2) are assured (see [1, 2, 5]).

3. Lagrangian quadrature. Since the existence results hold with the general Caratheodory conditions, we assume the following conditions in what follows:

(a) The map $(t, X) \to P(t, X)(\eta, \xi)$ of $I \times \tilde{\mathcal{A}}$ to $\mathbb{C}$ is continuous for arbitrary $\eta, \xi \in \mathbb{D}\underline{\otimes}\mathbb{E}$.

(b) P is Lipschitzian with continuous Lipschitz function $K^p_{\eta\xi}(t)$ on $[t_0, T] = I$, for $\eta, \xi \in \mathbb{D}\underline{\otimes}\mathbb{E}$.

(c) The unique exact solution $X(t)$ of (1.3) is such that the map $t \to \langle \eta, X(t)\xi \rangle := X_{\eta\xi}(t)$ is of class $C^1(I)$ for arbitrary $\eta, \xi \in \mathbb{D}\underline{\otimes}\mathbb{E}$.

Suppose that $t_1, t_2, t_3, \ldots, t_N$ are N distinct points of $[t_0, T]$ for some positive integer N. Then we seek an interpolating approximation to the value $X(t)$ of the exact solution in the form

$$X(t) \cong \sum_{p=1}^{N} L_p(t) X(t_p), \tag{3.1}$$

where

$$\begin{aligned} L_p(t) &= \prod_{\substack{k=1\\k\neq p}}^{N} \left[(t - t_k)/(t_p - t_k)\right], \qquad p = 1, 2, \ldots, N, \ \ t \neq t_p \\ &= T_N(t)/[(t - t_p)T'_N(t_p)], \end{aligned} \tag{3.2}$$

and

$$T_N(t) = (t - t_1)(t - t_2)\cdots(t - t_N).$$

We define the quadrature coefficient $a^{(n)}_{pk}$ by

$$\frac{d^n}{dt^n} L_k(t)|_{t=t_p} = a^{(n)}_{pk}. \tag{3.3}$$

At the nodes t_p, $p = 1, 2, \ldots, N$, the Lagrangian interpolants satisfy $L_k(t_p) = a^{(0)}_{pk} = \delta_{pk}$, where δ_{pk} is the Kronecker delta. This remains true under any linear transformation $t = au + b$. In most applications, we shall require the transformation of the interval $I = [t_0, T]$ of t into, say, the usually convenient interval $[-1, 1]$ of u by writing $u = [2t - T - t_0]/(T - t_0)$.

The following lemma, which is due to Olaofe [20, 21], will be useful in what follows.

Lemma 3.1.

$$\frac{\partial}{\partial t_m} L_p(t) = -a^{(1)}_{mp} L_m(t), \quad \textit{where}$$
$$a^{(1)}_{mp} = L'_p(t_m).$$

We shall write $X(t_p)$ for the value of the exact solution $X(t)$ at t_p and X_p as its approximate value at t_p. For nonnodal point $t \in [t_0, T]$, we shall write X_t for the approximate value of $X(t)$. The global error at t_p is given by $\|e_p\|_{\eta\xi} = \|X(t_p) - X_p\|_{\eta\xi}$ and the local error at any point t is given by $\|e_t\|_{\eta\xi} = \|X(t) - X_t\|_{\eta\xi}$ for arbitrary $\eta, \xi \in \mathbb{D}\underline{\otimes}\mathbb{E}$.

Ordinarily, (3.1) may be used to interpolate $\langle \eta, X(t)\xi\rangle$ when $\langle \eta, X(t_p)\xi\rangle$ are given for $p = 1, 2, \ldots, N$. During computation, X_p approximates $X(t_p)$ so that (3.1) becomes

$$\langle \eta, X(t)\xi\rangle \cong \sum_{p=1}^{N} L_p(t)\langle \eta, X_p\xi\rangle.$$

DEFINITION 3.2. *Let $L, R : \mathbb{N}\times\tilde{\mathcal{A}} \mapsto sesq[\mathbb{D}\underline{\otimes}\mathbb{E}]$; then the numbers $|L(N, X(t))(\eta, \xi)|$ and $|R(N, X(t))(\eta, \xi)|$ are called local truncation error and round-off error, respectively, if the exact representation for $\langle \eta, X(t)\xi\rangle$ is given by*

$$\langle \eta, X(t)\xi\rangle = \sum_{p=1}^{N} L_p(t)\langle \eta, X(t_p)\xi\rangle + L(N, X(t))(\eta, \xi) \tag{3.4}$$

and the computed representation by

$$\langle \eta, X_t\xi\rangle = \sum_{p=1}^{N} L_p(t)\langle \eta, X_p\xi\rangle + R(N, X(t))(\eta, \xi).$$

The complex valued map $X_{\eta\xi} : [t_0, T] \mapsto \mathbb{C}$ is defined by $X_{\eta\xi}(t) = \langle \eta, X(t)\xi\rangle$ for arbitrary $\eta, \xi \in \mathbb{D}\underline{\otimes}\mathbb{E}$, where $X(t)$ is the exact solution of problem (1.3).

Putting $\langle \eta, X(t)\xi\rangle - \langle \eta, X_t\xi\rangle := \langle \eta, e_t\xi\rangle$, the global error $\|e_t\|_{\eta\xi}$ satisfies

$$\|e_t\|_{\eta\xi} \le \sum_{p=1}^{N} |L_p(t)|\|e_p\|_{\eta\xi} + |L(N, X(t))(\eta, \xi)| + |R(N, X(t))(\eta, \xi)|.$$

At the nodal point t_k, $L(N, X(t_k))(\eta, \xi) = 0$.

Next, we shall describe the notion of the Chebyshev minimax best approximation of continuous complex valued functions.

Let $C[t_0, T]$ denote the space of continuous complex valued functions on $[t_0, T]$ equipped with the uniform norm

$$\|f\|_\infty = \max_{[t_0,T]} |f(t)|, \qquad f \in C[t_0, T].$$

For each positive integer N, let U_N denote the linear subspace of $C[t_0, T]$ consisting of polynomials of degree at most N.

For $f \in C[t_0, T]$, we define the distance from f to U_N by

$$\mathrm{dist}(f, U_N) = \min_{v\in U_N} \|f - v\|_\infty.$$

Then we refer to the element $u \in U_N$ as a best approximation of f if and only if

$$\|f - u\|_\infty = \mathrm{dist}(f, U_N).$$

We now derive the quadrature algorithm which will ensure that the Lagrangian interpolating projection is a best approximation in the minimax sense for a suitable choice of nodal points $t_1,\ t_2, \ldots, t_N$ in $[t_0, T]$.

By the Chebyshev minimax theory, it is well known that for any real valued continuous function f on $[t_0, T]$ and for each positive integer N, there is a polynomial P_N of degree N which is the best approximation to f in the minimax sense.

To this end, suitable points $t_1, t_2, \ldots, t_N$ in $[t_0, T]$ can be found at which the error function

$$R_N(t) = f(t) - P_N(t)$$

has alternate equal and opposite constant values (see, for example, Fox and Parker [8], Powell [26]).

This condition holds if and only if

$$\frac{\partial}{\partial t_j}[R_N(t)]^2 = 0, \qquad j = 1, 2, \ldots, N.$$

Consequently, we have the following result.

THEOREM 3.3. *Let $X(t)$ be the exact solution of problem (1.3) and let $\eta, \xi \in \mathbb{D}\underline{\otimes}\mathbb{E}$. Then for a suitable choice of nodes $t_1, t_2, \ldots, t_N$ in $[t_0, T]$, the Lagrangian interpolating projection $X_{\eta\xi}(t) \cong \sum_{p=1}^{N} L_p(t) X_{\eta\xi}(t_p)$ satisfies*

$$\max_{[t_0,T]} \left| X_{\eta\xi}(t) - \sum_{p=1}^{N} L_p(t) X_{\eta\xi}(t_p) \right| = dist(X_{\eta\xi}, U_N)$$

if and only if

$$X'_{\eta\xi}(t_m) = \sum_{p=1}^{N} a_{mp}^{(1)} X_{\eta\xi}(t_p), \quad m = 1, 2, \ldots, N,$$

for each nonnodal point $t \in [t_0, T]$ and $\forall \eta, \xi \in \mathbb{D}\underline{\otimes}\mathbb{E}$.

Proof. For $\eta, \xi \in \mathbb{D}\underline{\otimes}\mathbb{E}$, we have from (3.4)

$$|L(N, X(t))(\eta, \xi)| = \left| \langle \eta, X(t)\xi \rangle - \sum_{p=1}^{N} L_p(t) \langle \eta, X(t_p)\xi \rangle \right|, \tag{3.5}$$

where $L_p(t) = L_p(t; t_1, t_2, \ldots, t_N)$ is defined by (3.2) and depends on the nodes t_p, $p = 1, 2, \ldots, N$, for each $t \in [t_0, T]$. We now put

$$\langle \eta, X(t)\xi \rangle := X_{\eta\xi}(t) = U_{\eta\xi}(t) + iV_{\eta\xi}(t)$$

for some real valued functions $t \mapsto U_{\eta\xi}(t)$ and $t \mapsto V_{\eta\xi}(t)$. Then we have

$$L(N, X(t))(\eta, \xi) = \left(U_{\eta\xi}(t) - \sum_{p=1}^{N} L_p(t) U_{\eta\xi}(t_p) \right) + i \left(V_{\eta\xi}(t) - \sum_{p=1}^{N} L_p(t) V_{\eta\xi}(t_p) \right),$$

so that

$$|L(N, X(t))(\eta, \xi)|^2 = \left(U_{\eta\xi}(t) - \sum_{p=1}^{N} L_p(t) U_{\eta\xi}(t_p) \right)^2 + \left(V_{\eta\xi}(t) - \sum_{p=1}^{N} L_p(t) V_{\eta\xi}(t_p) \right)^2.$$

Since the maps $U_{\eta\xi}(t)$ and $V_{\eta\xi}(t)$ are continuous on $[t_0, T]$, by the Chebyshev minimax theory $\max_{[t_0,T]} |L(N, X(t))(\eta, \xi)|$ will be minimum for a suitable choice of nodes $t_1, t_2, \ldots, t_N$ if and only if

$$\frac{\partial}{\partial t_m} |L(N, X(t))(\eta, \xi)|^2 = 0, \quad m = 1, 2, \ldots, N. \tag{3.6}$$

But

$$\begin{aligned}
&\frac{\partial}{\partial t_m}|L(N,X(t))(\eta,\xi)|^2\\
&=\frac{\partial}{\partial t_m}\left(U_{\eta\xi}(t)-\sum_{p=1}^{N}L_p(t)U_{\eta\xi}(t_p)\right)^2+\frac{\partial}{\partial t_m}\left(V_{\eta\xi}(t)-\sum_{p=1}^{N}L_p(t)V_{\eta\xi}(t_p)\right)^2\\
&=2\left(U_{\eta\xi}(t)-\sum_{p=1}^{N}L_p(t)U_{\eta\xi}(t_p)\right)\left(-\frac{\partial}{\partial t_m}\sum_{p=1}^{N}L_p(t)U_{\eta\xi}(t_p)\right)\\
&+2\left(V_{\eta\xi}(t)-\sum_{p=1}^{N}L_p(t)V_{\eta\xi}(t_p)\right)\left(-\frac{\partial}{\partial t_m}\sum_{p=1}^{N}L_p(t)V_{\eta\xi}(t_p)\right), \qquad (3.7)\\
&m=1,2,\ldots,N.
\end{aligned}$$

Since $|L(N,X(t))(\eta,\xi)|$ does not varnish identically, (3.7) is zero if and only if

$$-\frac{\partial}{\partial t_m}\sum_{p=1}^{N}L_p(t)U_{\eta\xi}(t_p)=0$$

and

$$-\frac{\partial}{\partial t_m}\sum_{p=1}^{N}L_p(t)V_{\eta\xi}(t_p)=0\ ,\quad m=1,2,\ldots,N.$$

These equations imply that

$$-\frac{\partial}{\partial t_m}\sum_{p=1}^{N}L_p(t)X_{\eta\xi}(t_p)=0. \tag{3.8}$$

From (3.8) and in view of Lemma 3.1,

$$\left[\sum_{p=1}^{N}\left(-\frac{\partial}{\partial t_m}L_p(t)\right)X_{\eta\xi}(t_p)+\sum_{p=1}^{N}-L_p(t)\frac{\partial}{\partial t_m}(X_{\eta\xi}(t_p))\right]=0$$

or

$$L_m(t)\left[\sum_{p=1}^{N}a_{mp}^{(1)}X_{\eta\xi}(t_p)-X'_{\eta\xi}(t_m)\right]=0.$$

Since $L_m(t)\neq 0$, consequently we have

$$X'_{\eta\xi}(t_m)=\sum_{p=1}^{N}a_{mp}^{(1)}X_{\eta\xi}(t_p),\quad m=1,2,\ldots,N. \qquad \square \tag{3.9}$$

Remark 3.4.

(i) Equation (3.9) when applied at each of the nodes transforms the quantum stochastic differential equation (1.3) to a purely algebraic system given by

$$\sum_{p=1}^{N}a_{mp}^{(1)}X_{\eta\xi}(t_p)=P(t_m,X(t_m))(\eta,\xi),\quad m=1,2,3,\ldots,N,$$

with initial condition transforming to

$$\sum_{p=1}^{N} L_p(t_0) X_{\eta\xi}(t_p) \cong X_{\eta\xi}(t_0).$$

(ii) If we write

$$X_{\eta\xi}(t) = U_{\eta\xi}(t) + iV_{\eta\xi}(t)$$

and

$$P(t, X(t))(\eta, \xi) = \mathrm{Re}[P(t, X(t))(\eta, \xi)] + i\mathrm{Im}[P(t, X(t))(\eta, \xi)]$$

for some real valued functions $t \to U_{\eta\xi}(t)$, $t \to V_{\eta\xi}(t)$, $t \to \mathrm{Re}[P(t, X(t))(\eta, \xi)]$, $t \to \mathrm{Im}[P(t, X(t))(\eta, \xi)]$, then the system of the algebraic equations in (i) above may be written in two parts, real and imaginary, as follows:

$$\sum_{p=1}^{N} a_{mp}^{(1)} U_{\eta\xi}(t_p) = \mathrm{Re}[P(t_m, X(t_m))(\eta, \xi)],$$
$$\sum_{p=1}^{N} L_p(t_0) U_{\eta\xi}(t_p) \cong U_{\eta\xi}(t_0)$$

and

$$\sum_{p=1}^{N} a_{mp}^{(1)} V_{\eta\xi}(t_p) = \mathrm{Im}[P(t_m, X(t_m))(\eta, \xi)],$$
$$\sum_{p=1}^{N} L_p(t_0) V_{\eta\xi}(t_p) \cong V_{\eta\xi}(t_0), \quad m = 1, 2, 3, \ldots, N.$$

We observe that each of the systems of equations consists of $N+1$ algebraic equations in N unknowns. We shall discuss the methods of their solution in section 5.

(iii) The quadrature coefficients $a_{mp}^{(s)}$ defined by (3.3) satisfy

$$a_{mp}^{(s)} = \sum_{k=1}^{N} a_{mk}^{(s-q)}\ a_{kp}^{(q)}$$

$\forall q$ such that $q = 1, 2, \ldots, s-1$, with $s < N$ (see [20, 21]).

The numerical solution of the systems in (i) and (ii) above is a simple task for modern electronic computers. This is accomplished by using standard programs with minimal computing time and storage for the $N \times N$ matrix $[a_{ij}^{(1)}]$.

However, it is important to note that the computational complexity of Lagrange polynomials is very large due to extreme number of multiplications and additions. In practice, relatively low order quadrature is needed (that is, $3 \le N \le 15$, say) so that the total amount of storage and time needed on the computer is quite low.

Next, we present a corollary to the last theorem.

Corollary 3.5. *Assume that for each pair of* $\eta, \xi \in \mathbb{D}\underline{\otimes}\mathbb{E}$ *the exact solution* $X(t)$ *of problem* (1.3) *is such that*

$$X_{\eta\xi}(\cdot) \in C^{s+1}[t_0, T], \quad s = 0, 1, 2, \ldots;$$

then the representation

$$X^{(s)}_{\eta\xi}(t) \cong \sum_{p=1}^{N} L_p(t) X^{(s)}_{\eta\xi}(t_p)$$

satisfies

$$\max_{[t_0,T]} \left| X^{(s)}_{\eta\xi}(t) - \sum_{p=1}^{N} L_p(t) X^{(s)}_{\eta\xi}(t_p) \right| = dist(X^{(s)}_{\eta\xi}, U_N)$$

if and only if

$$X^{(s+1)}_{\eta\xi}(t_m) = \sum_{p=1}^{N} a^{(s+1)}_{mp} X_{\eta\xi}(t_p).$$

Proof. Using the inner product form of (3.1), if $X_{\eta\xi}(t)$ is replaced by its derivative, then by Theorem 3.3

$$X'_{\eta\xi}(t) \cong \sum_{p=1}^{N} L_p(t) X'_{\eta\xi}(t_p)$$

is a best approximation of $X'_{\eta\xi}(t)$ provided that

$$X''_{\eta\xi}(t_m) = \sum_{p=1}^{N} a^{(1)}_{mp} X'_{\eta\xi}(t_p), \qquad m = 1, 2, \ldots, N, \tag{3.10}$$

where, by (3.9), we have

$$X'_{\eta\xi}(t_p) = \sum_{k=1}^{N} a^{(1)}_{pk} X_{\eta\xi}(t_k).$$

Substituting in (3.10) for $X'_{\eta\xi}(t_p)$, we have

$$\begin{aligned} X''_{\eta\xi}(t_m) &= \sum_{p=1}^{N} a^{(1)}_{mp} \sum_{k=1}^{N} a^{(1)}_{pk} X_{\eta\xi}(t_k) \\ &= \sum_{p=1}^{N} \sum_{k=1}^{N} a^{(1)}_{mp} a^{(1)}_{pk} X_{\eta\xi}(t_k) \\ &= \sum_{p=1}^{N} \sum_{k=1}^{N} a^{(1)}_{mk} a^{(1)}_{kp} X_{\eta\xi}(t_p). \end{aligned}$$

Hence

$$X''_{\eta\xi}(t_m) = \sum_{p=1}^{N} a^{(2)}_{mp} X_{\eta\xi}(t_p),$$

where

$$a_{mp}^{(2)} = \sum_{k=1}^{N} a_{mk}^{(1)} a_{kp}^{(1)},$$

by Remark 3.4(iii) above.

It follows in general that the representation

$$X_{\eta\xi}^{(s)}(t) \cong \sum_{p=1}^{N} L_p(t) X_{\eta\xi}^{(s)}(t_p)$$

is a best approximation of $X_{\eta\xi}^{(s)}(t)$ if and only if

$$X_{\eta\xi}^{(s+1)}(t_m) = \sum_{p=1}^{N} a_{mp}^{(1)} X_{\eta\xi}^{(s)}(t_p),$$

i.e., if and only if

$$X_{\eta\xi}^{(s+1)}(t_m) = \sum_{p=1}^{N} a_{mp}^{(s+1)} X_{\eta\xi}(t_p),$$

where q can take any values from 1 to $s-1$, with $s < N$. ☐

4. Lagrangian interpolants and computation of quadrature coefficients. In this section, we compute the Lagrangian interpolants $L_p(t)$ and the quadrature coefficients $a_{ij}^{(k)}$. These computations are independent of $\eta, \xi \in \mathbb{D}\underline{\otimes}\mathbb{E}$ and are carried out in the same manner as in the classical context. For several benefits of orthogonality, we choose the set $\{L_p(t)\}$ to be orthogonal on the interval $[t_0, T]$ with respect to the weight function $w(t) = 1$ as in [20, 21]. Consequently, the interval $[t_0, T]$ for t may be transformed to the appropriate interval $[-1, 1]$ for u by the equation

$$u = \frac{2t - T - t_0}{T - t_0}.$$

In this case, the product polynomial $T_N(u)$ associated with $L_p(u)$ transformed to $u \in [-1, 1]$ is the Legendre polynomial $P_N(u)$ of degree N on $[-1, 1]$.

Consequently, we consider for the computation of the quadrature coefficients $a_{ij}^{(k)}$, the case of the unit weight function in the interval $[-1, 1]$ with the product polynomial $T_N(u)$ of $L_p(u)$ as the Legendre polynomial $P_N(u)$ of degree N.

Since $a_{mp}^{(1)} = L_p'(u_m)$ for $u_m \in [-1, 1]$, $m = 1, 2, \ldots, N$, it is easy to show that

$$a_{pq}^{(1)} = \frac{P_N'(u_p)}{(u_p - u_q) P_N'(u_q)}, \qquad p \neq q \tag{4.1}$$

and

$$a_{pp}^{(1)} = \frac{P_N''(u_p)}{2P_N'(u_p)} = \frac{u_p}{1 - u_p^2}, \tag{4.2}$$

where

$$P_N(u) = \frac{1}{2^N N!} D^N (u^2 - 1)^N, \qquad D = \frac{d}{du}, \quad u \in [-1, 1].$$

By considering the properties of Legendre polynomials (see [21, 29]), it can readily be shown that

$$a_{pq}^{(1)} = -a_{N+1-p,N+1-q}^{(1)}.$$

By using (4.1) and (4.2), the constants $a_{pq}^{(1)}$ for any value of N, say $N = 3, 4, \ldots, 15$, can be easily calculated. These values are then used as input data for a given problem once N has been chosen.

5. Bounds for the local and global truncation errors, convergence, and application to quantum stochastic differential equation. We first establish the existence of a bound for the local truncation error by assuming some smoothness conditions of the map $t \mapsto \langle \eta, X(t)\xi\rangle$ for each pair of $\eta, \xi \in \mathbb{D}\underline{\otimes}\mathbb{E}$. We then use this bound to show that $|L(N, X(t))(\eta, \xi)| \to 0$ as $N \to \infty$.

To this end, we assume that the maps $(\mu E)(\cdot, X(\cdot))(\eta, \xi)$, $(\gamma F)(\cdot, X(\cdot))(\eta, \xi)$, $(\sigma G)(\cdot, X(\cdot))(\eta, \xi)$, and $H(\cdot, X(\cdot))(\eta, \xi)$ which define the map $P(\cdot, X(\cdot))(\eta, \xi)$ in (2.3) are of class $C^{N-1}[t_0, T]$ to ensure that the map $X_{\eta\xi}(\cdot)$ is of class $C^N[t_0, T]$.

Theorem 5.1. *Assume that X is the exact solution of problem* (1.3) *such that $X_{\eta\xi}(\cdot)$ is of class $C^N[t_0, T]$ for each pair $\eta, \xi \in \mathbb{D}\underline{\otimes}\mathbb{E}$ and N, a positive integer. Then the local truncation error $L(N, X(t))(\eta, \xi)$ satisfies*

$$|L(N, X(t))(\eta, \xi)| \le \max_{t\in[t_0,T]} \left[\frac{|X_{\eta\xi}^{(N)}(t)|}{N!} \left| t^N - \sum_{p=1}^{N} L_p(t) t_p^N \right| \right]. \tag{5.1}$$

Proof. By (3.4) we have

$$\begin{aligned} L(N, X(t))(\eta, \xi) &= \langle \eta, X(t)\xi\rangle - \sum_{p=1}^{N} L_p(t) \langle \eta, X(t_p)\xi\rangle, \\ &= X_{\eta\xi}(t) - \sum_{p=1}^{N} L_p(t) X_{\eta\xi}(t_p). \end{aligned} \tag{5.2}$$

We now put

$$X_{\eta\xi}(t) = U_{\eta\xi}(t) + iV_{\eta\xi}(t),$$

where $t \to U_{\eta\xi}(t)$ and $t \to V_{\eta\xi}(t)$ are real valued functions for $\eta, \xi \in \mathbb{D}\underline{\otimes}\mathbb{E}$. Since $X_{\eta\xi}(\cdot)$ is of class $C^N[t_0, T]$, then $U_{\eta\xi}(\cdot)$ and $V_{\eta\xi}(\cdot)$ are also of class $C^N[t_0, T]$.

By Taylor's theorem, we can write, for $0 \le t \le 1$, $0 < \theta < 1$,

$$U_{\eta\xi}(t) = \sum_{p=1}^{N-1} \frac{t^p}{p!} U_{\eta\xi}^{(p)}(0) + \frac{t^N}{N!} U_{\eta\xi}^{(N)}(\theta t)$$

and

$$V_{\eta\xi}(t) = \sum_{p=1}^{N-1} \frac{t^p}{p!} V_{\eta\xi}^{(p)}(0) + \frac{t^N}{N!} V_{\eta\xi}^{N}(\theta t). \tag{5.3}$$

Substituting in (5.2), we get

$$
\begin{aligned}
&L(N,X(t))(\eta,\xi)\\
&=\sum_{j=1}^{N-1}\frac{t^j}{j!}U_{\eta\xi}^{(j)}(0)+\frac{t^N}{N!}U_{\eta\xi}^{(N)}(\theta t)+i\left[\sum_{j=1}^{N-1}\frac{t^j}{j!}V_{\eta\xi}^{(j)}(0)+\frac{t^N}{N!}V_{\eta\xi}^{(N)}(\theta t)\right]\\
&\quad-\sum_{p=1}^{N}L_p(t)\left[\sum_{j=1}^{N-1}\frac{t_p^j}{j!}U_{\eta\xi}^{(j)}(0)+\frac{t_p^N}{N!}U_{\eta\xi}^{(N)}(\theta t_p)\right]\\
&\quad-i\sum_{p=1}^{N}L_p(t)\left[\sum_{j=1}^{N-1}\frac{t_p^j}{j!}V_{\eta\xi}^{(j)}(0)+\frac{t_p^N}{N!}V_{\eta\xi}^{(N)}(\theta t_p)\right]\\
&=\sum_{j=1}^{N-1}\frac{t^j}{j!}U_{\eta\xi}^{(j)}(0)-\sum_{p=1}^{N}L_p(t)\sum_{j=1}^{N-1}\frac{t_p^j}{j!}U_{\eta\xi}^{(j)}(0)\\
&\quad+i\left[\sum_{j=1}^{N-1}\frac{t^j}{j!}V_{\eta\xi}^{(j)}(0)-\sum_{p=1}^{N}L_p(t)\sum_{j=1}^{N-1}\frac{t_p^j}{j!}V_{\eta\xi}^{(j)}(0)\right]\\
&\quad+\frac{t^N}{N!}\left(U_{\eta\xi}^{(N)}(\theta t)+iV_{\eta\xi}^{(N)}(\theta t)\right)-\frac{1}{N!}\sum_{p=1}^{N}L_p(t)t_p^N\left(U_{\eta\xi}^{(N)}(\theta t_p\right)\\
&\quad-\frac{i}{N!}\sum_{p=1}^{N}L_p(t)t_p^N V_{\eta\xi}^{(N)}(\theta t_p).
\end{aligned}
$$

By the exactness theorem of Lagrangian interpolation formula (see Stroud [29]), the last equation becomes

$$
\begin{aligned}
&L(N,X(t))(\eta,\xi)\\
&=\frac{1}{N!}t^N[U_{\eta\xi}^{(N)}(\theta t)+iV_{\eta\xi}^{(N)}(\theta t)]-\frac{1}{N!}\sum_{p=1}^{N}L_p(t)t_p^N[U_{\eta\xi}^{(N)}(\theta t_p)+iV_{\eta\xi}^{(N)}(\theta t_p)]\\
&=\frac{1}{N!}\left[t^N X_{\eta\xi}^{(N)}(\theta t)-\sum_{p=1}^{N}L_p(t)t_p^N X_{\eta\xi}^{(N)}(\theta t_p)\right]. \qquad (5.4)
\end{aligned}
$$

If we employ the integral form of the remainder in (5.3) we get

$$
\begin{aligned}
X_{\eta\xi}(t)&=\sum_{j=0}^{N-1}\frac{t^j}{j!}U_{\eta\xi}^{(j)}(0)+i\sum_{j=0}^{N-1}\frac{t^j}{j!}V_{\eta\xi}^{(j)}(0)\\
&\quad+\frac{1}{(N-1)!}\int_0^t(t-r)^{N-1}(U_{\eta\xi}^{(N)}(r)+iV_{\eta\xi}^{(N)}(r))dr. \qquad (5.5)
\end{aligned}
$$

Using (5.5) in (3.4), we get

$$
\begin{aligned}
&L(N,X(t))(\eta,\xi)\\
&=X_{\eta\xi}(t)-\sum_{p=1}^{N}L_p(t)X_{\eta\xi}(t_p)
\end{aligned}
$$

$$= \sum_{j=0}^{N-1} \frac{t^j}{j!} U_{\eta\xi}^{(j)}(0) + i \sum_{j=0}^{N-1} \frac{t^j}{j!} V_{\eta\xi}^{(j)}(0) + \frac{1}{(N-1)!} \int_0^t (t-r)^{N-1} X_{\eta\xi}^{(N)}(r) dr$$

$$- \sum_{p=1}^{N} L_p(t) \left[\sum_{j=0}^{N-1} \frac{t_p^j}{j!} U_{\eta\xi}^{(j)}(0) + i \sum_{j=0}^{N-1} \frac{t_p^j}{j!} V_{\eta\xi}^{(j)}(0) \right]$$

$$- \frac{1}{(N-1)} \sum_{p=1}^{N} L_p(t) \int_0^{t_p} (t_p - r)^{N-1} X_{\eta\xi}^{(N)}(r) dr$$

$$= \frac{1}{(N-1)!} \int_0^t (t-r)^{N-1} X_{\eta\xi}^{(N)}(r) dr$$

$$- \frac{1}{(N-1)!} \sum_{p=1}^{N} L_p(t) \int_0^{t_p} (t_p - r)^{N-1} X_{\eta\xi}^{(N)}(r) dr, \tag{5.6}$$

by the exactness of the Lagrangian interpolation formula for polynomials of degree less than or equal to $N-1$.

From (5.6), if we put $r = ts$, then

$$L(N, X(t))(\eta, \xi)$$

$$= \frac{1}{(N-1)!} \int_0^1 (1-s)^{N-1} t^N X_{\eta\xi}^{(N)}(ts) ds$$

$$- \frac{1}{(N-1)!} \sum_{p=1}^{N} L_p(t) \int_0^1 (1-s)^{N-1} t_p^N X_{\eta\xi}^{(N)}(t_p s) ds$$

$$= \frac{1}{(N-1)!} \int_0^1 (1-s)^{N-1} \left\{ t^N X_{\eta\xi}^{(N)}(ts) - \sum_{p=1}^{N} L_p(t) t_p^N X_{\eta\xi}^{(N)}(t_p s) \right\} ds.$$

Now by (5.4)

$$N! L(N, X(t))(\eta, \xi) = \left[t^N X_{\eta\xi}^{(N)}(ts) - \sum_{p=1}^{N} L_p(t) t_p^N X_{\eta\xi}^{(N)}(t_p s) \right],$$

where $0 < s = \theta < 1$.

Hence, on account of (5.6),

$$L(N, X(t))(\eta, \xi)$$

$$= \frac{1}{(N-1)!} \int_0^t (t-r)^{N-1} X_{\eta\xi}^{(N)}(r) dr$$

$$- \frac{1}{(N-1)!} \sum_{p=1}^{N} L_p(t) \int_0^t (t_p - r)^{N-1} X_{\eta\xi}^{(N)}(r) dr$$

$$- \frac{1}{(N-1)!} \sum_{p=1}^{N} L_p(t) \int_t^{t_p} (t_p - r)^{N-1} X_{\eta\xi}^{(N)}(r) dr$$

$$= \frac{1}{(N-1)!} \sum_{p=1}^{N} \int_{t_p}^{t} L_p(t) (t_p - r)^{N-1} X_{\eta\xi}^{(N)}(r) dr$$

$$= -\frac{1}{N!} X_{\eta\xi}^{(N)}(\theta t) \sum_{p=1}^{N} L_p(t)(t_p - t)^N.$$

Therefore

$$|L(N, X(t))(\eta, \xi)| \leq \frac{1}{N!} \max_{[t_0,T]} \left[\left| X_{\eta\xi}^{(N)}(t) \right| \left| t^N - \sum_{p=1}^{N} L_p(t) t_p^N \right| \right].$$

Consequently, $|L(N, X(t))(\eta, \xi)| \longrightarrow 0$ as $N \to \infty$ for arbitrary $\eta, \xi \in \mathbb{D}\underline{\otimes}\mathbb{E}$ and $L(N, X(t_k))(\eta, \xi) = 0, \quad k = 1, 2, \ldots, N.$ □

Remark 5.2.

(i) Upon application of (3.9) at each of the N nodes to (1.3), the following system of difference equations is obtained:

$$\sum_{p=1}^{N} a_{mp}^{(1)} X_{\eta\xi}(t_p) = P(t_m, X(t_m))(\eta, \xi), \quad m = 1, 2, \ldots, N. \tag{5.7}$$

The initial condition yields the quadrature equation

$$X_{\eta\xi}(t_0) = \sum_{p=1}^{N} L_p(t_0) X_{\eta\xi}(t_p). \tag{5.8}$$

Equations (5.7) and (5.8) lead to a system of $N+1$ equations in N unknown approximate nodal values $X_{\eta\xi,p}, \ p = 1, 2, \ldots, N$, which are to be determined.

It has been shown (see Olaofe [20, 21]) that the rank of the $N \times N$ matrix $[a_{ij}^{(1)}]$ is exactly $N-1$. The matrix is singular. Hence (5.8) is chosen together with any $N-1$ equations of (5.7). The remaining Nth equation of (5.7) is regarded as superfluous. Since the remaining equation must also be satisfied by the computed solutions, the measure of accuracy of the numerical result is given by the residual

$$R_{r,\eta,\xi} = \sum_{p=1}^{N} a_{rp}^{(1)} X_{\eta,\xi,p} - P(t_r, X(t_r))(\eta, \xi) \tag{5.9}$$

at the superfluous node $t = t_r$.

The superfluous error equation (5.9) is a practical method of estimating the accuracy of the numerical solution given by the Lagrangian quadrature method.

(ii) By Definition 3.2 above, the global error $\|e_t\|_{\eta\xi}$ satisfies

$$\|e_t\|_{\eta\xi} \leq \sum_{p=1}^{N} |L_p(t)| \|e_p\|_{\eta\xi} + |L(N, X(t))(\eta, \xi)| + |R(N, X(t))(\eta, \xi)|.$$

If we now put

$$e_{\eta\xi} = \max\{\|e_p\|_{\eta\xi}, \ p = 1, 2, \ldots, N\},$$

then by inequality (5.1)

$$\|e_t\|_{\eta\xi} \\ \leq e_{\eta\xi} \sum_{p=1}^{N} |L_p(t)| + \frac{1}{N!} \max_{[t_0,T]} \left[|X_{\eta\xi}^{(N)}(t)| \left| t^N - \sum_{p=1}^{N} L_p(t) t_p^N \right| \right] + |R(N, X(t))(\eta, \xi)|.$$

By continuity of the Lagrangian interpolant, the first term above is bounded. The global error increases due to truncation error, as well as due to round-off error. The truncation error decreases as N increases. The actual behavior of the round-off error will be determined in the investigation of numerical stability of the quadrature method.

6. Examples of quantum stochastic differential equations and numerical experiments. We first present some examples of the quantum stochastic integral equation (2.2) which satisfy the Lipschitz condition in the sense of this paper.

(a) Consider

$$X(t) = \mathbf{I} + \int_0^t ((L(s)X(s) + \lambda \exp[pA(s) + qA^+(s) + rs])d\Lambda(s)$$
$$+ U(s)X(s)dA(s) + V(s)X(s)dA^+(s) + (Z(s)X(s) + Q(s))ds), \qquad t \in [0,T], \tag{6.1}$$

where

$$A(t) = A_f(t), \quad A^+(t) = A_f^+(t), \quad \Lambda_\pi(t) = \Lambda(t),$$

$$f(t) \equiv g(t) \equiv \pi(t) \equiv 1, \quad \mathcal{R} = Y = \mathbb{C}.$$

$L,\ U,\ V,\ Z$ are continuous complex valued functions on $[0,T]$, $Q:[0,T] \to \tilde{\mathcal{A}}$ is an adapted process and $\lambda,\ p,\ q,\ r$ are complex constants.

Consequently,

$$\mathcal{R} \otimes \Gamma(L^2_Y(\mathbb{R}_+)) \equiv \Gamma(L^2_Y(\mathbb{R}_+))$$

and

$$\mathbb{D}\underline{\otimes}\mathbb{E} \equiv \mathbb{E}, \qquad \mathcal{A} = L^+(\mathbb{E}, \Gamma(L^2_Y(\mathbb{R}_+))).$$

The coefficients

$$E(t,X(t)) = L(t)X(t) + \lambda \exp(pA(t) + qA^+(t) + rt),$$

$$F(t,X(t)) = U(t)X(t), \ \ G(t,X(t)) = V(t)X(t),$$

$$H(t,X(t)) = Z(t)X(t) + Q(t), \qquad X(t) \in \tilde{\mathcal{A}},$$

are Lipschitzian with Lipschitz functions

$$K^E_{\eta\xi}(t) = |L(t)|, \ \ K^F_{\eta\xi}(t) = |U(t)|, \ \ K^G_{\eta\xi}(t) = |V(t)|, \ \ K^H_{\eta\xi}(t) = |Z(t)|,$$

respectively. That is, for each $M \in \{E,\ F,\ G,\ H\}$

$$\|M(t,x) - M(t,y)\|_{\eta\xi} \le K^M_{\eta\xi}(t)\|x-y\|_{\eta\xi}, \qquad x,y \in \tilde{\mathcal{A}}$$

and belongs to $L^2_{loc}([0,T] \times \tilde{\mathcal{A}})$ for $X \in L^2_{loc}(\tilde{\mathcal{A}})$.

For $\lambda = 0$, $Q(t) \equiv 0$, the existence, uniqueness, and the conditions for the solution of (6.1) to be unitary in the strong topology are well known (see, for example, Hudson [11]).

However, our present work is useful for computations of discrete approximations of the solution in the weak sense.

For $\eta, \xi \in \mathbb{E}$, such that $\eta = e(\alpha),\ \xi = e(\beta),\ \alpha, \beta \in L^2_Y(\mathbb{R}_+)$, (6.1) is equivalent to the initial value problem

$$\begin{aligned} &\frac{d}{dt}\langle \eta, X(t)\xi\rangle = P(t, X(t))(\eta, \xi),\\ &X(t_0) = \mathbf{I}, \qquad t \in [0, 1], \end{aligned} \tag{6.2}$$

where, by (2.3),

$$\begin{aligned} P(t, X(t))(\eta, \xi) &= [\bar{\alpha}(t)\beta(t)L(t) + \beta(t)U(t) + \bar{\alpha}(t)V(t) + Z(t)]\langle \eta, X(t)\xi\rangle \\ &+ \lambda\bar{\alpha}(t)\beta(t)\langle \eta, [\exp(pA(t) + qA^+(t) + rt)]\xi\rangle + \langle \eta, Q(t)\xi\rangle. \end{aligned} \tag{6.3}$$

By the Campbell–Hausdorff formula (see Meyer [17, p. 135]),

$$\langle \eta, [\exp(pA(t) + qA^+(t) + rt)]\xi\rangle = e^{(r+\frac{1}{2}pq)t}e^{p\int_0^t \beta(s)ds}e^{q\int_0^t \bar{\alpha}(s)ds}e^{\langle \alpha, \beta\rangle}.$$

Hence, the map $(t, x) \to P(t, x)(\eta, \xi)$ is Lipschitzian with Lipschitz function

$$K^P_{\eta\xi}(t) = |\bar{\alpha}(t)\beta(t)L(t) + \beta(t)U(t) + \bar{\alpha}(t)V(t) + Z(t)|.$$

We note that for arbitrary $h \in L^2_{\mathbb{C}}(\mathbb{R}_+)$, if we set

$$\lambda = 0, Y(t) \equiv 0, \quad V(t) = h(t), \;\; U(t) = -\bar{h}(t), \;\; Z(t) \equiv 0,$$

and

$$Q(t) = -\frac{1}{2}|h(t)|^2\mathbf{I}$$

in (6.1), then the solution $X(t)$ is the well-known Weyl operator $W(h\chi_{[0,t]})$ associated with h, where χ is the indicator function on $[0, T]$ and $\mathbf{I}$ is the identity operator on the Fock space (see [12]).

(b) Again, consider

$$\begin{aligned} X(t) = \mathbf{I} + \int_0^t &([X(s)A^2(s) + \lambda A^2(s)]d\Lambda(s) + X(s)A(s)dA(s) \\ &+ X(s)A(s)dA^+(s) + X(s)A(s)ds). \end{aligned} \tag{6.4}$$

Equation (6.4) is equivalent to

$$\begin{aligned} \frac{d}{dt}\langle \eta, X(t)\xi\rangle = &\left[\bar{\alpha}(t)\beta(t)\int_0^t \beta(s)ds + \beta(t) + \bar{\alpha}(t) + 1\right]\langle \eta, X(t)A(t)\xi\rangle \\ &+\lambda\bar{\alpha}(t)\beta(t)\langle \eta, A^2(t)\xi\rangle, \qquad t \in [0, T], \\ \langle \eta, X(0)\xi\rangle = \langle \eta, \xi\rangle&, \end{aligned} \tag{6.5}$$

where $A(t)$ is the annihilation process satisfying

$$A(t)\xi = \int_0^t \beta(s)ds\xi, \qquad \xi = e(\beta).$$

The power of $A(t)$ is that of composition of operators.

Equation (6.5) is Lipschitzian with Lipschitz function

$$K^P_{\eta\xi}(t) = \left| \int_0^t \beta(s)ds \left[\bar{\alpha}(t)\beta(t) \int_0^t \beta(\tau)d\tau + \beta(t) + \bar{\alpha}(t) + 1 \right] \right|. \tag{6.6}$$

We remark that for $\lambda = 0$, (6.4) is a special case of the stochastic evolution equation

$$X(t) = X(t_0) + \int_{t_0}^t X(s)[L_1 d\Lambda(s) + L_2 dA(s) + L_3 dA^+(s) + L_4 ds] \tag{6.7}$$

introduced in [13], with time dependent evolution operators $L_j(t)$ given by

$$L_1(t) = A^2(t), \qquad L_2(t) = L_3(t) = L_4(t) = A(t)$$

acting on $\Gamma(L^2(\mathbb{R}_+))$.

It is well known that there exists a unique and unitary solution of (6.7) in the strong topology if the operators $L_j,\ j = 1, \ldots, 4$, fulfil certain conditions (see [13, 33]).

If we now choose the exponential vectors η, ξ, where $\eta = e(\alpha)$, $\xi = e(\beta)$ such that $\alpha(t) = 0,\ \beta(t) = e^t$, then from (6.5)

$$\begin{aligned} &\frac{d}{dt}\langle \eta, X(t)\xi \rangle = (e^t + 1)\langle \eta, X(t)A(t)\xi \rangle, \\ &\langle \eta, X(0)\xi \rangle = \langle \eta, \xi \rangle = e^{\langle \alpha,\beta \rangle} = 1, \qquad t \in [0, 1], \end{aligned} \tag{6.8}$$

with exact solution

$$\langle \eta, X(t)\xi \rangle = \exp\left[\frac{1}{2}e^{2t} - t - \frac{1}{2} \right]. \tag{6.9}$$

Again by choosing $\alpha(t) = it$, $\beta(t) = -it$, we have from (6.5)

$$\begin{aligned} &\frac{d}{dt}\langle \eta, X(t)\xi \rangle = \left(\frac{it^4}{2} - 2it + 1 \right) \langle \eta, X(t)A(t)\xi \rangle - \lambda t^2 \langle \eta, A^2(t)\xi \rangle, \\ &\langle \eta, X(0)\xi \rangle = \langle \eta, \xi \rangle = e^{-\frac{1}{2}}, \qquad t \in [0, 1], \end{aligned} \tag{6.10}$$

with exact solution for $\lambda = 0$ given by

$$\langle \eta, X(t)\xi \rangle = \exp\left[\frac{t^7}{28} - \frac{t^4}{4} - \frac{it^3}{6} - \frac{1}{2} \right].$$

(c) Let $\mathcal{R} = \mathbb{C}$, and $u \in L^\infty_{loc,\mathbb{C}}(\mathbb{R}_+)$, a fixed locally bounded function. For a real constant $l \geq 0$ and $f(t) = l^{\frac{1}{2}}(e^{iu(t)} - 1)$, we consider

$$\begin{aligned} &dX(t) = X(t)\left(d\Lambda_{e^{iu}-I} - dA_{e^{-iuf}} + dA^+_f - \frac{1}{2}|f(t)|^2 dt \right), \qquad t \geq 0, \\ &X(0) = X_0. \end{aligned}$$

The last equation is driven by the gauge, annihilation, and creation operators of strengths $e^{iu} - I$, e^{-iuf}, and f, respectively. For $\eta = e(\alpha),\ \ \xi = e(\beta)$, its equivalent form is given by

$$\begin{aligned} &\frac{d}{dt}\langle \eta, X(t)\xi \rangle = P(t, X(t))(\eta, \xi), \\ &\langle \eta, X(0)\xi \rangle = \langle \eta, X_0\xi \rangle, \end{aligned}$$

where

$$P(t, X(t))(\eta, \xi)$$
$$= \left[\beta(t)\left(\bar{\alpha}(t)e^{iu(t)} - \overline{e^{-iu(t)f(t)}}\right) + \bar{\alpha}(t)(f(t) - \beta(t)) - \frac{1}{2}|f(t)|^2\right]\langle \eta, X(t)\xi\rangle$$

is Lipschitzian with Lipschitz function

$$K^P_{\eta\xi}(t) = \left|\left[\beta(t)\left(\bar{\alpha}(t)e^{iu(t)} - \overline{e^{-iu(t)f(t)}}\right) + \bar{\alpha}(t)(f(t) - \beta(t)) - \frac{1}{2}|f(t)|^2\right]\right|.$$

If we set

$$V_u(t) = \exp\left(i\int_0^t l\sin u(s)ds\right)X(t),$$

then by Proposition 6.2 in [13],

$$\frac{d}{dt}\langle \eta, V_u(t)\xi\rangle = (e^{iu(t)} - 1)(\bar{\alpha}(t) + l^{\frac{1}{2}})(\beta(t) + l^{\frac{1}{2}})\langle \eta, V_u(t)\xi\rangle,$$
$$V_u(0) = X(0).$$

The equation is also Lipschitzian with Lipschitz function

$$K_{\eta\xi,u}(t) = |(e^{iu(t)} - 1)(\bar{\alpha}(t) + l^{\frac{1}{2}})(\beta(t) + l^{\frac{1}{2}})|.$$

Numerical examples.

Example 1. We apply the Lagrangian quadrature method to solve (6.8) by using 5 zeros of the Legendre polynomial $P_5(u)$ of degree 5 in the interval $[-1, 1]$ as discretization points. Thus, we perform the integration in 5 nodes in each of the step subintervals [0, 0.1], [0.1, 0.3], [0.3, 0.5], [0.5, 0.7], [0.7, 0.9], and [0.9, 1] of the interval [0, 1].

By employing the linear transformation

$$u = \frac{2t - T - t_0}{T - t_0},$$

we convert (6.8) in the variable t in each of the subintervals to the variable $u \in [-1, 1]$.

For $\eta = e(\alpha), \xi = e(\beta)$, (6.8) transforms to the algebraic equations given by

$$\sum_{p=1}^{5} a^{(1)}_{mp} X_{\eta\xi,p} = P(u_m, X_m)(\eta, \xi)$$

$$= \frac{1}{2}(T - t_0)(e^{(T-t_0)u_m + T + t_0} - 1)X_{\eta\xi,m}, \qquad m = 1, 2, 3, 4, 5, \tag{6.11}$$

on each of the subintervals

$$[t_0, T] \in \{[0, 0.1],\ [0.1, 0.3],\ [0.3, 0.5],\ [0.5, 0.7],\ [0.7, 0.9],\ [0.9, 1.0]\}.$$

The initial value for the first subinterval [0, 0.1] transforms to

$$\sum_{p=1}^{5} L_p(-1)X_{\eta\xi,p} = 1. \tag{6.12}$$

Here, $X_{\eta\xi,p}$ approximates $X_{\eta\xi}(u_p)$, $u_p \in [-1,1]$, $p = 1,2,\ldots,5$, and the initial value for the subsequent intervals are given by

$$\sum_{p=1}^{5} L_p(1) X_{\eta\xi,p},$$

where

$$X_{\eta\xi,p}, \qquad p = 1,2,\ldots,5,$$

are the computed values from the immediate preceding interval.

Equations (6.11) and (6.12) are 6 linear algebraic equations in 5 unknowns:

$$X_{\eta\xi,p}, \qquad p = 1,2,\ldots,5.$$

To solve for the unknowns, we treat as superfluous (i.e., we ignore) any one of the internal nodal equations appearing in (6.11) and combine the remaining equations with (6.12). The superfluous equation is then used to calculate the residual error of the quadrature method.

In this example, we treat as superfluous the equation given by (6.11) for $m = 1$, for each of the subintervals $[t_0, T]$, i.e.,

$$\sum_{p=1}^{5} a_{1p}^{(1)} X_{\eta\xi,p} = \frac{1}{2}(T - t_0)(e^{(T-t_0)u_1 + T + t_0} - 1) X_{\eta\xi,1}. \tag{6.13}$$

The absolute residual error is then given by

$$\left| \sum_{p=1}^{5} a_{1p}^{(1)} X_{\eta\xi,p} - \frac{1}{2}(T - t_0)(e^{(T-t_0)u_1 + T + t_0} - 1) X_{\eta\xi,1} \right|, \tag{6.14}$$

which is the error for which the quadrature solutions fail to satisfy the superfluous equation.

The quadrature coefficients $a_{mp}^{(1)}$, $m, p = 1,2,\ldots,5$, appearing in (6.11) are given by

$$a_{pp}^{(1)} = \frac{u_p}{1 - u_p^2}, \quad a_{pq}^{(1)} = \frac{P_5'(u_p)}{(u_p - u_q) P_5'(u_q)}, \qquad p \neq q, \tag{6.15}$$

and the Lagrangian polynomial $L_p(u)$ of degree 4 is given by

$$L_p(u) = \prod_{\substack{k=1 \\ k \neq p}}^{5} \left[(u - u_k)/(u_p - u_k) \right], \qquad p = 1,2,\ldots,5, \tag{6.16}$$

where u_p, $p = 1,\ldots,5$, are the five zeros of Legendre polynomial $P_5(u)$ of degree 5.

By using Mathcad computational software on a Samtron personal computer, we generate the following table of values.

p	u_p	$L_p(-1)$	$L_p(1)$
1	0.000000	0.533333186	0.5333331845
2	0.538469	−0.267941412	−0.8931568153
3	−0.538469	−0.893156815	−0.2679414133
4	0.906180	0.076358457	1.5514065859
5	−0.906180	1.551406587	0.0763584582

The quadrature coefficient matrix is given by

$$[\mathbf{a}_{\mathbf{ij}}^{(\mathbf{1})}] =$$

j / i	1	2	3	4	5
1	0.000000000	1.435391432	−1.435391432	−0.301166462	0.301166462
2	−2.402748080	0.758352423	0.9285585610	0.960247980	−0.244414903
3	2.402748080	−0.928558561	−0.758352423	0.244414903	−0.960247980
4	4.043565172	−7.70199715	−1.960413273	5.067049385	0.551766757
5	−4.043565172	1.960413273	7.701997150	−0.551766757	−5.067049385

We solve the system of linear equations in each of the six subintervals on a Mega-Image personal computer by using a software program developed with a single numerical precision or word size to obtain the following results.

(a) For the subinterval $[0, 0.1], u = 20t - 1,\ t = \frac{1}{20}(1+u)$.

u_p	t_p	Computed value $\langle \eta, X_p \xi \rangle$	Exact value $\langle \eta, X(t_p)\xi \rangle$	Absolute error $\|X_p - X(t_p)\|_{\eta\xi}$
0.000000	0.05000000	1.0025800697	1.0025888040	8.734×10^{-6}
0.538469	0.07692345	1.0062434016	1.0062521690	8.767×10^{-6}
−0.538469	0.02307655	1.0005324619	1.0005409610	8.499×10^{-6}
0.906180	0.09530900	1.0009733201	1.0097366680	3.467×10^{-6}
−0.906180	0.00469100	1.0000188125	1.0000220750	3.262×10^{-6}
1.000000	0.10000000	1.0107586190	1.0107588440	2.240×10^{-7}

Residual error for the subinterval $= 3.260 \times 10^{-7}$.

(b) For the subinterval $[0.1, 0.3]$, $u = 10t - 2$, $t = \frac{1}{10}(2+u)$.

u_p	t_p	Computed value $\langle \eta, X_p \xi \rangle$	Exact value $\langle \eta, X(t_p)\xi \rangle$	Absolute error $\|X_p - X(t_p)\|_{\eta\xi}$
0.000000	0.2000000	1.0469640346	1.046982638	1.8603×10^{-5}
0.538469	0.2538469	1.0798827100	1.079913195	3.0485×10^{-5}
−0.538469	0.1461531	1.0238766814	1.023884223	7.542×10^{-6}
0.906180	0.2906180	1.1090291990	1.109052081	2.2882×10^{-5}
−0.906180	0.1093820	1.0129658746	1.012970158	4.284×10^{-6}
1.000000	0.3000000	1.117440811	1.117461282	2.0471×10^{-5}.

Residual error for the subinterval $= 3.2754 \times 10^{-5}$.

(c) For the subinterval [0.3, 0.5], $u = 10t - 4$, $t = \frac{1}{10}(u+4)$.

u_p	t_p	Computed value $\langle \eta, X_p \xi \rangle$	Exact value $\langle \eta, X(t_p)\xi \rangle$	Absolute error $\|X_p - X(t_p)\|_{\eta\xi}$
0.0000000	0.4000000	1.2370270174	1.237100660	7.3643×10^{-5}
0.5384690	0.4538469	1.3302350684	1.330364540	1.29471×10^{-4}
−0.5384690	0.3461531	1.1653070910	1.165333229	2.5198×10^{-5}
0.9061800	0.4906180	1.4094136881	1.409532054	1.18366×10^{-4}
−0.9061800	0.3093820	1.1262680223	1.126295482	2.7460×10^{-5}
1.0000000	0.5000000	1.4319787790	1.432098590	1.19811×10^{-4}

Residual error for the subinterval $= 1.40497 \times 10^{-4}$.

(d) For the subinterval [0.5, 0.7], $u = 10t - 6, t = \frac{1}{10}(u + 6)$.

u_p	t_p	Computed value $\langle \eta, X_p\xi \rangle$	Exact value $\langle \eta, X(t_p)\xi \rangle$	Absolute error $\Vert X_p - X(t_p)\Vert_{\eta\xi}$
0.0000000	0.6000000	1.7503830996	1.750774850	3.91751×10^{-4}
0.5384690	0.6538469	2.0029050837	2.003651551	7.46467×10^{-4}
−0.5384690	0.5461530	1.5594732874	1.559581521	1.08234×10^{-4}
0.9062800	0.6906180	2.2232129379	2.223963183	7.50245×10^{-4}
−0.9061800	0.5093820	1.4555795392	1.455722886	1.43347×10^{-4}
1.0000000	0.7000000	2.2870345930	2.287821337	7.86744×10^{-4}

Residual error for the subinterval = 7.96578×10^{-4}.

(e) For the subinterval [0.7, 0.9], $u = 10t - 8,\ \ t = \frac{1}{10}(u + 8)$.

u_p	t_p	Computed value $\langle \eta, X_p\xi \rangle$	Exact value $\langle \eta, X(t_p)\xi \rangle$	Absolute error $\Vert X_p - X(t_p)\Vert_{\eta\xi}$
0.000000	0.8000000	3.2399942574	3.243056380	3.062123×10^{-3}
0.538469	0.8538469	4.0660804136	4.072535340	6.454926×10^{-3}
−0.538469	0.7461531	2.6572106604	2.657848547	6.378870×10^{-4}
0.906280	0.8906180	4.8380606659	4.845144775	7.084109×10^{-3}
−0.906180	0.7093820	2.3542432100	2.355193899	9.506890×10^{-4}
1.000000	0.9000000	5.0699378020	5.077523954	7.586152×10^{-3}

Residual error for the subinterval = 6.543177×10^{-3}.

(f) For the subinterval $[0.9, 1.0]$, $u = 20t - 19, t = \frac{1}{20}(u + 19)$.

u_p	t_p	Computed value $\langle \eta, X_p\xi \rangle$	Exact value $\langle \eta, X(t_p)\xi \rangle$	Absolute error $\Vert X_p - X(t_p)\Vert_{\eta\xi}$
0.000000	0.95000000	6.6285640837	6.638906201	1.0342117×10^{-2}
0.538469	0.97692345	7.7628308886	7.775406243	1.2575355×10^{-2}
−0.538469	0.92307655	5.7152021899	5.723748457	8.546268×10^{-3}
0.906280	0.99530900	8.6982084176	8.712154152	1.3945735×10^{-2}
−0.906180	0.90469100	5.1921131115	5.199930037	7.816926×10^{-3}
1.000000	1.00000000	8.961388104	8.975763940	1.4375836×10^{-2}

Residual error for the subinterval = 1.235407×10^{-3}.

Example 2. We consider the simple Fock space $\Gamma(L^2_Y(\mathbb{R}_+))$, where $Y = \mathcal{R} = \mathbb{C}, f = g \equiv 1$, and its $L^2(\Omega, \mathcal{F}, W)$ realization, where $(\Omega, \mathcal{F}, W)$ is a Wiener space. Each random variable X is identified with the operator of multiplication by X so that $Q(t) = A(t) + A^+(t) = w(t)$ is the evaluation of the Brownian path w at time t. In this case, it has been shown that quantum stochastic integrals of adapted Brownian functional F such that $\int_{t_0}^t E_w[F(s,\cdot)^2]ds < \infty$ exist (see [1, 2]). Here E_w is the expected value function.

For exponential vectors $\eta = e(\alpha)$ and $\xi = e(\beta)$, where $\alpha,\ \beta$ are purely imaginary valued functions in $L^2_{\mathbb{C}}(\mathbb{R}_+)$, the equivalent form (1.3) of the quantum analogue of the classical Ito stochastic differential equation

$$\begin{aligned} &dX(t,w) = H(t,X(t))dt + F(t,X(t))dW(t), \\ &X(t_0) = X_0, \qquad t \in [t_0, T], \end{aligned} \tag{6.17}$$

is given by

$$\frac{d}{dt}E_w[X(t,w)z(w)] = E_w[\beta(t)F(t,X(t))z(w)] + E_w[\bar{\alpha}(t)F(t,X(t))z(w)] + E_w[H(t,X(t))z(w)],$$

$$E_w[X(t_0)z(w)] = E_w[X_0 z(w)], \quad \text{almost all} \quad t \in [t_0,T], \tag{6.18}$$

where

$$z(w) = \exp\left\{\int_0^\infty(-\alpha(s)+\beta(s))dw(s) - \frac{1}{2}\int_0^\infty(\alpha^2(s)+\beta^2(s))ds\right\} \tag{6.19}$$

(see [1, 2] for details).

By considering the linear functionals $F(t,x) = bx$, $H(t,x) = ax$, where a, b are real constants, we apply the Lagrangian quadrature method to solve (6.18). Thus, as in Example 1, we perform the integration in 5 nodes in each of the step subintervals $[0,0.1]$, $[0.1,0.3]$, $[0.3,0.7]$, and $[0.7,1]$ of the interval $[0,1]$.

For $a = \frac{3}{2}$, $b = 1$, $\alpha(t) = \beta(t) = i$, problem (6.18) after the necessary transformation to the variable $u \in [-1,1]$ becomes

$$\frac{d}{du}E_w[X(u)z(w)] = \frac{3}{4}(T-t_0)E_w[X(u)z(w)],$$

$$X(-1) = X_0, \qquad u \in [-1,1] \tag{6.20}$$

on each of the subintervals $[t_0,T] \in \{[0,0.1],\ [0.1,0.3],\ [0.3,0.7],\ [0.7,1]\}$.

Equation (6.20) transforms to the algebraic equations given by

$$\sum_{p=1}^{5} a_{mp}^{(1)} E_w[X_p(w)z(w)] = P(u_m,X_m)(\eta,\xi)$$

$$= \frac{3}{4}(T-t_0)E_w[X_m(u)z(w)], \qquad m = 1,2,3,4,5, \tag{6.21}$$

with the initial value for the first subinterval $[0,0.1]$ transforming to

$$\sum_{p=1}^{5} L_j(-1)E_w[X_p(w)z(w)] = E_w[z(w)] = e, \tag{6.22}$$

where

$$\langle \eta, X(u)\xi\rangle \cong \sum_{p=1}^{5} L_p(u)E_w[X_p(w)z(w)], \qquad p = 1,2,\ldots,5.$$

The initial values for the subsequent intervals are given by

$$\sum_{p=1}^{5} L_p(1)E_w[X_p(w)z(w)],$$

where

$$E_w[X_p(w)z(w)], \qquad p = 1,2,\ldots,5,$$

are the computed values from the immediate preceding interval.

Again, we treat as superfluous the equation given by (6.21) for $m = 1$ for each of the subintervals $[t_0, T]$. This yields the absolute residual error given by

$$\left|\sum_{p=1}^{5} a_{1p}^{(1)} E_w[X_p(w)z(w)] - \frac{3}{4}(T - t_0)E_w[X_1(w)z(w)]\right|. \tag{6.23}$$

We solve the system of linear equations in each of the four subintervals to obtain the following results.

(a) For the subinterval $[0, 0.1], u = 20t - 1,\ t = \frac{1}{20}(1 + u)$.

u_p	t_p	Computed value $E_w[X_pz(w)]$	Exact value $\exp(1 + \frac{3}{2}t)$	Absolute error
0.000000	0.05000000	2.9299678213	2.929992901	2.5080×10^{-5}
0.538469	0.07692345	3.0507182716	3.050743023	2.4752×10^{-5}
−0.538469	0.02307655	2.8139973688	2.814022136	2.4768×10^{-5}
0.906180	0.09530900	3.1360391076	3.136048287	9.1800×10^{-6}
−0.906180	0.00469100	2.7374670005	2.737476471	9.4710×10^{-6}
1.000000	0.10000000	3.1581933020	3.158192910	3.9200×10^{-7}

Residual error for the subinterval $= 3.01843 \times 10^{-6}$.

(b) For the subinterval $[0.1, 0.3]$, $u = 10t - 2$, $t = \frac{1}{10}(2 + u)$.

u_p	t_p	Computed value $E_w[X_pz(w)]$	Exact value $\exp(1 + \frac{3}{2}t)$	Absolute error
0.000000	0.2000000	3.6692651703	3.669296668	3.1498×10^{-5}
0.538469	0.2538469	3.9779328927	3.977964873	3.1918×10^{-5}
−0.538469	0.1461531	3.3845492094	3.384579418	3.0209×10^{-5}
0.906180	0.2906180	4.2035278057	4.203539885	1.2080×10^{-5}
−0.906180	0.1093820	3.2029409383	3.202952370	1.1432×10^{-5}
1.000000	0.3000000	4.2631144550	4.263114515	6.0000×10^{-8}.

Residual error for the subinterval $= 4.8735 \times 10^{-6}$.

(c) For the subinterval $[0.3, 0.7]$, $u = 5t - 2.5$, $t = \frac{1}{5}(2.5 + u)$.

u_p	t_p	Computed value $E_w[X_pz(w)]$	Exact value $\exp(1 + \frac{3}{2}t)$	Absolute error
0.000000	0.50000000	5.7545129246	5.754602676	8.9752×10^{-5}
0.538469	0.60769380	6.7633539045	6.763501302	1.47398×10^{-4}
−0.538469	0.39230620	4.8961586520	4.896199540	4.0888×10^{-5}
0.906180	0.68123600	7.5521976331	7.552313952	1.16319×10^{-4}
−0.906180	0.31876400	4.3847871371	4.384808705	2.1568×10^{-5}
1.000000	0.70000000	7.767798135	7.767901106	1.02971×10^{-4}

Residual error for the subinterval $= 1.156212 \times 10^{-4}$.

(d) For the subinterval $[0.7, 1.0]$, $u = \frac{20}{3}t - \frac{17}{3}$, $t = \frac{3}{20}(u + \frac{17}{3})$.

u_p	t_p	Computed value $E_w[X_p z(w)]$	Exact value $\exp(1 + \frac{3}{2}t)$	Absolute error
0.000000	0.85000000	9.7276915337	9.727919013	2.2748×10^{-4}
0.538469	0.93077035	10.8980614837	10.980879410	2.6458×10^{-4}
−0.538469	0.76922965	8.6177335813	8.617926194	1.92613×10^{-4}
0.906280	0.98592700	11.9277999780	11.928022930	2.2296×10^{-4}
−0.906180	0.71407300	7.9334830988	7.933620589	1.37491×10^{-4}
1.000000	1.00000000	12.1822979700	12.182493960	1.959906×10^{-4}

Residual error for the subinterval $= 2.53749 \times 10^{-5}$.

Remark 6.1.

(i) Suppose that the numerical integration is to be performed for the interval $t_0 \leq t \leq T$, and to satisfy a prescribed error margin or tolerance. If the length $T - t_0$ of the interval is sufficiently small and the number N of nodal points is sufficiently large, one application of the quadrature algorithm may be adequate. However, usually in practice, this will not be the case, and if N is conveniently fixed, the interval $[t_0, T]$ has to be divided into subintervals $[t_0, t_1], [t_1, t_2] \cdots [t_{q-1}, t_q]$, say. Thus starting with the first subinterval $[t_0, t_1]$ and the prescribed initial condition, numerical quadrature integration is performed on N nodes in $[t_0, t_1]$. If the residual error at the superfluous node is sufficiently small, then integration proceeds into the next interval $[t_1, t_2]$ with a computed initial condition obtained by the use of extrapolation formula

$$X_{\eta\xi}(t_1) = \sum_{p=1}^{N} L_p(t_1) X_{\eta\xi}(t_p),$$

where $X_{\eta\xi}(t_p), p = 1, 2, \ldots, N$, are the computed values for $[t_0, t_1]$. Otherwise the length $t_1 - t_0$ of the subinterval $[t_0, t_1]$ is subdivided until the first subinterval produces the desired results. However, we could increase the number of nodes to achieve the same purpose. However, without loss of generality it is assumed that N is fixed. Integration can proceed in this manner until the interval $[t_q, T]$ is reached. Finally, we remark that if the residual error at any stage $[t_k, t_{k+1}]$ is much smaller than the tolerance level, then the succeeding interval may be chosen by doubling the size of the interval $[t_k, t_{k+1}]$. Stability of this method will be addressed elsewhere.

(ii) The numerical experiments given in Examples 1 and 2 above show that the Lagrangian quadrature method is a method of high accuracy. We have convergence to exact values at each of the discretization point to a minimum number of one decimal place in Example 1 and three decimal places in Example 2. The accumulated error at the final time instant $t = 1$ is 1.4375836×10^{-2} in Example 1 with six subintervals of [0, 1] and 1.959906×10^{-4} in Example 2 with four subintervals of $[0, 1]$.

In comparison with the Euler scheme and a 2-step scheme which were employed in [2] to solve the same model problem of Example 2, we notice that the Euler and the 2-step scheme produced accumulated errors of 0.78042828 and 0.11070989, respectively, at the final time $t = 1$ with steplength of $h = 2^{-4}$. This shows that the quadrature method of this paper is more accurate than said schemes.

(iii) As a measure of the computational complexity of the quadrature scheme, we estimate the arithmetic floating point operations required for implementation as follows: By (6.15), computation of each diagonal element $a_{pp}^{(1)}$ of the $N \times N$ quadrature coefficient matrix $[a_{pq}^{(1)}]$ requires 1 multiplication, 1 addition, and 1 division. For each of the other entries, $a_{pq}^{(1)}$, $p \neq q$, a maximum of $4N-5$ multiplications (i.e., if the values of the derivative of the Legendre polynomial $P_N(u)$ of degree N are evaluated directly), $2N-1$ additions and 1 division are required. The Lagrangian interpolant $L_p(u)$ defined by (6.16) requires $2(N-1)$ additions, $2(N-2)$ multiplications, and 1 division for each $p = 1, 2, \ldots, N$, where N is the number of nodes.

As a result of the above estimates, the Lagrangian quadrature scheme has a large number of multiplication, addition, and division floating point operations. This could be calculated by employing the standardized weights for each type of floating point operation as in [32, p. 193]. Since the number depends directly on the total nodal points N, it is necessary that the total nodes be restricted to the range $1 \leq N \leq 15$ for each of the subdivisions of the entire interval $[t_0, T]$ in order to minimize the CPU time. For the total nodes $N = 5$ employed in Examples 1 and 2 above, very short CPU times were returned by the personal computer for each of the fragments of the numerical computations reported in Examples 1 and 2 above.

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REFERENCES

[1] E.O. Ayoola, *On Numerical Procedures for Solving Lipschitzian Quantum Stochastic Differential Equations*, Ph.D. thesis, University of Ibadan, Ibadan, Nigeria, 1998.

[2] E.O. Ayoola, *Converging multistep methods for weak solution of quantum stochastic differential equations*, Stochastic Anal. Appl., 18 (2000), pp. 525–554.

[3] C. Barnett, R.F. Streater, and I.F. Wilde, *The Ito-Clifford integral* II—*Stochastic differential equations*, J. Lond. Math. Soc. (2), 27 (1983), pp. 373–384.

[4] R.E. Bellman, B.G. Kashef, and J. Casti, *Differential quadrature: A technique for the rapid solution of nonlinear partial differential equations*, J. Comput. Phys., 10 (1972), pp. 40–52.

[5] G.O.S. Ekhaguere, *Lipschitzian quantum stochastic differential inclusions*, Internat. J. Theoret. Phys., 31 (1992), pp. 2003–2034.

[6] G.O.S. Ekhaguere, *Quantum stochastic differential inclusions of hypermaximal monotone type*, Internat. J. Theoret. Phys., 34 (1995), pp. 323–353.

[7] G.O.S. Ekhaguere, *Quantum stochastic evolutions*, Internat. J. Theoret. Phys., 35 (1996), pp. 1909–1946.

[8] L. Fox and I.B. Parker, *Chebyshev Polynomials in Numerical Analysis*, Oxford University Press, London, 1972.

[9] P. Glockner, *Quantum stochastic differential equations on *-bialgebra,* Math. Proc. Cambridge Philos. Soc., 109 (1991), pp. 571–595.

[10] A.S. Holevo, *Exponential formulae in quantum stochastic calculus*, Proc. Roy. Soc. Edinburgh Sect. A, 126 (1996), pp. 375–389.

[11] R.L. Hudson, *Quantum stochastic calculus and some of its applications*, in Contemporary Stochastic Analysis, G.O.S. Ekhaguere, ed., World Scientific, River Edge, NJ, 1991, pp. 31–70.

[12] R.L. Hudson and K.R. Parthasarathy, *Construction of quantum diffusion*, in Quantum Probability and Applications to the Quantum Theory of Irreversible Processes, Lecture Notes in Math. 1055, L. Accardi, A. Frigerio, and V. Gorini, eds., Springer-Verlag, Berlin, 1982, pp. 173–198.

[13] R.L. Hudson and K.R. Parthasarathy, *Quantum Ito's formulae and stochastic evolutions*, Comm. Math. Phys., 93 (1984), pp. 301–324.

[14] P.E. Kloeden and E. Platen, *Numerical Solutions of Stochastic Differential Equations*, 2nd ed., Springer-Verlag, New York, 1995.

[15] P.E. Kloeden, E. Platen, and H. Schurz, *Numerical Solution of SDE through Computer Experiments*, Springer-Verlag, New York, 1994.
[16] S.I. Marcus, *Modelling and approximation of stochastic differential equations driven by semimartingales*, Stochastics, 4 (1981), pp. 223–245.
[17] P. Meyer, *Quantum Probability for Probabilists*, Lecture Notes in Math. 1538, Springer-Verlag, Berlin, 1993.
[18] G.N. Milstein, *Numerical Integration for Stochastic Differential Equations*, Kluwer, Dordrecht, The Netherlands 1995.
[19] N.J. Newton, *Asymptotically efficient Runge–Kutta methods for a class of Itô and Stratonovich equations*, SIAM J. Appl. Math, 51 (1991), pp. 542–567.
[20] G.O. Olaofe, *Quadrature solutions of the double eigenvalue problem*, in Proceedings of the 7th Biennial Numerical Analysis Conference, Dundee, 1979.
[21] G.O. Olaofe, *Quadrature solution of ordinary and partial differential equations*, in Proceedings of the 2nd International Conference on Numerical Analysis, University of Benin, Nigeria, 1986.
[22] G.O. Olaofe and J.C. Mason, *Quadrature solution of nonlinear ordinary and partial differential equations*, in Proceedings of Sixth International Conference on Mathematical Modelling, Washington University, St. Louis, 1987.
[23] K.R. Parthasarathy, *Boson stochastic calculus*, Pramana J. Phys., 25 (1985), pp. 457–465.
[24] K.R. Parthasarathy, *An Introduction to Quantum Stochastic Calculus*, Monogr. Math. 85, Birkhäuser Verlag, Basel, 1992.
[25] W.P. Petersen, *Numerical Simulation of Ito Stochastic Differential Equations on Supercomputers*, IMA Vol. Math. Appl. 7, Springer-Verlag, New York, 1987, pp. 215–228.
[26] M.J.D. Powell, *Approximation Theory and Methods*, Cambridge University Press, Cambridge, UK, 1981.
[27] P. Protter, *Approximations of solution of stochastic differential equations driven by semimartingale*, Ann. Probab., 13 (1985), pp. 716–743.
[28] W. Rümelin, *Numerical treatment of stochastic differential equations*, SIAM J. Numer. Anal., 19 (1982), pp. 604–613.
[29] A.H. Stroud, *Numerical Quadrature and Solution of Ordinary Differential Equations*, Springer-Verlag, New York, 1974.
[30] A.H. Stroud and D. Secrest, *Gaussian Quadrature Formulas*, Prentice-Hall, Englewood Cliffs, NJ, 1966.
[31] K. Twardowska, *Wong-Zakai approximations for stochastic differential equations*, Acta Appl. Math., 43 (1996), pp. 317–359.
[32] C.W. Ueberhuber, *Numerical Computation* I*, Methods, Software and Analysis*, Springer-Verlag, Berlin, Heidelberg, New York, 1997.
[33] G.F. Vincent-Smith, *Unitary quantum stochastic evolutions*, Proc. London Math. Soc. (3), 63 (1991), pp. 401–425.
[34] W. von Waldenfels, *Ito solution of the linear quantum stochastic differential equation describing light emission and absorption*, in Quantum Probability and Applications to the Quantum Theory of Irreversible Processes, Lecture Notes in Math. 1055, L. Accardi, A. Frigerio, and V. Gorini, eds., Springer-Verlag, Berlin, 1982, pp. 384–411.
[35] W. von Waldenfels, *Stratonovich solution of a quantum stochastic differential equation describing light emission and absorption*, in Stochastic Aspects of Classical and Quantum Systems, Lecture Notes in Math. 1109, S. Albeverio, P. Combe, and M. Sirugue-Collin, eds., Springer-Verlag, Berlin, 1983, pp. 155–176.
[36] W. von Waldenfels, *Spontaneous light emission described by a quantum stochastic differential equation*, in Quantum Probability and Applications, II, Lecture Notes in Math. 1136, L. Accardi and W. von Waldenfels, eds., Springer-Verlag, Berlin, 1984, pp. 516–534.
[37] W. Wagner and E. Platen, *Approximation of Ito Integral Equations*, preprint, ZIMM, Acad. Sci. GDR, Berlin, 1978.
[38] D.M. Young and R.T. Gregory, *A Survey of Numerical Mathematics*, Addison-Wesley, Reading, MA, 1972.

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Error Estimates for Discretized Quantum Stochastic Differential Inclusions

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ABSTRACT

This paper is concerned with the error estimates involved in the solution of a discrete approximation of a quantum stochastic differential inclusion (QSDI). Our main results rely on certain properties of the averaged modulus of continuity for multivalued sesquilinear forms associated with QSDI. We obtained results concerning the estimates of the Hausdorff distance between the set of solutions of the QSDI and the set of solutions of its discrete approximation. This extends the results of Dontchev and Farkhi Dontchev, A.L.; Farkhi, E.M. (Error estimates for discretized differential inclusions. *Computing* **1989**, *41*, 349–358) concerning classical differential inclusions to the present noncommutative quantum setting involving inclusions in certain locally convex space.

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1. INTRODUCTION

In continuation of our work in Refs.[3,4] concerning numerical procedures for quantum stochastic differential inclusion (QSDI), this paper is concerned with the development, analysis and error estimates involved in a one—step discrete scheme for solving quantum stochastic differential inclusion:

$$\begin{aligned} dX(t) &\in E(t, X(t))\, d\wedge_\pi(t) + F(t, X(t))\, dA_f(t) + G(t, X(t))\, dA_g{}^+(t) \\ &\quad + H(t, X(t))\, dt, \quad \text{almost all } t \in [0, T] \\ X(0) &= X^0 \end{aligned} \tag{1.1}$$

The coefficients E, F, G, H in (1.1) are elements of $L_{\mathrm{loc}}{}^2([0, T] \times \tilde{\mathcal{A}})_{\mathrm{mvs}}$, and the integrators $\wedge_\pi, A_f, A_g{}^+ : [0, T] \to \tilde{\mathcal{A}}$ are the gauge, annihilation and creation processes. The locally convex space $\tilde{\mathcal{A}}$ is the completion of the linear space $L^+(\mathbb{D}\underline{\otimes}\mathbb{E}, \mathcal{R} \otimes \Gamma(L_\gamma{}^2(\mathbb{R}_+)))$ in the Hausdorff topology generated by the family of seminorms $\{\|x\|_{\eta\xi} = |\langle \eta, x\xi\rangle| : x \in \tilde{\mathcal{A}}, \eta, \xi \in \mathbb{D}\underline{\otimes}\mathbb{E}\}$. A solution $X(t)$ of (1.1) is a densely defined linear operator lying in $Ad(\tilde{\mathcal{A}})_{\mathrm{wac}}$ (see Refs.[3,4,8–10] for details).

For arbitrary $\eta, \xi \in \mathbb{D}\underline{\otimes}\mathbb{E}$, the equivalent form of (1.1) is the first order initial value nonclassical inclusion given by

$$\begin{aligned} &\frac{d}{dt}\langle \eta, X(t)\xi\rangle \in P(t, X(t))(\eta, \xi) \\ &X(0) = X^0, \quad \text{almost all } t \in [0, T] \end{aligned} \tag{1.2}$$

where $(\eta, \xi) \to P(t, X(t))(\eta, \xi)$ is a multivalued sesquilinear form on $\mathbb{D}\underline{\otimes}\mathbb{E}$ with values in the field of complex numbers. The explicit form of the map $P(t, x)(\eta, \xi)$ is described as follows.

For $(t, x) \in \mathbb{R}_+ \times \tilde{\mathcal{A}}$, $\eta, \xi \in \mathbb{D}\underline{\otimes}\mathbb{E}$ such that $\eta = c \otimes e(\alpha)$, $\xi = d \otimes e(\beta)$, $c, d \in \mathbb{D}$, $\alpha, \beta \in L_{\gamma,\mathrm{loc}}{}^\infty(\mathbb{R}_+)$, define

$$P_{\alpha\beta} : [0, T] \times \tilde{\mathcal{A}} \longrightarrow 2^{\tilde{\mathcal{A}}}$$

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by

$$P_{\alpha\beta}(t,x) = \mu_{\alpha\beta}(t)E(t,x) + \nu_{\beta}(t)F(t,x) + \sigma_{\alpha}(t)G(t,x) + H(t,x)$$

where

$$\mu_{\alpha\beta}(t) = \langle \alpha(t), \pi(t)\beta(t)\rangle_{\gamma}, \quad \nu_{\beta}(t) = \langle f(t), \beta(t)\rangle_{\gamma},$$
$$\sigma_{\alpha}(t) = \langle \alpha(t), g(t)\rangle_{\gamma}$$

This leads to the multifunction P: $[0, T] \times \tilde{\mathcal{A}} \to 2^{\mathrm{sesq}(\mathbb{D}\underline{\otimes}\mathbb{E})}$ defined by

$$P(t,x)(\eta,\xi) := \langle \eta, P_{\alpha\beta}(t,x)\xi\rangle = \{\langle \eta, Z(t,x)\xi\rangle : Z(t,x) \in P_{\alpha\beta}(t,x)\} \tag{1.3}$$

In what follows, we employ the basic function spaces, the set theoretic operations and the Hausdorff metric as in Refs.[3,4,8–10].

The plan for the rest of the paper is as follows: Sec. 2 contains preliminary statements and basic results in respect of the modulus of continuity of multivalued sesquilinear forms associated with QSDI (1.1). The main result of this paper concerning the error estimates of the discretized inclusion is established in Sec. 3. This extends and compliments the results of Ref.[6] concerning similar discretizations of classical differential inclusions.

2. PRELIMINARY STATEMENTS AND ESTIMATES

We present in this section, some notations, definitions, and estimates which we will employ in the sequel. Without loss of generality, we consider inclusion (1.1) and (1.2) defined only on the interval $[0, 1] \subseteq \mathbb{R}_+$. Any other interval $[0, T]$, for $T > 0$ may be converted into $[0, 1]$ by an appropriate regular transformation. In what follows, unless otherwise indicated, $\eta, \xi \in \mathbb{D}\underline{\otimes}\mathbb{E}$ such that $\eta = c \otimes e(\alpha)$, $\xi = d \otimes e(\beta)$, $c, d \in \mathbb{D}$, $\alpha, \beta \in L_{\gamma,\mathrm{loc}}^{\infty}(\mathbb{R}_+)$.

Let P: $[0, 1] \times \tilde{\mathcal{A}} \to 2^{\mathrm{sesq}(\mathbb{D}\underline{\otimes}\mathbb{E})}$ be a multivalued sesquilinear form satisfying the following conditions:

$\mathcal{S}_{(i)}$ For every $x \in \tilde{\mathcal{A}}$, $t \in [0, 1]$, $P(t,x)(\eta,\xi)$ is nonempty, compact, and convex subset of $\mathbb{C}$ the complex field.

$\mathcal{S}_{(ii)}$ The map $P(t,\cdot)(\eta,\xi)$ is locally Lipschitzian uniformly in $t \in [0, 1]$, with Lipschitz constants $K_{\eta,\xi}$.

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$\mathcal{S}_{(iii)}$ There exists positive constants L and A such that

$$|P(t, x)(\eta, \xi)| \leq L\|x\|_{\eta\xi} + A$$

for all $x \in \tilde{\mathcal{A}}$, $t \in [0, 1]$, and for all $\eta, \xi \in \mathbb{D}\underline{\otimes}\mathbb{E}$.

Here,

$$|P(t, x)(\eta, \xi)| = \sup_{Z_{\eta\xi} \in P(t,x)(\eta,\xi)} |Z_{\eta\xi}|$$

We note that the map $t \to P(t, x)(\eta, \xi)$ is measurable in [0, 1] for every fixed element $x \in \tilde{\mathcal{A}}$ (see Ref.[8]) since the map is locally integrable on $\mathbb{R}_+$.

In our subsequent analysis, we shall need the following Theorem which is the basic existence result of (1.1) due to Ref.[8]

Theorem 2.1

Let θ be a positive number and $I = [t_0, T] \subseteq \mathbb{R}_+$. Assume that the following conditions hold.

(a) $Z: I \mapsto \tilde{\mathcal{A}}$ is an arbitrary process lying in $Ad(\tilde{\mathcal{A}})_{\text{wac}}$ and there exists positive function $W_{\eta\xi}(t)$ satisfying

$$\mathbf{d}\left(\frac{d}{dt}\langle \eta, Z(t)\xi\rangle, P(t, Z(t)(\eta, \xi)\right) \leq W_{\eta\xi}(t)$$

(b) Each of the maps E, F, G, H is Lipschitzian from $Q_{Z,\theta}$ to $(\text{clos}(\tilde{\mathcal{A}}), \tau_H)$ where

$$Q_{Z,\theta} = \{(t, x) \in I \times \tilde{\mathcal{A}}: \|x - Z(t)\|_{\eta\xi} \leq \theta, \forall \eta, \\ \xi \in \mathbb{D}\underline{\otimes}\mathbb{E}, \text{ and } \|x_0 - Z(t_0)\|_{\eta\xi} \leq \theta\}$$

(c) For arbitrary $\eta, \xi \in \mathbb{D}\underline{\otimes}\mathbb{E}$, $t \in I$,

$$E_{\eta\xi}(t) = \|x_0 - Z(t_0)\|_{\eta\xi} \exp\left(\int_{t_0}^{t} ds K_{\eta\xi}{}^{P}(s)\right) \\ + \int_{t_0}^{t} ds W_{\eta\xi}(s) \exp\left(\int_{t_0}^{t} dr K_{\eta\xi}{}^{P}(r)\right)$$

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If in addition, E, F, G, H are continuous from $I \times \tilde{\mathcal{A}}$ to $(\text{clos}(\tilde{\mathcal{A}}), \tau_H)$, then there exist a solution Φ of inclusion (1.1) and a subset $J \subseteq I$ such that

$$\|\Phi(t) - Z(t)\|_{\eta\xi} \leq E_{\eta\xi}(t), \quad t \in J$$

and

$$\left|\frac{d}{dt}\langle \eta, \Phi(t)\xi\rangle - \frac{d}{dt}\langle \eta, Z(t)\xi\rangle\right| \leq K_{\eta\xi}{}^{P}(t)E_{\eta\xi}(t) + W_{\eta\xi}(t)$$

for almost all $t \in J$ where $J = \{t \in I: E_{\eta\xi}(t) \leq \theta\}$ and $K_{\eta\xi}{}^{P}: I \mapsto (0, \infty)$ is the Lipschitz function for P lying in $L_{\text{loc}}{}^{1}(I)$.

Remark 2.2

(i) In this paper, we assume that the coefficients E, F, G, H appearing in (1.1) are Lipschitzian uniformly in $t \in [0, 1]$. Consequently, the map $(t, x) \to P(t, x)(\eta, x)(\eta, \xi)$ is also Lipschitzian in x, uniformly in $t \in [0, 1]$. That is for $(t, x), (t', y) \in [0, 1] \times \tilde{\mathcal{A}}$,

$$\rho(P(t, x)(\eta, \xi), P(t', y)(\eta, \xi)) \leq K_{\eta\xi}\|x - y\|_{\eta\xi}$$

(ii) By Theorem 2.1, the set of solutions of (1.1) is nonempty and the values of the solutions are contained in the set

$$Q = A \exp(K_{\eta\xi})B \subseteq \theta B$$

where

$$B = \{x \in \tilde{\mathcal{A}}: \|x\|_{\eta\xi} \leq 1, \forall \eta, \xi \in \mathbb{D}\underline{\otimes}\mathbb{E}\}$$

and $\theta > 0$ is some positive fixed number. This can be shown by setting $Z(t) \equiv 0$ and $Z(0) = X^0$ in Theorem 2.1. Then by $\mathcal{S}_{(\text{iii})}$ above,

$$\mathbf{d}(0, P(t, 0)(\eta, \xi)) \leq A$$

Setting $W_{\eta\xi}(t) \equiv A$ in Theorem 2.1, we conclude that there exists a solution $\Phi \in Ad(\tilde{\mathcal{A}})_{\text{wac}}$ of (1.1) such that

$$\|\Phi(t)\|_{\eta\xi} \leq A \exp(K_{\eta\xi})$$

for all $\eta, \xi \in \mathbb{D}\underline{\otimes}\mathbb{E}$. Obviously, $J \times Q \subseteq Q_{0,\theta}$.

Following a similar procedure as in Ref.[6], we now introduce the notion of the modulus of continuity for multivalued sesquilinear forms associated with QSDI. Consider a multivalued sesquilinear form

$$F\colon [0, 1] \longrightarrow 2^{\mathrm{sesq}(\mathbb{D}\underline{\otimes}\mathbb{E})}$$

such that $F(t)(\eta, \xi)$ is compact for all $t \in [0, 1]$, $\eta, \xi \in \mathbb{D}\underline{\otimes}\mathbb{E}$.

Definition 2.3

(i) The local modulus of continuity for $F(\cdot)(\eta, \xi)$ is defined by

$$\omega(F; t, h, \eta, \xi) = \sup\left\{ \rho(F(s)(\eta, \xi), F(u)(\eta, \xi)); \; s, u \in \left[t - \frac{h}{2}, t + \frac{h}{2}\right] \bigcap [0, 1] \right\}$$

(ii) The L_p averaged modulus of continuity for $F(\cdot)(\eta, \xi)$ is defined by

$$\tau(F; h)(\eta, \xi)_p = \left\{ \int_0^1 [\omega(F; t, h, \eta, \xi)]^p \, dt \right\}^{1/p}, \quad 1 \leq p \leq \infty$$

We note that the above modulus is well defined since the map

$$t \longrightarrow \omega(F; t, h, \eta, \xi)$$

is measurable and bounded on [0, 1]. For $p = 1$, we simply write

$$\tau(F; h)(\eta, \xi)_1 = \tau(F; h)(\eta, \xi)$$

(iii) For $0 \leq t_1 \leq t_2 \leq \cdots \leq t_k \leq 1$, the p-variation of the map $t \to F(t)(\eta, \xi)$ is defined by

$$W_p(F; \eta, \xi) = \sup\left\{ \sum_{i=1}^{k} [\rho(F(t_{i+1})(\eta, \xi), F(t_i)(\eta, \xi))] \right\}^p.$$

Consequent upon the definitions above, we have the following results whose proofs follow similar arguments as in Ref.[6].

Theorem 2.4

(i) For $h_1 < h_2$, we have

$$\tau(F; h_1)(\eta, \xi)_p \leq \tau(F; h_2)(\eta, \xi)_p$$

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(ii) $\tau(F_1 + F_2; h)(\eta, \xi)_p \leq \tau(F_1; h)(\eta, \xi)_p + \tau(F_2; h)(\eta, \xi)_p$, where

$$(F_1 + F_2)(t)(\eta, \xi) = F_1(t)(\eta, \xi) + F_2(t)(\eta, \xi)$$

the algebraic sum of two mulivalued sesquilinear forms:

$$F_1, F_2: [0, 1] \longrightarrow 2^{\mathrm{sesq}(\mathbb{D}\underline{\otimes}\mathbb{E})}$$

(iii) For any positive integer k,

$$\tau(F; kh)(\eta, \xi) \leq k\tau(F; h)(\eta, \xi)$$

(iv) If F has bounded p-variation $W_p(F; \eta, \xi)$, then

$$\tau(F; h)(\eta, \xi)_p \leq [W_p(F; \eta, \xi)]^{1/p} h^{1/p}$$

(v) $\lim_{h\to 0} \tau(F; h)(\eta, \xi) = 0$ if and only if the map $t \to F(t)(\eta, \xi)$ is Hausdorff continuous at almost all $t \in [0, 1]$.

Next, we fix a partition $0 < t_1 < t_2 \cdots t_N = 1$ of the interval $[0, 1]$ where $t_{i+1} - t_i = h = 1/N$ and define the averaged modulus of continuity for the map $(t, x) \to P(t, x)(\eta, \xi)$. This is needed for the comparison of the set of solutions of (1.1) with the set of solutions of the associated discrete inclusion

$$\begin{aligned} &\langle \eta, X_{i+1}\xi\rangle \in \langle \eta, X_i\xi\rangle + hP(t_i, X_i)(\eta, \xi) \\ &\quad X_0 = X^0, \quad i = 0, 1, \ldots, N-1 \end{aligned} \tag{2.1}$$

In (2.1), $\{X_i\}_{i=0}{}^N$ is a discrete set of members of $\tilde{\mathcal{A}}$ that approximates $\{X(t_i)\}$, $X(t)$ being an exact solution of (1.1). By Theorem 6.3 in Ref.[8], inclusion (2.1) is equivalent to the discrete inclusion given by

$$\begin{aligned} X_{i+1} \in X_i + \int_{t_i}^{t_{i+1}} &(E(t_i, X_i)\, d\wedge_\pi(s) + F(t_i, X_i)\, dA_f(s) \\ &+ G(t_i, X_i)\, dA_g{}^+(s) + H(t_i, X_i)\, ds) \end{aligned}$$

for approximating QSDI (1.1) in the space $\tilde{\mathcal{A}}$.

Definition 2.5

For fixed $x \in Q$, $t \in [0, 1]$, we adopt the following notations and definitions as in Ref.[6].

(i) $\omega(P; t, x, h, \eta, \xi) = \sup\{\rho(P(s, x)(\eta, \xi), P(u, x)(\eta, \xi)); s, u \in [t - (h/2), t + (h/2)] \cap [0, 1]\}$,

where $\rho(\cdot\,\cdot)$ is the Hausdorff metric on $2^{\mathbb{C}}$.

(ii) Define the map

$$\Omega(P;\ \cdot\ , h, \eta, \xi) := \sup\{\omega(P;\ \cdot\ , x, h, \eta, \xi), x \in Q\}$$

Then, the map $t \to \Omega(P; t, h, \eta, \xi)$ is measurable, bounded, and therefore integrable on [0, 1].

(iii) The averaged modulus of continuity for the map P is defined by

$$\tau(P; h)(\eta, \xi) = \int_0^1 \Omega(P; t, h, \eta, \xi)\, dt$$

We remark here that Theorem 2.4 (i)–(v) hold for the map $P(t, x)(\eta, \xi)$ appearing in (1.2) provided that in (iv), $P(\ \cdot\ , x)(\eta, \xi)$ has bounded p-variation uniformly in $x \in Q \subseteq \mathcal{A}$, and in (v), $P(\ \cdot\ , x)(\eta, \xi)$ is Hausdorff continuous almost everywhere in [0, 1] uniformly in $x \in Q$.

3. ERROR ESTIMATES

This section is devoted to the establishment of error estimates involved in solving the discretized inclusion (2.1) in place of (1.2). In what follows, let $\eta, \xi \in \mathbb{D}\underline{\otimes}\mathbb{E}$ be arbitrary such that $\eta = c \otimes e(\alpha)$, $\xi = d \otimes e(\beta)$, $c, d \in \mathbb{D}$, $\alpha, \beta \in L_\gamma^2(\mathbb{R}_+)$. In solving the discretized inclusion (2.1), one may choose $X_{\eta\xi,i+1} := \langle \eta, X_{i+1}\xi \rangle$ arbitrarily from the right hand side of (2.1). However, we employ a definite approach which ensures that these solutions possess certain properties. To this end, we introduce the notion of Lipschitzian quantum stochastic processes.

Definition 3.1

A stochastic process $X : [0, 1] \to \tilde{\mathcal{A}}$ will be said to be Lipschitzian on [0, 1] if there exist constants $L_{\eta\xi} > 0$ such that

$$\|X(t_1) - X(t_2)\|_{\eta\xi} \leq L_{\eta\xi}|t_1 - t_2|, \forall t_1, t_2 \in [0, 1] \tag{3.1}$$

We remark that the set of such processes denoted by $L(\tilde{\mathcal{A}})$, is not empty.

The annihilation process $A : [0, 1] \to \tilde{\mathcal{A}}$ given by

$$A(t)\xi = \left(\int_0^t \beta(s)\, ds\right)\xi, \quad \xi = e(\beta),\ \beta \in L_{\mathbb{C},\mathrm{loc}}{}^\infty(\mathbb{R}_+)$$

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satisfies, for $t_1, t_2 \in [0, 1]$,

$$\begin{aligned}\|A(t_1) - A(t_2)\|_{\eta\xi} &= |\langle\eta, A(t_1)\xi\rangle - \langle\eta, A(t_2)\xi\rangle| \\ &\leq |\langle\eta, \xi\rangle| |\int_{t_2}^{t_1} \beta(s)\, ds| \\ &\leq M|\langle\eta, \xi\rangle| |t_1 - t_2| \\ &= L_{\eta\xi}|t_1 - t_2|\end{aligned}$$

where

$$M = \sup_{[0,1]} |\beta(s)|, \quad L_{\eta\xi} = M|\langle\eta, \xi\rangle|$$

Let $\tilde{X} \in L(\tilde{\mathcal{A}}) \cap Ad(\tilde{\mathcal{A}})_{\text{wac}}$. We construct the solution X^h of the discrete inclusion (2.1) as follows: $X_0{}^h = X^0$ and for $i = 0, 1, 2 \cdots N - 1$,

$$\begin{aligned}X_{\eta\xi,i+1}{}^h &:= \langle\eta, X_{i+1}{}^h\xi\rangle \\ &= \text{proj}(\langle\eta, \tilde{X}(t_{i+1})\xi\rangle, \langle\eta, X_i{}^h\xi\rangle + hP(t_i, X_i{}^h)(\eta, \xi))\end{aligned} \tag{3.2}$$

where proj(a, A) is the unique element of A closest to the element $a \in \mathbb{C}$. The existence and uniqueness of proj(a, A) are assured by the convexity and compactness of $A \subseteq \mathbb{C}$. Consequently, we have the following results.

Proposition 3.2

Assume that the map $(t, x) \to P(t, x)(\eta, \xi)$ appearing in (1.2) satisfies the conditions $\mathcal{S}_{(\text{i})} - \mathcal{S}_{(\text{iii})}$. Then there exists constant $C = C(\eta, \xi) > 0$ such that for all $N \geq 2$,

$$\max_{0\leq i\leq N} \|X_i^h - \tilde{X}(t_i)\|_{\eta\xi} \leq C\Bigg[\|\tilde{X}(0) - X^0\|_{\eta\xi} + \int_0^1 \mathbf{d}\left(\frac{d}{dt}\langle\eta, \tilde{X}(t)\xi\rangle, P(t, \tilde{X}(t))(\eta, \xi)\right) dt + \tau(P; h)(\eta, \xi) + h\Bigg]$$

Proof.

We first show that for any integrable sesquilinear form valued map $f : [a, b] \to \text{sesq}(\mathbb{D}\underline{\otimes}\mathbb{E})$ and any compact and convex set $A \subseteq \mathbb{C}$,

$$\mathbf{d}\left(\int_a^b f(t)(\eta, \xi)\, dt, (b - a)A\right) \leq \int_a^b \mathbf{d}(f(t)(\eta, \xi), A)\, dt \tag{3.3}$$

Since

$$\int_a^b \operatorname{proj}(A, f(t)(\eta, \xi))\, dt \;\in\; (b-a)A$$

then

$$\begin{aligned}\mathbf{d}\left(\int_a^b f(t)(\eta, \xi)\, dt, (b-a)A\right) &\le \left|\int_a^b f(t)(\eta, \xi)\, dt - \int_a^b \operatorname{proj}(A, f(t)(\eta, \xi))\right| \\ &\le \int_a^b \left|f(t)(\eta, \xi) - \operatorname{proj}(A, f(t)(\eta, \xi))\right| dt \\ &= \int_a^b \mathbf{d}(f(t)(\eta, \xi), A)\, dt\end{aligned}$$

Also, for any compact subsets A, B of $\mathbb{C}$, $y \in \mathbb{C}$, we shall employ the well known inequality:

$$\mathbf{d}(y, A) \le \mathbf{d}(y, B) + \rho(A, B) \tag{3.4}$$

Next, for $i = 1, 2 \ldots, N$, we employ the notation

$$C_{\eta\xi,i}{}^{h} := \mathbf{d}(\langle \eta, \tilde{X}(t_i)\xi\rangle, \langle \eta, \tilde{X}(t_{i-1})\xi\rangle + hP(t_{i-1}, \tilde{X}(t_{i-1}))(\eta, \xi))$$

Using (3.3) and (3.4), we obtain

$$\begin{aligned}\sum_{i=1}^{N} C_{\eta\xi,i}{}^{h} &= \sum_{i=1}^{N} \mathbf{d}(\langle \eta, \tilde{X}(t_i)\xi\rangle - \langle \eta, \tilde{X}(t_{i-1})\xi\rangle, hP(t_{i-1}, \tilde{X}(t_{i-1}))(\eta, \xi)) \\ &\le \sum_{i=1}^{N} \int_{t_i}^{t_{i+1}} \mathbf{d}\left(\frac{d}{dt}\langle \eta, \tilde{X}(t)\xi\rangle, P(t_{i-1}, \tilde{X}(t_{i-1}))(\eta, \xi)\right) dt \\ &\le \sum_{i=1}^{N} \int_{t_i}^{t_{i+1}} \left[\mathbf{d}\left(\frac{d}{dt}\langle \eta, \tilde{X}(t)\xi\rangle, P(t, \tilde{X}(t))(\eta, \xi)\right)\right. \\ &\quad \left. + \rho(P(t, \tilde{X}(t))(\eta, \xi), P(t_{i-1}, \tilde{X}(t_{i-1}))(\eta, \xi))\right] dt \qquad (3.5) \\ &\quad + \sum_{i=1}^{N} \int_{t_i}^{t_{i+1}} \rho(P(t, \tilde{X}(t))(\eta, \xi), P(t_{i-1}, \tilde{X}(t_{i-1}))(\eta, \xi))\, dt \qquad (3.6)\end{aligned}$$

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The last term in (3.6) satisfies

$$\sum_{i=1}^{N}\int_{t_i}^{t_{i+1}} \rho(P(t,\tilde{X}(t))(\eta,\xi), P(t_{i-1},\tilde{X}(t_{i-1}))(\eta,\xi))\,dt$$
$$\leq \sum_{i=1}^{N}\int_{t_i}^{t_{i+1}} \rho(P(t,\tilde{X}(t))(\eta,\xi), P(t_i,\tilde{X}(t_i))(\eta,\xi))\,dt$$
$$+\sum_{i=1}^{N}\int_{t_i}^{t_{i+1}} \rho(P(t_i,\tilde{X}(t_i))(\eta,\xi), P(t_{i-1},\tilde{X}(t_{i-1}))(\eta,\xi))\,dt$$
$$\leq \sum_{i=1}^{N}\int_{t_i}^{t_{i+1}} \rho(P(t,\tilde{X}(t))(\eta,\xi), P(t_i,\tilde{X}(t_i))(\eta,\xi))\,dt$$
$$+\sum_{i=1}^{N}\int_{t_i}^{t_{i+1}} K_{\eta\xi}\|\tilde{X}(t_i)-\tilde{X}(t_{i-1})\|_{\eta\xi}\,dt \leq \tau(P;h)(\eta,\xi)$$
$$+K_{\eta\xi}L_{\eta\xi}h$$

Consequently, from (3.6),

$$\sum_{i=1}^{N} C_{\eta\xi,i}{}^{h} \leq \int_0^1 \mathbf{d}\left(\frac{d}{dt}\langle\eta,\tilde{X}(t)\xi\rangle, P(t,\tilde{X}(t))(\eta,\xi)\right)dt$$
$$+K_{\eta\xi}L_{\eta\xi}h + \tau(P;h)(\eta,\xi) \tag{3.7}$$

where $K_{\eta\xi}$, $L_{\eta\xi}$ are the Lipschitz constants for the map $P(t,\cdot)(\eta,\xi)$ and the process $\tilde{X}$ respectively.

Next, we define $D_{\eta\xi,i}{}^{h}$ by

$$D_{\eta\xi,i}{}^{h} := \mathbf{d}(\langle\eta,\tilde{X}(t_i)\xi\rangle, \langle\eta,X_{i-1}{}^{h}\xi\rangle + hP(t_{i-1},X_{i-1}{}^{h})(\eta,\xi)),$$
$$i=1,2\ldots,N$$

Again, by inequality (3.4),

$$D_{\eta\xi,i}{}^{h} \leq \mathbf{d}(\langle\eta,\tilde{X}(t_i)\xi\rangle, \langle\eta,\tilde{X}(t_{i-1})\xi\rangle + hP(t_{i-1},\tilde{X}(t_{i-1}))(\eta,\xi))$$
$$+\rho(\langle\eta,X_{i-1}{}^{h}\xi\rangle + hP(t_{i-1},X_{i-1}{}^{h})(\eta,\xi), \langle\eta,\tilde{X}(t_{i-1})\xi\rangle$$
$$+hP(t_{i-1},\tilde{X}(t_{i-1}))(\eta,\xi))$$

$$\leq C_{\eta\xi,i}{}^h + \|\tilde{X}(t_{i-1}) - X_{i-1}{}^h\|_{\eta\xi} + h\rho(P(t_{i-1}, X_{i-1}{}^h)(\eta, \xi),$$
$$P(t_{i-1}, \tilde{X}(t_{i-1}))(\eta, \xi))$$
$$\leq C_{\eta\xi,i}{}^h + (1 + hK_{\eta\xi})\|\tilde{X}(t_{i-1}) - X_{i-1}{}^h\|_{\eta\xi}$$

But by (3.2), the construction of solution $X_{\eta\xi,i}{}^h$ of the discrete inclusion (2.1) satisfies

$$D_{\eta\xi,i}{}^h = |\langle \eta, \tilde{X}(t_i)\xi\rangle - X_{\eta\xi,i}{}^h| = \|\tilde{X}(t_i) - X_i{}^h\|_{\eta\xi}$$

Therefore, setting

$$D^h_{\eta\xi,0} = \|\tilde{X}(0) - X^0\|_{\eta\xi}$$

we have

$$D_{\eta\xi,i}{}^h \leq C_{\eta\xi,i}{}^h + (1 + hK_{\eta\xi})D_{\eta\xi,i-1}{}^h, \quad i = 1, 2\ldots, N$$

By the discrete version of the well known Gronwall inequality (see Dontchev and Farhki,[6] for example) we get

$$\max_{0\leq i\leq N} D_{\eta\xi,i}{}^h \leq e^{K_{\eta\xi}}\left(\sum_{i=1}^{N} C_{\eta\xi,i}{}^h + D_{\eta\xi,0}{}^h\right)$$

Hence, by (3.7), the conclusion of Proposition (3.2) follows from the last inequality, where the constant $C = \exp(K_{\eta\xi})$.

Remark 3.3

If the quantum stochastic process $\tilde{X}$ is a solution of (1.2), then

$$\|\tilde{X}(0) - X^0\|_{\eta\xi} = 0, \quad \mathbf{d}\left(\frac{d}{dt}\langle \eta, \tilde{X}(t)\xi\rangle, P(t, \tilde{X}(t))(\eta, \xi)\right) = 0$$

for almost all $t \in [0, 1]$. Furthermore, by Theorem 2.3 (iv),

$$\tau(P; h)(\eta, \xi) \leq hW(P; \eta, \xi)$$

where $W(P; \eta, \xi)$ is the variation of the map P introduced in Definition 2.2. In particular, if $P(\cdot, x)(\eta, \xi)$ has a bounded variation uniformly in $x \in Q$, then $\tau(P; h)(\eta, \xi) = O(h)$. Consequently, Proposition 3.2 yields

$$\max_{0\leq i\leq N} \|\tilde{X}(t_i) - X_i{}^h\|_{\eta\xi} \leq e^{K_{\eta\xi}}[1 + W(P; \eta, \xi)]h$$

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The main result of this paper depends partially on the next lemma. To this end, we introduce the following notation.

$$R_{\eta\xi} := \sup_N \{|P(t_i, X_i^N)(\eta, \xi)| : X_{\eta\xi}{}^N = \{X_{\eta\xi,i}{}^N\} \text{ solves (2.1)},$$
$$t_i \in [0, 1]\} \tag{3.8}$$

Lemma 3.4
Let $R_{\eta\xi}$ satisfy (3.8). Then for every solution

$$X_{\eta\xi}{}^N := \{X_{\eta\xi,i}{}^N = \langle \eta, X_i^N \xi \rangle,\ i = 0, 1, 2 \ldots, N\}$$

of (2.1), there exists a solution $\Phi(\cdot)$ of (1.1) such that

$$\max_{0 \le i \le N} \|\Phi(t_i) - X_i^N\|_{\eta\xi} \le \exp(K_{\eta\xi})\big(R_{\eta\xi} K_{\eta\xi} h + \tau(P; h)(\eta, \xi)\big) \tag{3.9}$$

Proof.
Let a stochastic process $Y^N : [0, 1] \to \tilde{\mathcal{A}}$ be defined as follows:

$$Y^N(t) = X_i^N + \frac{1}{h}(t - t_i)(X_{i+1}{}^N - X_i^N),\ t_i \le t \le t_{i+1},$$
$$i = 0, 1, 2 \ldots, N - 1$$

The associated piecewise linear matrix element $Y_{\eta\xi}{}^N(t) := \langle \eta, Y^N(t)\xi \rangle$ is given by

$$Y_{\eta\xi}{}^N(t) = X_{\eta\xi,i}{}^N + \frac{1}{h}(t - t_i)(X_{\eta\xi,i+1}{}^N - X_{\eta\xi,i}{}^N) \tag{3.10}$$

where $\{X_{\eta\xi,i}{}^N\}$ is a set of grid solutions of (2.1). Clearly, $Y^N \in Ad(\tilde{\mathcal{A}})_{\text{wac}}$.
By construction we have

$$\frac{d}{dt} Y_{\eta\xi}{}^N(t) = \frac{1}{h}(X_{\eta\xi,i+1}{}^N - X_{\eta\xi,i}{}^N) \in P(t_i, X_i^N)(\eta, \xi) \tag{3.11}$$

Next, we estimate

$$\mathbf{d}\left(\frac{d}{dt} Y_{\eta\xi}{}^N(t), P(t, Y^N(t))(\eta, \xi)\right)$$

By inequality (3.4),

$$\begin{aligned}
&\mathbf{d}\left(\frac{d}{dt}Y_{\eta\xi}{}^{N}(t), P(t, Y^{N}(t))(\eta, \xi)\right)\\
&\quad\le \mathbf{d}\left(\frac{d}{dt}Y_{\eta\xi}{}^{N}(t), P(t, X_i^{N})(\eta, \xi)\right)\\
&\qquad + \rho(P(t, Y^{N}(t))(\eta, \xi), P(t, X_i^{N})(\eta, \xi))\\
&\quad\le \mathbf{d}\left(\frac{d}{dt}Y_{\eta\xi}{}^{N}(t), P(t, X_i^{N})(\eta, \xi)\right)\\
&\qquad + K_{\eta\xi}|Y_{\eta\xi}{}^{N}(t) - X_{\eta\xi,i}{}^{N}|\\
&\quad\le \mathbf{d}\left(\frac{d}{dt}Y_{\eta\xi}{}^{N}(t), P(t, X_i^{N})(\eta, \xi)\right) + K_{\eta\xi}R_{\eta\xi}h \qquad (3.12)
\end{aligned}$$

The last inequality follows from (3.8) and (3.10). Again, by employing (3.11) and (3.4), we have

$$\begin{aligned}
&\mathbf{d}\left(\frac{d}{dt}Y_{\eta\xi}{}^{N}(t), P(t, X_i^{N})(\eta, \xi)\right)\\
&\quad\le \rho(P(t_i, X_i^{N})(\eta, \xi), P(t, X_i^{N})(\eta, \xi))\\
&\quad\le \omega(P; X_i^{N}, t, 2h, \eta, \xi)\\
&\quad\le \sup_{x\in Q}\omega(P; x, t, 2h, \eta, \xi) = \Omega(P; t, 2h, \eta, \xi)
\end{aligned}$$

Applying Theorem 2.1 by setting

$$W_{\eta\xi}(t) = K_{\eta\xi}R_{\eta\xi}h + \Omega(P; t, 2h, \eta, \xi)$$

we conclude that there exists a solution $\Phi(\cdot)$ of (1.1) satisfying inequality (3.9).

Next, we present our main result. To this end, we introduce the following sets of vectors in the space $\mathbb{C}^{N+1}$.

Denote by $S(X^0)$ the subset of $Ad(\tilde{\mathcal{A}})_{\text{wac}}$ consisting of the trajectory bundle of (1.1). Then, we define the following sets.

$$R^{N}(X^0)(\eta, \xi) := \{\Phi_{\eta\xi}{}^{N} = (\Phi_{\eta\xi}(t_i), i = 0, 1, 2 \ldots, N),\ \text{such that } \Phi(\cdot) \in S(X^0)\}$$

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$$D^N(X^0)(\eta,\xi) := \{X_{\eta\xi}{}^N = (X_{\eta\xi,i}, i = 0, 1, 2\ldots, N), \text{ such that } X_{\eta\xi}{}^N \text{ is a solution of the discretized inclusion (2.1)}\}$$

For $U, V \in \mathbb{C}^{N+1}$, we employ the maximum norm

$$|U - V| := \max_{0 \leq i \leq N} |U_i - V_i|$$

and the associated Hausdorff metric ρ on $2^{\mathbb{C}^{N+1}}$.

Consequently, the following theorem has been established by the combination of Proposition 3.2 and Lemma 3.4.

Theorem 3.5

Assume that the QSDI (1.2) satisfies conditions $\mathcal{S}_{(\mathrm{i})} - \mathcal{S}_{(\mathrm{iii})}$. Then there exist positive constants depending only on η, ξ such that

$$\rho(R^N(X^0)(\eta, \xi), D^N(X^0)(\eta, \xi)) \leq C[\tau(P; h)(\eta, \xi) + h] \tag{3.13}$$

for all $N \geq 2$.

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REFERENCES

1. Aubin, J.P.; Cellina, A. *Differential Inclusions*; Springer-Verlag, 1984.
2. Aubin, J.P.; Frankowska, H. *Set-Valued Analysis*; Birkhauser, 1990.
3. Ayoola, E.O. Exponential formula for the reachable sets of quantum stochastic differential inclusions. Stoch. Anal. Appl., **2003**, *21* (3), 515–543.
4. Ayoola, E.O. Construction of approximate attainability sets for Lipschitzian quantum stochastic differential inclusions. Stoch. Anal. Appl. **2001**, *19* (3), 461–471.

5. Ayoola, E.O. On convergence of one-step schemes for weak solutions of quantum stochastic differential equations. Acta. Appl. Math. **2001**, *67* (1), 19–58.
6. Dontchev, A.L.; Farkhi, E.M. Error estimates for discretized differential inclusions. Computing **1989**, *41*, 349–358.
7. Dontchev, A.; Lempio, F. Difference methods for differential inclusions: a survey. SIAM Rev. **1992**, *34* (2), 263–294.
8. Ekhaguere, G.O.S. Lipschitzian quantum stochastic differential inclusions. Internat. J. Theoret. Phys. **1992**, *31* (11), 2003–2034.
9. Ekhaguere, G.O.S. Quantum stochastic differential inclusions of hypermaximal monotone type. Internat. J. Theoret. Phys. **1995**, *34* (3), 323–353.
10. Ekhaguere, G.O.S. Quantum stochastic evolutions. Internat. J. Theoret. Phys. **1996**, *35* (9), 1909–1946.
11. Hudson, R.L.; Parthasarathy, K.R. Quantum Ito's formulae and stochastic evolutions. Commun. Math. Phys. **1984**, *93*, 301–324.
12. Komarov, V.A.; Pevchikh, K.E. A method of approximating attainability sets for differential inclusions with a specified accuracy. Comput. Maths. Math. Phys. **1991**, *31* (1), 109–112.
13. Veliov, V. Second order discrete approximation to linear differential inclusions. SIAM J. Num. Anal. **1992**, *29*, 439–451.
14. Wolenski, P.R. A uniqueness theorem for differential inclusions. J. Differ. Eq. **1990**, *84*, 165–182.
15. Wolenski, P.R. The exponential formula for the reachable set of a Lipschitz differential inclusion. SIAM J. Control Optim. **1990**, *28* (5), 1148–1161.

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CARATHEODORY SOLUTION OF QUANTUM STOCHASTIC DIFFERENTIAL INCLUSIONS

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ABSTRACT. This work is concerned with the existence of solution of Quantum stochastic differential inclusions in the sense of Caratheodory. The multivalued stochastic process involved which is non-convex is Scorza-Dragoni lower semicontinuous (SD-l.s.c.) hence giving rise to a directionally continuous selection. The Quantum stochastic differential inclusion is driven by annihilation, creation and gauge operators.

Keywords and phrases: Lower semicontinuous multifunctions, Scorza Dragoni's property, quantum stochastic processes.
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1. INTRODUCTION

The vast applications of differential inclusions in control theory, economic model, evolution inclusions to mention a few, had made the study of differential inclusions of great interest [1], [8], [18]. Likewise, the quantum stochastic differential inclusions which is a multivalued generalization of quantum stochastic differential equation of Hudson and Parthasarathy has vast applications. This extension was first done in [9] in which the existence of solutions of Lipschitzian quantum stochastic differential inclusions was established. The study of solution set of this problem was done in [2], [3] and references cited there. The case of discontinuous quantum stochastic differential inclusions has application in the study of optimal quantum stochastic control [15]. The quantum stochastic calculus is driven by quantum stochastic processes called annihilation, creation and gauge arising from quantum field operators.
A multivalued map that is lower semicontinuous and convex-valued has continuous selection by Michael selection theorem, but if the convexity is dropped the continuous selection does not exist. But for a differential inclusion with lower semicontinuous multifunction that is not convex-valued, there is an analogue of Michael selection

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theorem called the directionally continuous selection [4] which gave rise to a class of discontinuous differential equations. A more general case of this selection for infinite dimensional space is found in [5].

The quantum stochastic differential inclusions considered in this work has its coefficients to be multivalued stochastic processes that have a special form of lower semicontinuity called Scorza-Dragoni lower semicontinuous case. It is noteworthy that the Scorza-Dragoni property is a multivalued generalization of Lusin property[14]. The directionally continuous selection of the Scorza-Dragoni of the multifunction gave rise to a class of quantum stochastic differential equations considered in [16] which have solutions in the sense of Caratheodory. Apart from the application of this work in quantum stochastic control, another motivation for the work is the application of the results in the study of non-convex quantum stochastic evolution inclusions which shall be considered in a later work.

In section 2 we give preliminaries which are essential for the work and we prove the main results in section 3.

2. PRELIMINARY

In what follows, if U is a topological space, we denote by $\mathrm{clos}(U)$, the collection of all non-empty closed subsets of U.

To each pair (D, H) consisting of a pre-Hilbert space D and its completion H, we associate the set $L_w^+(D, H)$ of all linear maps x from D into H, with the property that the domain of the operator adjoint contains D. The members of $L_w^+(D, H)$ are densely-defined linear operators on H which do not necessarily leave D invariant and $L_w^+(D, H)$ is a linear space when equipped with the usual notions of addition and scalar multiplication.

To H corresponds a Hilbert space $\Gamma(H)$ called the boson Fock space determined by H. A natural dense subset of $\Gamma(H)$ consists of linear space generated by the set of exponential vectors(Guichardet, [12]) in $\Gamma(H)$ of the form

$$e(f) = \bigoplus_{n=0}^{\infty} (n!)^{-\frac{1}{2}} \bigotimes^{n} f, \ \ f \in H$$

where $\bigotimes^0 f = 1$ and $\bigotimes^n f$ is the n-fold tensor product of f with itself for $n \geq 1$.

In what follows, $\mathbb{D}$ is some pre-Hilbert space whose completion is $\mathcal{R}$ and γ is a fixed Hilbert.

$L^2_\gamma(\mathbb{R}_+)$(resp. $L^2_\gamma([0,t))$, resp. $L^2_\gamma([t,\infty))$ $t \in \mathbb{R}_+$) is the space of

square integrable γ-valued maps on $\mathbb{R}_+$ (resp.$[0,t)$, resp.$[t,\infty)$).
The inner product of the Hilbert space $\mathcal{R}\otimes\Gamma(L^2_\gamma(\mathbb{R}_+))$ will be denoted by $\langle .,.\rangle$ and $\parallel . \parallel$ the norm induced by $\langle .,.\rangle$.
Let $\mathbb{E},\mathbb{E}_t$ and $\mathbb{E}^t$, $t>0$ be linear spaces generated by the exponential vectors in Fock spaces $\Gamma(L^2_\gamma(\mathbb{R}_+)),\Gamma(L^2_\gamma([0,t)))$ and $\Gamma(L^2_\gamma([t,\infty)))$ respectively ;

$$\mathcal{A}\equiv L^+_w(\mathbb{D}\underline{\otimes}\mathbb{E},\mathcal{R}\otimes\Gamma(L^2_\gamma(\mathbb{R}_+)))$$
$$\mathcal{A}_t\equiv L^+_w(\mathbb{D}\underline{\otimes}\mathbb{E}_t,\mathcal{R}\otimes\Gamma(L^2_\gamma([0,t))))\otimes\mathbb{I}^t$$
$$\mathcal{A}^t\equiv \mathbb{I}_t\otimes L^+_w(\mathbb{E}^t,\Gamma(L^2_\gamma([t,\infty)))),\ t>0$$

where $\underline{\otimes}$ denotes algebraic tensor product and $\mathbb{I}_t$(resp.$\mathbb{I}^t$) denotes the identity map on $\mathcal{R}\otimes\Gamma(L^2_\gamma([0,t))))$(resp.$\Gamma(L^2_\gamma([t,\infty))))$, $t>0$
For every $\eta,\xi\in\mathbb{D}\underline{\otimes}\mathbb{E}$ define

$$\parallel x \parallel_{\eta,\xi}=\mid\langle\eta,x\xi\rangle\mid,\ x\in\mathcal{A}$$

then the family of seminorms

$$\{\parallel . \parallel_{\eta\xi}: \eta,\xi\in\mathbb{D}\underline{\otimes}\mathbb{E}\}$$

generates a topology τ_w , weak topology .
The completion of the locally convex spaces $(\mathcal{A},\tau_w)$,$(\mathcal{A}_t,\tau_w)$ and $(\mathcal{A}^t,\tau_w)$ are respectively denoted by $\widetilde{\mathcal{A}}$,$\widetilde{\mathcal{A}}_t$ and $\widetilde{\mathcal{A}}^t$.
We define the Hausdorff topology on clos$(\widetilde{\mathcal{A}})$ as follows:
For $x\in\widetilde{\mathcal{A}}$, $\mathcal{M},\mathcal{N}\in$ clos$(\widetilde{\mathcal{A}})$ and $\eta,\xi\in\mathbb{D}\underline{\otimes}\mathbb{E}$, define

$$\rho_{\eta\xi}(\mathcal{M},\mathcal{N})\equiv\max(\delta_{\eta\xi}(\mathcal{M},\mathcal{N}),\delta_{\eta\xi}(\mathcal{N},\mathcal{M}))$$

where

$$\delta_{\eta\xi}(\mathcal{M},\mathcal{N})\equiv\sup_{x\in\mathcal{M}}\mathbf{d}_{\eta\xi}(x,\mathcal{N})\text{ and}$$
$$\mathbf{d}_{\eta\xi}(x,\mathcal{N})\equiv\inf_{y\in\mathcal{N}}\parallel x-y\parallel_{\eta\xi}.$$

The Hausdorff topology which shall be employed in what follows, denoted by, τ_H , is generated by the family of pseudometrics $\{\rho_{\eta\xi}(.):\eta,\xi\in\mathbb{D}\underline{\otimes}\mathbb{E}\}$
Moreover, if $\mathcal{M}\in clos(\widetilde{\mathcal{A}})$, then $\parallel\mathcal{M}\parallel_{\eta\xi}$ is defined by

$$\parallel\mathcal{M}\parallel_{\eta\xi}\equiv\rho_{\eta\xi}(\mathcal{M},\{0\});$$

for arbitrary $\eta, \xi \in \mathbb{D}\underline{\otimes}\mathbb{E}$.
For $A, B \in clos(\mathbb{C})$ and $x \in \mathbb{C}$, a complex number, define

$$d(x, B) \equiv \inf_{y \in B} \mid x - y \mid$$
$$\delta(A, B) \equiv \sup_{x \in A} d(x, B)$$
$$\text{and } \rho(A, B) \equiv \max(\delta(A, B), \delta(B, A)).$$

Then ρ is a metric on $clos(\mathbb{C})$ and induces a metric topology on the space.
Let $I \subseteq \mathbb{R}_+$. A *stochastic process* indexed by I is an $\widetilde{\mathcal{A}}$-valued measurable map on I.
A stochastic process X is called *adapted* if $X(t) \in \widetilde{\mathcal{A}}_t$ for each $t \in I$.
We write $\text{Ad}(\widetilde{\mathcal{A}})$ for the set of all adapted stochastic processes indexed by I.
Definition 1: A member X of $\text{Ad}(\widetilde{\mathcal{A}})$ is called
(i) weakly absolutely continuous if the map $t \mapsto \langle \eta, X(t)\xi \rangle$, $t \in I$ is absolutely continuous for arbitrary $\eta, \xi \in \mathbb{D}\underline{\otimes}\mathbb{E}$
(ii) locally absolutely p-integrable if $\| X(.) \|_{\eta\xi}^{p}$ is Lebesgue - measurable and integrable on $[0, t) \subseteq I$ for each $t \in I$ and arbitrary $\eta, \xi \in \mathbb{D}\underline{\otimes}\mathbb{E}$.
We denote by $\text{Ad}(\widetilde{\mathcal{A}})_{wac}$(resp.$L^p_{loc}(\widetilde{\mathcal{A}})$) the set of all weakly, absolutely continuous(resp. locally absolutely p-integrable) members of $\text{Ad}(\widetilde{\mathcal{A}})$.
Stochastic integrators: Let $L^\infty_{\gamma,loc}(\mathbb{R}_+)$ [resp.$L^\infty_{B(\gamma),loc}(\mathbb{R}_+)$] be the linear space of all measurable , locally bounded functions from $\mathbb{R}_+$ to γ [resp. to B(γ) , the Banach space of bounded endomorphisms of γ]. If $f \in L^\infty_{\gamma,loc}(\mathbb{R}_+)$ and $\pi \in L^\infty_{B(\gamma),loc}(\mathbb{R}_+)$, then πf is the member of $L^\infty_{\gamma,loc}(\mathbb{R}_+)$ given by $(\pi f)(t) = \pi(t) f(t)$, $t \in \mathbb{R}_+$.
For $f \in L^2_\gamma(\mathbb{R})_+$ and $\pi \in L^\infty_{B(\gamma),loc}(\mathbb{R}_+)$; the annihilation , creation and gauge operators, $a(f), a^+(f)$ and $\lambda(\pi)$ in $L^+_w(\mathbb{D}, \Gamma(L^2_\gamma(\mathbb{R})_+))$ respectively, are defined as:

$$a(f)\mathbf{e}(g) = \langle f, g \rangle_{L^2_\gamma(\mathbb{R}_+)} \mathbf{e}(g)$$
$$a^+(f)\mathbf{e}(g) = \frac{d}{d\sigma}\mathbf{e}(g + \sigma f) \mid_{\sigma=0}$$
$$\lambda(\pi)\mathbf{e}(g) = \frac{d}{d\sigma}\mathbf{e}(e^{\sigma\pi} f) \mid_{\sigma=0}$$

$g \in L^2_\gamma(\mathbb{R})_+$
For arbitrary $f \in L^\infty_{\gamma,loc}(\mathbb{R}_+)$ and $\pi \in L^\infty_{B(\gamma),loc}(\mathbb{R}_+)$, they give rise

to the operator-valued maps A_f, A_f^+ and Λ_π defined by:

$$A_f(t) \equiv a(f\chi_{[0,t)})$$
$$A_f^+(t) \equiv a^+(f\chi_{[0,t)})$$
$$\Lambda_\pi(t) \equiv \lambda(\pi\chi_{[0,t)})$$

$t \in \mathbb{R}_+$, where χ_I denotes the indicator function of the Borel set $I \subseteq \mathbb{R}_+$. The maps A_f, A_f^+ and Λ_π are stochastic processes , called annihilation, creation and gauge processes , respectively, when their values are identified with their amplifications on $\mathcal{R}\otimes\Gamma(L^2_\gamma(\mathbb{R}_+))$. These are the stochastic integrators in Hudson and Parthasarathy[13] formulation of boson quantum stochastic integration.
For processes $p, q, u, v \in L^2_{loc}(\widetilde{\mathcal{A}})$, the quantum stochastic integral:

$$\int_{t_0}^{t} \big(p(s)d\Lambda_\pi(s) + q(s)dA_f(s) + u(s)dA_g^+(s) + v(s)ds\big),\ t_0, t \in \mathbb{R}_+$$

is interpreted in the sense of Hudson-Parthasarathy[13] The definition of Quantum stochastic differential Inclusions follows as in [9]. A relation of the form

$$\begin{aligned} dX(t) &\in E(t, X(t))d\Lambda_\pi(t) + F(t, X(t))dA_f(t) \\ &\quad + G(t, X(t))dA_g^+(t) + H(t, X(t))dt \text{ almost all } t \in I \qquad (1) \\ X(t_0) &= x_0 \end{aligned}$$

is called Quantum stochastic differential inclusions(QSDI) with coefficients E, F, G, H and initial data (t_0, x_0).
Equation(1) is understood in the integral form:

$$\begin{aligned} X(t) \in x_0 + \int_{t_0}^{t} &\big(E(s, X(s))d\Lambda_\pi(s) + F(s, X(s))dA_f(s) \\ &+G(s, X(s))dA_g^+(s) + H(s, X(s))ds\big),\ t \in I \end{aligned}$$

called a stochastic integral inclusion with coefficients E, F, G, H and initial data (t_0, x_0)
An equivalent form of (1) has been established in [9], Theorem 6.2

as :

$$\begin{aligned}(\mu E)(t,x)(\eta,\xi) &= \{\langle \eta, \mu_{\alpha\beta}(t)p(t,x)\xi\rangle : p(t,x) \in E(t,x)\}\\ (\nu F)(t,x)(\eta,\xi) &= \{\langle \eta, \nu_{\beta}(t)q(t,x)\xi\rangle : q(t,x) \in F(t,x)\}\\ (\sigma G)(t,x)(\eta,\xi) &= \{\langle \eta, \sigma_{\alpha}(t)u(t,x)\xi\rangle : u(t,x) \in G(t,x)\}\\ \mathbb{P}(t,x)(\eta,\xi) &= (\mu E)(t,x)(\eta,\xi) + (\nu F)(t,x)(\eta,\xi)\\ &\quad + (\sigma G)(t,x)(\eta,\xi) + H(t,x)(\eta,\xi)\\ H(t,x)(\eta,\xi) &= \{v(t,x)(\eta,\xi) : v(.,X(.))\\ &\quad \text{is a selection of } H(.,X(.)) \forall\ X \in L^2_{loc}(\widetilde{\mathcal{A}})\}\end{aligned} \tag{2}$$

Then Problem (1) is equivalent to

$$\begin{aligned}\frac{d}{dt}\langle \eta, X(t)\xi\rangle &\in \mathbb{P}(t,X(t))(\eta,\xi)\\ X(t_0) &= x_0\end{aligned} \tag{3}$$

for arbitrary $\eta,\xi \in \mathbb{D}\underline{\otimes}\mathbb{E}$, almost all $t \in I$. Hence the existence of solution of (1) implies the existence of solution of (3) and vice-versa.

As explained in [9], for the map $\mathbb{P}$,

$$\mathbb{P}(t,x)(\eta,\xi) \neq \widetilde{\mathbb{P}}(t,\langle \eta, x\xi\rangle)$$

for some complex-valued multifunction $\widetilde{\mathbb{P}}$ defined on $I\times\mathbb{C}$ for $t \in I$, $x \in \widetilde{\mathcal{A}}$, $\eta,\xi \in \mathbb{D}\underline{\otimes}\mathbb{E}$.

Definition 2: For an arbitrary $\eta,\xi \in \mathbb{D}\underline{\otimes}\mathbb{E}$, let $M > 0$, we define a set $\Gamma^M_{\eta\xi}$, as

$$\Gamma^M_{\eta\xi} = \{(t,x) \in I \times \widetilde{\mathcal{A}} :\mid \langle \eta, x\xi\rangle \mid\leq Mt\}$$

Let $(t_0,x_0) \in I\times\widetilde{\mathcal{A}}$ and $\epsilon > 0$. For an arbitrary $\eta,\xi \in \mathbb{D}\underline{\otimes}\mathbb{E}$, $(t_0,x_0) \in I \times \widetilde{\mathcal{A}}$ and $\delta > 0$, the family of conical neighbourhoods;

$$\Gamma^M_{\eta\xi}((t_0,x_0),\delta) = \{(t,x) \in I \times \widetilde{\mathcal{A}} :\| x - x_0 \|_{\eta\xi}\leq M(t-t_0),$$
$$t_0 \leq t < t_0 + \delta\}$$

generates a topology, τ^+, which satisfies the following property:
(P) For every pair of sets $A \subset B$, with A closed and B open(in the original topology), there exists a set C, closed-open with respect to τ^+, such that $A \subset C \subset B$.

This topology follows from [5] and the references cited there.

Definition 3: (i) For an arbitrary pair $\eta,\xi \in \mathbb{D}\underline{\otimes}\mathbb{E}$ a map $\Phi : I \times \widetilde{\mathcal{A}} \to \widetilde{\mathcal{A}}$ will be said to be $\Gamma^M_{\eta\xi}$-continuous(directionally continuous or τ^{+}-continuous) at a point $(t_0, x_0) \in I \times \widetilde{\mathcal{A}}$, if for every $\epsilon > 0$ there exists $\delta > 0$ such that

$$\| \Phi(t,x) - \Phi(t_0,x_0) \|_{\eta\xi} \leq \epsilon \text{ if } t_0 \leq t \leq t_0 + \delta \text{ and } \| x - x_0 \|_{\eta\xi} \leq M(t - t_0)$$

(ii) For an arbitrary $\eta, \xi \in \mathbb{D}\underline{\otimes}\mathbb{E}$, $S \subset \widetilde{\mathcal{A}}$, a sesquilinear-form valued map $\Psi : S \to 2^{sesq(\mathbb{D}\underline{\otimes}\mathbb{E})^2}$ will be said to be lower semicontinuous on S if for every closed subset C of $\mathbb{C}$ the set $\{s \in S : \Psi(s)(\eta, \xi) \subset C\}$ is closed in S.

We remark that if E, F, G, H are lower semicontinuous on S, then the sesquilinear-form valued $\mathbb{P}$ is lower semicontinuous on S.

A multivalued generalization of Lusin property which is called Scorza - Dragoni property [14] employed in [6] is used to define the form of lower semicontinuity in this work. The well-known Lusin property is the following.

Definition 4:(Lusin's property)Let X and Y be two separable metric spaces and let $f : I \times X \to Y$ be function such that

(i)$t \to f(t,u)$ is measurable for every $u \in X$

(ii) $u \to f(t,u)$ is continuous for almost every $t \in I$, $I \subseteq \mathbb{R}_+$.

Then, for each $\epsilon > 0$, there exists a closed set $A \subseteq I$ such that $\lambda(I \setminus A) < \epsilon$, (λ is the Lebesgue measure on $\mathbb{R}$) and the restriction of f to $A \times X$ is continuous.

Definition 5: A sequilinear- form valued map $\Psi : [0,T] \times \widetilde{\mathcal{A}} \to 2^{sesq(\mathbb{D}\underline{\otimes}\mathbb{E})^2}$ is Scorza-Dragoni lower semicontinuous (SD-l.s.c.) on $[0,T] \times \widetilde{\mathcal{A}}$ if there exists a sequence of disjoint compact sets $J_n \subset [0,T]$, with meas$([0,T] \setminus \bigcup_{n \in \mathbb{N} J_n}) = 0$ such that Ψ is lower semicontinuous on each set $J_n \times \widetilde{\mathcal{A}}$.

If Ψ is lower semicontinuous and convex-valued then by Michael selection theorems, there exists continuous selection of Ψ. But if the convexity is removed and Ψ is not decomposable valued multifunction then the existence of continuous selection is not guaranteed. However, a non-convex analogue of Michael selection is Directional continuous selection result in [4] and for infinite dimensional space in [5]. We established in this work that such selection exists for SD-lsc multivalued stochastic process.

For an arbitrary $\eta, \xi \in \mathbb{D}\underline{\otimes}\mathbb{E}$, if $\Psi \in \mu E, \nu F, \sigma G, H$ appearing in (1) are SD-lsc then the map $(t,x) \to \mathbb{P}(t,x)(\eta,\xi)$ is SD-lsc.

A quantum stochastic differential inclusion will be said to be SD-lower semicontinuous if the coefficients are SD-lsc.

3. MAIN RESULTS

Theorem 1: For almost all $t \in I$, $\eta, \xi \in \mathbb{D}\underline{\otimes}\mathbb{E}$. Suppose the following holds:
(i) The maps $X \to \Psi(t,X)(\eta,\xi)$, $\Psi \in \{\mu E, \nu F, \sigma G, H\}$ are non-empty lower semicontinuous multivalued stochastic processes
(ii)The maps $t \to \Psi(t,X)(\eta,\xi)$ are closed
(iii) τ^+ is a topology on $I \times \widetilde{\mathcal{A}}$ with property (P).
Then the sesquilinear form valued multifunction, $(t, X(t)) \to \mathbb{P}(t, X(t))(\eta,\xi)$

$$\mathbb{P}(t,X(t))(\eta,\xi) = (\mu E)(t,X(t))(\eta,\xi) + (\nu F)(t,X(t))(\eta,\xi) + (\sigma G)(t,X(t))(\eta,\xi) + H(t,X(t))(\eta,\xi)$$

admits a τ^+-continuous selection.
Proof: $\mathbb{P}$ is non-empty , since each of $\Psi \in \{\mu E, \nu F, \sigma G, H\}$ is non-empty.
Therefore, $\mathbb{P}$ is a non-empty lower semicontinuous sesquilinear form-valued multifunction.
We shall employ a similar procedure as in the proof of Theorem 3.2 in [5] to construct a τ^+-continuous ϵ-approximate selections P_ϵ of $\mathbb{P}$, hence by inductive hypothesis we obtain a τ^+-continuous selection P of $\mathbb{P}$.
Let $\epsilon > 0$ be fixed , since $X \to \mathbb{P}(t,X)(\eta,\xi)$ is lower semicontinuous , for every $X(t) \in \widetilde{\mathcal{A}}$, we choose point $y_{\eta\xi,X}(t) \in \mathbb{P}(t,X(t))(\eta,\xi)$ and neighbourhood U_X of $X(t)$ such that

$$\inf_{y_{\eta\xi,\mathbb{P}}(t)\in\mathbb{P}(t,X(t'))(\eta,\xi)} \mid y_{\eta\xi,X}(t) - y_{\eta\xi,\mathbb{P}}(t) \mid < \epsilon \ \forall \ X(t') \in U_X \qquad (4)$$

Now , let $(V_\alpha)_{\alpha\in\beta^\epsilon}$ be a local finite open refinement of $(U_X)_{X(t)\in\widetilde{\mathcal{A}}}$, with $V_\alpha \subset U_{X_\alpha}$, and let $(W_\alpha)_{\alpha\in\beta^\epsilon}$ be another open refinement such that $cl(W_\alpha) \subset V_\alpha$ for all $\alpha \in \beta^\epsilon$. By property (P), for each α , we can choose a set Z_α , clopen w.r.t. τ^+, such that

$$cl(W_\alpha) \subset int(Z_\alpha) \subset cl(Z_\alpha) \subset V_\alpha \qquad (5)$$

Then $(Z_\alpha)_\alpha$ is a local finite τ^+ clopen covering of $\widetilde{\mathcal{A}}$. Let $\preceq$ be a well-ordering of the set β^ϵ, define for each $\alpha \in \beta^\epsilon$,

$$\Omega^\epsilon_\alpha = Z_\alpha \setminus (\bigcup_{\lambda<\alpha} Z_\lambda)$$

Set $\mathcal{O}^\epsilon = (\Omega^\epsilon_\alpha)$, $\alpha \in \beta^\epsilon$. By well-ordering , every $x \in \widetilde{\mathcal{A}}$ belongs to exactly one set $\Omega^\epsilon_{\overline{\alpha}}$ where $\overline{\alpha} = \min\{\alpha \in \beta^\epsilon : x \in Z_\alpha\}$. Hence , $\mathcal{O}^\epsilon$ is a partition of $\widetilde{\mathcal{A}}$. Moreover, since Z_α is locally finite(wrt τ and therefore wrt τ^+), the sets $\bigcup_{\lambda\leq\alpha} Z_\lambda$ are τ^+ clopen. Hence $\mathcal{O}^\epsilon$ is a τ^+ clopen disjoint covering of $\widetilde{\mathcal{A}}$ such that, $\{cl(\Omega^\epsilon_\alpha)\}$ refines $(V_\alpha)_\alpha$.

By setting $y^{\epsilon}_{\eta\xi,\alpha} = y_{\eta\xi,X_\alpha}$ and $P_\epsilon(t, X(t))(\eta, \xi) = y_{\eta\xi,X_\alpha}$, $\forall\alpha \in \beta^\epsilon$ we have τ^+ continuous function P_ϵ, which by (4), satisfies

$$\inf_{y_{\eta\xi,\mathbb{P}}(t)\in\mathbb{P}(t,X(t))(\eta,\xi)} \mid P_\epsilon(t, X(t))(\eta, \xi) - y_{\eta\xi,\mathbb{P}}(t) \mid < \epsilon$$

Therefore , there exists an ϵ-approximate selection P_ϵ of $\mathbb{P}$.
Since ϵ was arbitrarily chosen ,thus we have a τ^+-continuous selection P of $\mathbb{P}$. □

Theorem 2: Suppose the following holds for an arbitrary $\eta, \xi \in \mathbb{D}\underline{\otimes}\mathbb{E}$, $\Psi \in \{\mu E, \nu F, \sigma G, H\}$:
(i) $t \to \Psi(t, X(t))(\eta, \xi)$ are measurable for all $X \in \widetilde{\mathcal{A}}$
(ii) $X \to \Psi(t, X(t))(\eta, \xi)$ are SD-lower semicontinuous with respect to a seminorm $\| \, . \, \|_{\eta\xi}$, for almost all $t \in I$
(iii) Ψ are integrably bounded, that is, there exists $L^{\Psi}_{\eta\xi}(t) \in L^1(I)$ such that, a.e. $t \in I$, for all $X \in \widetilde{\mathcal{A}}$,

$$\inf_{y\in\Psi(t,x)(\eta,\xi)} \mid y \mid \leq L^{\Psi}_{\eta\xi}(t).$$

Then the SD-lower semicontinuous quantum stochastic differential inclusions

$$\begin{gathered}\frac{d}{dt}\langle \eta, X(t)\xi\rangle \in \mathbb{P}(t, X(t))(\eta, \xi) \\ X(t_0) = x_0\end{gathered} \tag{6}$$

has an adapted weakly absolutely continuous solution in the sense of Caratheodory.

Proof: Since for arbitrary $\eta, \xi \in \mathbb{D}\underline{\otimes}\mathbb{E}$, $\Psi \in \mu E, \nu F, \sigma G, H$ are SD-lower semicontinuous then $\mathbb{P}(t, x)(\eta, \xi)$ is SD-lower semicontinuous, $\forall x \in \widetilde{\mathcal{A}}$, a.e. $t \in I$. The sequence of disjoint compact sets $J_n = \bigcap_\Psi J^\Psi_n$ and $meas(I \setminus \cup_{n\in\mathbb{N}} J_n) = 0$ such that $\mathbb{P}(.,.)(\eta, \xi)$ restricted to $\Omega_n = J_n \times \widetilde{\mathcal{A}}$ is lower semicontinuous, with respect to $\| \, . \, \|_{\eta\xi}$.
Also, suppose $L_{\eta\xi} = 5 \max L^{\Psi}_{\eta\xi}(t)$, then a.e. $t \in I$,

$$\inf_{y\in\mathbb{P}(t,x)(\eta,\xi)} \mid y \mid \leq L_{\eta\xi}(t),$$

for all $X \in \widetilde{\mathcal{A}}$
For each $n \geq 1$, we can apply Theorem (1) and obtain τ^+-continuous selections $P_n \in \mathbb{P}$.
For an arbitrary selection g from $\mathbb{P}$, if we define

$$P(t, X)(\eta, \xi) = \begin{cases} P_n(t, X)(\eta, \xi) & \text{if } t \in J_n, \\ g(t, X)(\eta, \xi) & \text{if } t \notin \cup_{n\in\mathbb{N}} J_n \end{cases}$$

then P is a τ^+-continuous selection of $\mathbb{P}$, such that $| P(t,x)(\eta,\xi) | \leq L_{\eta\xi}(t) < L_{n,\eta\xi}$, for every $(t,X) \in I \times \widetilde{\mathcal{A}}$, $\eta,\xi \in \mathbb{D}\underline{\otimes}\mathbb{E}$.
Then by applying Lusin's property to each bound of $L_{n,\eta\xi}$, $n \in \mathbb{N}$ the set of solutions of τ^+-continuous quantum stochastic differential equations is the solution set of (6) in the sense of Caratheodory. □

REFERENCES

[1] J.P. Aubin and A. Cellina, *Differential Inclusions*, Springer- Verlag, Berlin, 1984.

[2] E.O. Ayoola, Continuous selections of solution sets of Lipschitzian quantum stochastic differential inclusions.*Int. J. Theor.Phys.*43, 10, 2041-2059, 2004.

[3] E.O. Ayoola, Topological properties of solution sets of Lipschitzian quantum stochastic differential inclusions.*Acta Appl.Math*, 100, 15-37, 2008.

[4] A. Bressan and A. Cortesi, Directionally continuous selections in Banach spaces, *Non linear Analysis. Theory, Method and Applications* 13, 987-992, 1989.

[5] A. Bressan and G. Colombo, Selections and representations of multifunctions in paracompact spaces *Studia Mathematical* 102, 209-216, 1992.

[6] A. Cernea and V. Staicu, Directionally continuous selections and nonconvex evolution inclusions *Set-Valued Analysis* 11, 9-20, 2003.

[7] K. Deimling, *Non linear Functional Analysis*, Springer-Verlag, Berlin, 1985.

[8] K. Deimling, *Multivalued differential equations*, Walter de Gruyter , 1992.

[9] G.O.S. Ekhaguere, Lipschitzian quantum stochastic differential inclusions *Int. J. Theor. Phys.*, 31, 11, 2003-2034, 1992.

[10] G.O.S. Ekhaguere, Quantum stochastic evolutions *Int. J. Theor. Phys.*, 35, 9, 1909-1946, 1996.

[11] G.O.S. Ekhaguere, Topological solutions of Non commutative stochastic differential equations *Stoch. Anal. and Appl.* 25, 9, 961-993, 2007.

[12] A. Guichardet, *Symmetric Hilbert spaces and related topics* Lecture Notes in Mathematics, 261, Springer-Verlag, Berlin, 1972.

[13] R.L. Hudson and K.R. Parthasarathy, Quantum Ito's formula and stochastic evolutions*Comm. Math. Phys.*, 93, 3, 301-323, 1984.

[14] A. Kucia, Scorza-Dragoni type theorems ,*Fundamental Mathematical* 138, 197-203, 1991.

[15] M.O. Ogundiran and E.O. Ayoola, Mayer problem for Quantum stochastic control *J. Math. Phys.*51, 1, 023521-023521-8, 2010.

[16] M.O. Ogundiran and E.O. Ayoola, Directionally continuous quantum stochastic differential equations *Far East J. Appl. Maths.* 57, 1, 33-48, 2011.

[17] M. Reed and B. Simon, *Methods of Modern Mathematical Physics I. Functional Analysis*, 2nd edn., Academic press, New York-London, 1980.

[18] A. Tolstonogov, *Differential inclusions in a Banach space*, Mathematics and its applications, Kluwer Academics Publishers, New York, 2000.

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Existence and Uniqueness of Solutions of a Class of Quantum Stochastic Evolution Equations

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Abstract. We study the existence and uniqueness of solutions of a class of Quantum Stochastic Evolution Equations (QSEEs) defined on a locally convex space whose topology is generated by a family of seminorms defined via the norm of the range space of the operator processes. These solutions are called strong solutions in comparison with the solutions of similar equations defined on the space of operator processes where the topology is generated by the family of seminorms defined via the inner product of the range space. The evolution operator generates a bounded semigroup. We show that under some more general conditions, the unique solution is stable. These results extend some existing results in the literature concerning strong solutions of quantum stochastic differential equations.

AMS Subject Classification: 58J65, 81S25, 60H10
Keywords and Phrases: Strong solutions, Stability, Bounded semigroup, General Lipschitz condition.

1 Introduction

Several results on weak forms of solutions of the following quantum stochastic differential equation have been studied. See [1, 3-7] and the

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references therein. The motivations for studying this class of equations have been discussed in the references.

$$\begin{aligned} dz(t) &= U(t,z(t))d\wedge_\pi(t) + V(t,z(t))dA_g(t) \\ &\quad + W(t,z(t))dA_{f^+}(t) + H(t,z(t))dt, \\ z(t_0) &= z_0,\ t \in I = [t_0, T] \end{aligned} \tag{1}$$

In Eq. (1), the coefficients U, V, W, and H lie in a certain class of stochastic processes defined in [1], while the gauge, creation, annihilation processes $\Lambda_\Pi, A_{f^+}, A_g$ and the Lebesgue measure t are well defined in [2] and the references therein. $z \in \tilde{\mathcal{B}}$ is a locally convex space.
Quantum stochastic differential equation (QSDE) (1) is understood in the framework of the Hudson and Parthasarathy [9] quantum stochastic calculus. It has found applications in many physical systems, especially those that have to do with quantum optics, quantum measure theory, quantum open systems and quantum dynamical systems (see [1-7]).
In [6], some properties of solutions of Eq. (1) were studied. Results on the existence and uniqueness of solutions of this class of equations were established in the space of the operator processes endowed with the weak topologies. In [7], quantum stochastic differential inclusions of hypermaximal monotone type were studied under some general conditions and the existence of solution of an evolution operator connected with these inclusions were established. Also, see [4] for some results on evolution inclusions where the multivalued map P_1 is of hypermaximal monotone type. Further studies were carried out by [5] on properties of solution sets of quantum stochastic differential inclusions of Eq. (1) under the weak topologies. However, Ayoola in [1] investigated some existence properties on the space when endowed with the strong topology under a more general Lipschitz condition on the coefficients (U, V, W, H). Some new results including stability results were obtained. The results in [1, 2] generalized some similar results in the classical setting. This paper is concerned with the study of the properties of solutions of an evolution equation defined on the space with the strong topology. In [3, 12] existence of mild solutions of evolution QSDEs was studied under the weak topologies. Evolution problems have found practical applications in virtually all fields of sciences. See the references [11, 13-15] for some applications of evolution problems. The results in the present work

extend some existing results on strong solutions of Eq. (1) and extend the solution space for which QSDE will be applicable. We will consider some applications in our subsequent work.

2 Preliminaries

In what follows, the following evolution equation is considered.

$$\begin{aligned} dz(t) &= A(t)z(t) + U(t,z(t))d\wedge_\pi(t) + V(t,z(t))dA_g(t) \\ &\quad + W(t,z(t))dA_{f^+}(t) + H(t,z(t))dt, \\ z(t_0) &= z_0,\ t \in I \end{aligned} \tag{2}$$

where A generates a bounded semigroup $\{S(t) : t \geq 0\}$. For details on semigroup and their applications , see the references [8,10]. We adopt in most cases the definitions and notations of the spaces used in this paper from the references [1-3]. $\tilde{\mathcal{B}}$ is the completion of the topological space $(\tilde{\mathcal{B}}, \tau)$, and τ is the topology generated by the family of seminorms $||\phi||_\xi = ||\phi\xi||, \xi \in I\!\!D\underline{\otimes} I\!\!E$, where $||.||$ is the norm of the space $\mathcal{R} \otimes \Gamma(L^2_\gamma(\mathbb{R}_+))$. The space $\mathcal{B}$ is the linear space of all linear operators on $\mathcal{R} \otimes \Gamma(L^2_\gamma(\mathbb{R}_+))$. $I\!\!D, I\!\!E$, and $\mathcal{R}$ are well defined in [1]. The notations and structures of the following spaces are from the references [1, 2]. $\mathcal{R}\otimes\Gamma(L^2_\gamma(\mathbb{R}_+))$, $Ad(\tilde{\mathcal{B}})_{ac}$, $L^p_{loc}(\tilde{\mathcal{B}})$, $L^2_\gamma(\mathbb{R}_+)$, $L(\tilde{\mathcal{B}})$, $I\!\!D\underline{\otimes} I\!\!E$, $\text{Fin}(I\!\!D\underline{\otimes} I\!\!E)$. $I\!\!D, I\!\!E$, and $\mathcal{R}$ is well defined in [1].

Definition 2.1.
(i) $\phi : I \to \tilde{\mathcal{B}}$ is a stochastic process indexed by $I = [0, T] \subseteq \mathbb{R}_+$.
(ii) If $\phi(t) \in \tilde{\mathcal{B}}_t$, $t \in I$, then ϕ is said to be adapted and we denote the set of all such stochastic processes by $Ad(\tilde{\mathcal{B}})$.
(iii) $\phi(t) \in Ad(\tilde{\mathcal{B}})_{ac}$ is said to be adapted, absolutely continuous.
(iv) $\phi(t) \in L^p_{loc}(\tilde{\mathcal{B}})$ is said to be locally, absolutely p-integrable, where $p \in (0, \infty)$.
(v) Since the evolution operator A generates a bounded semigroup $\{S(t)\}_{t\geq 0}$, for each $t \geq 0$, there exists a constant $M > 0$ such that $\|S(t)\|_\xi \leq M$.
(vi) Let $\theta \in Fin(I\!\!D\underline{\otimes} I\!\!E)$ and $z \in \tilde{\mathcal{B}}$ then, $||z||_\theta = max_{\xi\in\theta}||z||_\xi$, where the set $\{\|.\|_\theta : \theta \in Fin(I\!\!D\underline{\otimes} I\!\!E)\}$ is a family of seminorms on $\tilde{\mathcal{B}}$ and $Fin(I\!\!D\underline{\otimes} I\!\!E)$ denote the set of all finite subsets of $I\!\!D\underline{\otimes} I\!\!E$. Also see Definitions 2.5 and 2.6 in [2].

Definition 2.2.
A stochastic process $\phi \in L^2_{loc}(\tilde{\mathcal{B}})$ is called a strong solution of the problem (2) on I if it is absolutely continuous and satisfies

$$\begin{aligned} \phi(t) &= S(t)\phi_0 + \int_{t_0}^{t} S(t-s)[U(s,\phi(s))d\wedge_\pi(s) + V(s,\phi(s))dA_g(s) \\ &\quad + W(s,\phi(s))dA_{f^+}(s) + H(s,\phi(s))ds], \\ \phi(t_0) &= \phi_0,\ t \in I \end{aligned} \tag{3}$$

Definition 2.3.
$\Phi : I \times \tilde{\mathcal{B}} \to \tilde{\mathcal{B}}$ is Lipschitzian if

$$\|\Phi(t,y) - \Phi(t,z)\|_\xi \leq K^\Phi_\xi(t)\|y-z\|_{\theta_{\Phi(\xi)}}$$

is satisfied for each $\xi \in I\!\!D\underline{\otimes} I\!\!E$, where $y,z \in \tilde{\mathcal{B}}$, $\theta \in (I\!\!D\underline{\otimes} I\!\!E, Fin(I\!\!D\underline{\otimes} I\!\!E))$ and $K^\Phi_\xi : I \to (0,\infty)$ is a Lipschitz function lying in $L^1_{loc}(I)$. $I = [0,T] \subseteq \mathbb{R}_+$.

Remark 2.4. Theorem 2.2 and Remarks (a) - (c) in [1] hold in this case.

For the remaining part of this paper, $\xi \in I\!\!D\underline{\otimes} I\!\!E$ is arbitrary, except otherwise stated, the following result established in [1] will be used to establish the major results.

Theorem 2.5. (a) Let $p,q,u,v \in L^2_{loc}(\tilde{\mathcal{B}})$ and let $\mathbf{M}$ be their stochastic integral. If $\xi \in I\!\!D\underline{\otimes} I\!\!E$ where $\xi = d \otimes e(\beta), \alpha, \beta \in L^\infty_{\gamma,loc}(\mathbb{R}_+)$ and $t \geq 0$, then

$$\begin{aligned} <\eta, \mathbf{M}(t)\xi> &= \int_0^t <\eta, \{\alpha(s), \pi(s)\beta(s)>_\gamma p(s) \\ &+ <f(s), \beta(s)>_\gamma q(s) \\ &+ <\alpha(s), g(s)>_\gamma u(s) + v(s)\}\xi > ds. \end{aligned} \tag{4}$$

(b) Let
$K(T) = \sup_{0\leq s\leq T} \max\{|\langle \beta(s), \pi(s)\beta(s)\rangle|, |\langle f(s), \beta(s)\rangle|, |\langle \beta(s), g(s)\rangle|, ||\pi(s)\beta(s)||^2, ||g(s)||^2\}$.
Then for $T > 0$ and $0 \leq t \leq T$,

$$\begin{aligned} ||\mathbf{M}(t)\xi||^2 &\leq 6K(T)^2 \int_0^t e^{t-s}\{||p(s)\xi||^2 + ||q(s)\xi||^2 + ||u(s)\xi||^2 \\ &+ ||v(s)\xi||^2\}ds. \end{aligned} \tag{5}$$

(c) Let $0 \leq s \leq t \leq T$. Then

$$\begin{aligned} ||(\mathbf{M}(t) - \mathbf{M}(s))\xi||^2 &\leq 6K(T)^2 \int_0^t e^{t-\tau}\{||p(\tau)\xi||^2 + ||q(\tau)\xi||^2 \\ &+ ||u(\tau)\xi||^2 + ||v(\tau)\xi||^2\}d\tau. \end{aligned} \quad (6)$$

Note 2.6. $\mathbf{M}$ is absolutely continuous, hence, $\mathbf{M} \in L^2_{loc}(\tilde{\mathcal{B}})$.

3 Main Results

This section is dedicated to the main results on existence and uniqueness of strong solutions of (2). Subsequently, except otherwise stated, $t \in I = [t_0, T] \subseteq \mathbb{R}_+$ and $\xi \in I\!\!D \underline{\otimes} I\!\!E$ is arbitrary.

Theorem 3.1.
Suppose that the coefficients $U, V, W, H \in L^2_{loc}(I \times \tilde{\mathcal{B}})$ are Lipschitzian. Then for $(t_0, z_0) \in I \times \tilde{\mathcal{A}}$ there exists a unique strong solution φ of equation (2) satisfying $\varphi(t_0) = z_0$.

Proof. To prove the theorem, we make the following assumptions:
H_1. Let $\{\varphi_n(t)\}_{n \geq 0}$ be a sequence of successive approximations of $\varphi \in \tilde{\mathcal{B}}$ and
H_2. $\varphi_n(t),\ n \geq 1$ define an absolutely continuous process in $L^2_{loc}(\tilde{\mathcal{A}})$.
Let $T > t_0,\ t \in I$ be fixed. Then, we prove $H_1 - H_2$ as follows.
For $n \geq 0$, we have

$$\begin{aligned} \varphi_{n+1}(t) &= S(t)z_0 + \int_{t_0}^t S(t-s)[U(s, \varphi_n(s))d\wedge_\pi(s) \\ &+ V(s, \varphi(s))dA_g^+(s) + W(s, \varphi_n(s))dA_f(s) + H(s, \varphi_n(s))ds]. \end{aligned}$$

By hypothesis, $U(s, z_0), V(s, z_0), W(s, z_0), H(s, z_0) \in \tilde{\mathcal{B}}_s$ for $s \in [t_0, T]$ while
$U(., z_0), V(., z_0), W(., z_0), H(., z_0) \in L^2_{loc}(\tilde{\mathcal{B}})$.
Therefore, the quantum stochastic integral which defines $\varphi_1(t)$ exists for $t \in [t_0, T]$. By Theorem 2.5, $\varphi_1(t) \in L^2_{loc}(\tilde{\mathcal{B}})$.
Hence, it implies that each
$U(s, \varphi_n(s)), V(s, \varphi(s)), W(s, \varphi_n(s))$ and $H(s, \varphi_n(s) \in L^2_{loc}(\tilde{\mathcal{B}})$.
This proves assumptions $H_1 - H_2$.

Next, we show that the sequence of successive approximations converges as follows:

$$\begin{aligned}\| \varphi_{n+1}(t) - \varphi_n(t) \|_\xi &= \| \int_{t_0}^t S(t-s)[(U(s,\varphi_n(s)) \\ &\quad -U(s,\varphi_{n-1}(s)))d \wedge_\pi (s) \\ &\quad +(V(s,\varphi(s)) - V(s,\varphi_{n-1}(s)))dA_g^+(s) \\ &\quad +(W(s,\varphi_n(s)) - W(s,\varphi_{n-1}(s)))dA_f(s) \\ &\quad +(H(s,\varphi_n(s)) - H(s,\varphi_{n-1}(s)))ds]\|_\xi.\end{aligned}$$

By Theorem 2.5 and (v) of Definition 2.1, we get

$$\begin{aligned}\| \varphi_{n+1}(t) - \varphi_n(t) \|_\xi^2 &\le 6M^2K(T)^2 \int_{t_0}^t e^{t-s}\{||U(s,\varphi_n(s)) \\ &\quad -U(s,\varphi_{n-1}(s))||_\xi^2 \\ &\quad +||V(s,\varphi(s)) - V(s,\varphi_{n-1}(s))||_\xi^2 \\ &\quad +||W(s,\varphi_n(s)) - W(s,\varphi_{n-1}(s))||_\xi^2 \\ &\quad +||H(s,\varphi_n(s)) - H(s,\varphi_{n-1}(s))||_\xi^2\}ds. \qquad (7)\end{aligned}$$

By definition 2.3, we have

$$\| \mathbf{M}(s,\varphi_n(s)) - \mathbf{M}(s,\varphi_{n-1}(s)) \|_\xi \le K_\xi^{\mathbf{M}}(s) \| \varphi_n(s) - \varphi_{n-1}(s) \|_{\theta_{\mathbf{M}\xi}},$$

for each $\mathbf{M} \in \{U, V.W, H\}$. Thus, there exists $\xi_{\mathbf{M}}^1 \in \theta_{\mathbf{M}}(\xi)$ satisfying

$$\| \varphi_n(s) - \varphi_{n-1}(s) \|_{\theta_{\mathbf{M}\xi}} = \| \varphi_n(s) - \varphi_{n-1}(s) \|_{\xi_{\mathbf{M}}^1} .$$

Using (7), we obtain

$$\begin{aligned}\| \varphi_{n+1}(t) - \varphi_n(t) \|_\xi^2 &\le NC(T)L_\xi \int_{t_0}^t e^{t-s} \| \varphi_n(s) - \varphi_{n-1}(s) \|_{\xi_1}^2 \, ds \\ &= NC(T)L_\xi e^t \\ &\quad \times \int_{t_0}^t e^{-s} \| \varphi_n(s) - \varphi_{n-1}(s) \|_{\xi_1}^2 \, ds. \qquad (8)\end{aligned}$$

Where

$$\| \varphi_n(s) - \varphi_{n-1}(s) \|_{\xi_1} = \max_{\mathbf{M}\in\{U,V.W,H\}} \| \varphi_n(s) - \varphi_{n-1}(s) \|_{\xi_{\mathbf{M}}^1} . \qquad (9)$$

and $N = M^2,\ C(T) = 6K(T)^2$,

$$L_\xi = \text{ess}\sup_{s\in[0,T]}\left[K_\xi(s) = \sum_{\mathbf{M}\in\{U,V.W,H\}} K_\xi^{\mathbf{M}}(s)^2\right].$$

Continuing the iteration and replacing ξ_2 with ξ_1 in (9), yields

$$\begin{aligned}\| \varphi_{n+1}(t) - \varphi_n(t) \|_\xi^2 \leq{}& N^2C(T)^2 L_\xi L_{\xi_1} e^t \\ &\times \int_{t_0}^t \int_{t_0}^s e^{-s'} \| \varphi_{n-1}(e^{-s'}) - \varphi_{n-2}(e^{-s'}) \|_{\xi_2}^2 \, ds' ds \\ &\leq N^n C(T)^n \mathbf{M}(\xi)^n e^t \int_{t_0}^t ds_1 \int^{s_1} ds_2 ... \int_{t_0}^{s_{n-2}} ds_{n-1} \\ &\times \int_{t_0}^{s_{n-1}} e^{-s_n} \| \varphi_1(s_n) - \varphi_0(s_n) \|_\xi^2 \, ds_n, \qquad (10)\end{aligned}$$

where $\mathbf{M}_n(\xi) = \max\{L_{\xi,j}, j = 0, 1, ..., n-1\}$, $\mathbf{M}(\xi) = \sup_{n\in\mathbb{N}}\{\mathbf{M}_n(\xi)\}$, and $L_{\xi,j}, j = 0, 1, ..., n-1$ are positive real numbers.
Since the map $s \longrightarrow \|\varphi_1(s) - z_0\|_\xi$ is continuous on I, we obtain, $R_{\xi_n} = \sup_{s\in I} \|\varphi_1(s) - z_0\|_{\xi_n} < \infty$ and put $R_\xi = \sup_{n\in\mathbb{N}}\{R_{\xi_n}\}$ in (10) to get

$$\| \varphi_{n+1}(t) - \varphi_n(t) \|_\xi^2 \leq [NC(T)\mathbf{M}(\xi)]^n e^T \frac{T^n}{n!} R_\xi^2, n = 0, 1, 2, ...$$

For $n > k$ we get,

$$\begin{aligned}\| \varphi_{n+1}(t) - \varphi_{k+1}(t) \|_\xi &= \|\Sigma_{m=k+1}^n (\varphi_{m+1}(t) - \varphi_m(t))\|_\xi \\ &\leq \Sigma_{m=k+1}^n \|\varphi_{m+1}(t) - \varphi_m(t)\|_\xi \\ &\leq e^{\frac{T}{2}} R_\xi \sum_{m=k+1}^n \left(\frac{[NC(T)\mathbf{M}(\xi)]^m T^m}{m!}\right)^{\frac{1}{2}} < \infty.\end{aligned}$$

Showing that $\varphi_n(t)$ is a Cauchy sequence in $\tilde{\mathcal{B}}$ and converges uniformly to $\varphi(t)$.
Now since $\varphi_n(t)$ is adapted and absolutely continuous, the same is true for $\varphi(t)$.

Next, we show that $\varphi(t)$ satisfies Eq. (2).
Let $\varphi(t_0) = z_0$ and by (8), there exists $\xi \in I\!\!D\underline{\otimes}I\!\!E$ such that

$$\begin{aligned}
&|| \int_{t_0}^{t} S(t-s)[U(s,\varphi_n(s))d\wedge_\pi(s) + V(s,\varphi_n(s))dA_g^+(s) \\
&\quad + W(s,\varphi_n(s))dA_f(s) + H(s,\varphi_n(s))ds]||_\xi^2 \\
&- || \int_{t_0}^{t} S(t-s)[U(s,\varphi(s))d\wedge_\pi(s) \\
&\quad + V(s,\varphi(s))dA_g^+(s) + W(s,\varphi(s))dA_f(s) \\
&+ \quad H(s,\varphi(s))ds]||_\xi^2 \\
&= || \int_{t_0}^{t} S(t-s)(P(s,\varphi_n(s)) - P(s,\varphi(s)))ds||_\xi^2 \\
&\le NC(T)L_\xi e^t \times \int_{t_0}^{t} e^{-s} \parallel \varphi_n(s) - \varphi(s) \parallel_\xi^2 ds \to 0 \text{ as } n \to \infty.
\end{aligned}$$

Since $\varphi_n(s) \to \varphi(s)$ in $\tilde{\mathcal{B}}$ uniformly on $[t_0, T]$, we have

$$\begin{aligned}
\varphi(t) &= \lim_{n\to\infty} \varphi_{n+1}(t) \\
&= S(t)z_0 + \lim_{n\to\infty} (\int_{t_0}^{t} S(t-s)(U(s,\varphi_n(s))d\wedge_\pi(s) \\
&+ V(s,\varphi_n(s))dA_g^+(s) \\
&+ W(s,\varphi_n(s))dA_f(s) + H(s,\varphi_n(s))ds) \\
&= S(t)z_0 + \int_{t_0}^{t} S(t-s)(U(s,\varphi(s))d\wedge_\pi(s) \\
&+ V(s,\varphi(s))dA_g^+(s) \\
&+ W(s,\varphi(s))dA_f(s) + H(s,\varphi(s))ds), , t \in I.
\end{aligned}$$

This shows that $\varphi(t)$ is a solution of Eq. (2).

Uniqueness

Suppose that $y(t), t \in [t_0, T]$ is another adapted absolutely continuous solution with $y(t_0) = z_0$, then just as we established the above result, we obtain

$$\parallel \varphi(t) - y(t) \parallel_\xi^2 \le [NC(T)\mathbf{M}(\xi)]^n e^T \frac{T}{n!} \sup_{t\in I} \parallel \varphi(t) - y(t) \parallel_\xi^2 < \infty. \quad (11)$$

By the right hand side of Eq. (11), we conclude that for $n \in \mathbb{N}$,
$\| \varphi(t) - y(t) \|_\xi = 0$ and $\varphi(t) = y(t)$ on $I\!D \underline{\otimes} I\!E$, $t \in I$. Hence the solution is unique.

4 Stability

In this section, we show that under the condition (v) of Definition 2.1, the solution of Eq.(2) is stable.
(a) let the coefficients U, V, W, H satisfy the conditions of Theorem 3.1 and let $z(t), y(t)$, $t \in [t_0, T]$ be solutions of Eq. (2) such that $z(t_0) = z_0$ and $y(t_0) = y_0$, $z_0, y_0 \in \tilde{\mathcal{B}}$. The solution z(t) is stable under the changes in the initial condition over a finite time interval as follows:
(b) Let L_ξ, N and $C(T)$ be constants such that

$$L_\xi = \operatorname*{ess\,sup}_{s \in I} K_\xi(s),\ C(T) = 12K(T)^2 \text{ and } N = M^2$$

where $K(T)$ is as defined in Theorem 2.3 and $||S(t)||_\xi$ by (v) of Definition 2.1.
(c) Define the function $K_\xi(s)$ as

$$K_\xi(s) = \sum_{\mathbf{M} \in \{U, V.W, H\}} (K_\xi^{\mathbf{M}}(s))^2$$

Theorem 4.1. Let the conditions of Definition 2.1 hold and let $\epsilon > 0$ be given. Then there exists $\delta > 0$ such that if $\|z_0 - y_0\|_\xi < \delta$,
then $\|z(t) - y(t)\|_\xi < \epsilon$, $\forall\, t \in [0, T]$.

Proof:
Let $z_n(t), y_n(t),\ n = 0, 1, ...$ be the iterates corresponding to z_0, y_0 respectively. Let $z_0(t) = z_0$ and $y_0(t) = y_0$, $0 \le t \le T$. Then we get

$$\begin{aligned}
\| z_{n+1}(t) - y_{n+1}(t) \|_\xi \le {} & ||S(t-s)(z_0 - y_0)||_\xi \\
& + || \int_{t_0}^t S(t-s)[(U(s, z_n(s)) - U(s, y_n(s)))d \wedge_\pi (s) \\
& + (V(s, z(s)) - V(s, y_n(s)))dA_g^+(s) \\
& + (W(s, z_n(s)) - W(s, y_n(s)))dA_f(s) \\
& + (H(s, z_n(s)) - H(s, y_n(s)))ds]||_\xi
\end{aligned}$$

So that by applying Theorem 2.5 and condition (v) of Definition 2.1, we obtain

$$\begin{aligned}\| z_{n+1}(t)-y_{n+1}(t) \|_\xi^2 \leq & 2M^2||z_0 - y_0||_\xi^2 \\ & + 2M^2|| \int_{t_0}^t S(t-s)[(U(s,z_n(s)) - U(s,y_n(s)))d\wedge_\pi(s) \\ & + (V(s,z(s)) - V(s,y_n(s)))dA_g^+(s) \\ & + (W(s,z_n(s)) - W(s,y_n(s)))dA_f(s) \\ & + (H(s,z_n(s)) - H(s,y_n(s)))ds]||_\xi^2\end{aligned}$$

$$\begin{aligned}\| z_{n+1}(t) - y_{n+1}(t) \|_\xi^2 &\leq 2N||z_0 - y_0||_\xi^2 \\ &+ NC(T)\int_{t_0}^t e^{s-t}\{||U(s,z_n(s)) - U(s,y_n(s))||_\xi^2 \\ &+ ||V(s,z(s)) - V(s,y_n(s))||_\xi^2 \\ &+ ||(W(s,z_n(s)) - W(s,y_n(s))||_\xi^2 \\ &+ (H(s,z_n(s)) - H(s,y_n(s)))||_\xi^2\}ds.\end{aligned}$$

Since Definition 2.3 also holds for the coefficients U, V, W, H, we find elements $\xi_{\mathbf{M},1} \in \theta_{\mathbf{M}}(\xi) \in Fin(\mathbb{D}\underline{\otimes}\mathbb{E})$, $\mathbf{M} \in \{U, V, W, H\}$ such that

$$\begin{aligned}\| z_{n+1}(t)-y_{n+1}(t) \|_\xi^2 \leq & 2N||z_0 - y_0||_\xi^2 + NC(T) \\ & \times \int_{t_0}^t e^{t-s_1}[\sum_{\mathbf{M}\in\{U,V,W,H\}} K_\xi^{\mathbf{M}}(s_1)^2||z_n(s_1)) - y_n(s_1))||_{\xi_{\mathbf{M},1}}^2]ds_1 \\ & \leq 2N||z_0 - y_0||_\xi^2 \\ & + NC(T)L_\xi e^t \int_{t_0}^t e^{-s_1}||z_n(s_1) - y_n(s_1)||_\xi^2 ds_1. \qquad (12)\end{aligned}$$

where $\xi_1 \in \xi_{\mathbf{M},1} : \mathbf{M} \in \{U, V.W, H\}\}$ satisfies

$$\| \varphi_n(s) - \varphi_{n-1}(s) \|_{\xi_1}^2 = \max_{\mathbf{M}\in\{U,V.W,H\}} \| \varphi_n(s) - \varphi_{n-1}(s) \|_{\xi_{\mathbf{M},1}}^2,$$

where $s \in I$.
Also, if we have $\xi_2 \in I\!D\underline{\otimes} I\!E$ then,

$$\begin{aligned}
\| z_n(s_1) - y_n(s_1) \|_\xi^2 &\leq 2N \|(z_0 - y_0)\|_{\xi_1}^2 \\
&+ NC(T)L_{\xi_1} \int_{s_1}^t e^{s_1 - s_2} ||z_{n-1}(s_2) - y_{n-1}(s_2)||_{\xi_2}^2 ds_2.
\end{aligned}$$

By (12), we obtain for $t \in [0, T]$,

$$\begin{aligned}
\| z_{n+1}(t) - y_{n+1}(t) \|_\xi^2 &\leq 2N \|(z_0 - y_0)\|_\xi^2 \\
&+ 2NC(T) \|z_0 - y_0\|_\xi^2 L_\xi e^t \int_0^t e^{-s_1} ds_1 \\
&+ N^2 C(T)^2 L_\xi L_{\xi_1} e^t \\
&\times \int_0^t \int_0^{s_1} e^{-s_2} ||z_{n-1}(s_2) - y_{n-1}(s_2)||_{\xi_2}^2 ds_2 ds_1.
\end{aligned}$$

Continuous iterations yields,

$$\begin{aligned}
\| z_{n+1}(t) - y_{n+1}(t) \|_\xi^2 &\leq 2N \|z_0 - y_0\|_\xi^2 + 2NC(T) \|z_0 - y_0\|_{\xi_1}^2 L_\xi e^T t \\
&+ 2N^2 C(T)^2 \|z_0 - y_0\|_{\xi_2}^2 L_\xi L_{\xi_1} e^T \int_0^t \int_0^{s_1} ds_2 ds_1 \\
&+ 2N^3 C(T)^3 \|z_0 - y_0\|_{\xi_2}^2 L_\xi L_{\xi_1} L_{\xi_2} e^T \int_0^t \int_0^{s_1} \\
&\times \int_0^{s_2} \int_0^{s_1} ds_3 ds_2 ds_1 \\
&+ ... + N^{n+1} C(T)^{(n+1)} e^T L_\xi L_{\xi_1} L_{\xi_2} ... L_{\xi_n} \int_0^t \int_0^{s_1} ... \\
&\times \int_0^{s_n} ||z_0(s_{n+1}) - y_0(s_{n+1})||_{\xi_{n+1}}^2 ds_1 ds_2 ds_3 ... ds_{n+1}.
\end{aligned}$$

Now, by letting $\mathbf{K}(\xi) = \sup_{n \in \mathbb{N}} \{L_\xi, L_{\xi_1}, L_{\xi_2}, ..., L_{\xi_n}\}$,
$\eta_n \in \{\xi, \xi_1, \xi_2, ..., \xi_n, \xi_{n+1}\}$ so that if

$$\|z_0 - y_0\|_{\eta_n} = \max\{\|z_0 - y_0\|_{\xi_j}, j = 0, 1, ..., n+1\},$$

we obtain

$$\begin{aligned} \| z_{n+1}(t) - y_{n+1}(t) \|_\xi^2 &\leq 2e^T \|z_0 - y_0\|_{\eta_n}^2 \sum_{m=0}^{n+1} [NC(T)\mathbf{K}(\xi)]^m \frac{T^m}{m!} \\ &\leq 2 \|z_0 - y_0\|_{\eta_n}^2 e^{(NC(T)\mathbf{K}(\xi)+T)}. \end{aligned} \quad (13)$$

Thus, by taking the square root of both sides of (13) and letting $n \to \infty$, we obtain $\|z(t) - y(t)\|_\xi \leq \epsilon$.
Take $\delta = \epsilon[2e^{(NC(T)\mathbf{K}(\xi)T+T)}]^{-\frac{1}{2}}$, for all $t \in [0, T]$, and the desired result is obtained.
Remark 4.2 If $N < 1$, we obtain the results in [1].

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References

[1] E. O. Ayoola, A. W. Gbolagade, Further Results on Existence, Uniqueness and Stability of Strong Solutions of Quantum Stochastic Differential Equations, *Applied mathematics letters* 18(2005) 219-227.

[2] E. O. Ayoola, Existence and stability results for strong solutions of Lipschitzian quantum stochastic differential equations. *Stoch. Anal. and Appl.*, 20(2)(2002), 263-281.

[3] S. A. Bishop, E. O. Ayoola and G. J. Oghonyon, Existence of mild solution of impulsive quantum stochastic differential equation with nonlocal conditions, *Anal. Math. Phys.* 7(3)(2017), 255-265.

[4] S. A. Bishop and E. O. Ayoola, On topological properties of solution sets of non Lipschitzian quantum stochastic differential inclusions, *Anal. Math. Phys.* 2015, DOI 10.1007/s13324-015-0109-1.

[5] S. A. Bishop, On continuous selection sets of non Lipschitzian quantum stochastic evolution inclusions, *Int. J. of Stoch. Anal.* Vol. 2015, Article ID 834194, pp. 5, http://dx.doi.org/10.1155/2015/834194.

[6] G.O.S. Ekhaguere, Topological solutions of non commutative stochastic differential equations *Stoch. Anal. and Appl.* 25(9)(2007), 961-993.

[7] G.O.S. Ekhaguere, Quantum stochastic evolutions, *Int. J. of Theor. Phys.*, 35(9)(1996), 1909 - 1946.

[8] E. Hille, *Functional analysis and semi-groups*, Amer. Math. Soc., 1949, N.Y.

[9] R. L. Hudson and K. R. Parthasarathy, Quantum Ito's formulae and stochastic evolutions. *Comm. Math. Phys.*, 93(1984), 301-3249.

[10] S. V. Lototsky, B. L. Rozovskii, Wiener chaos solutions of linear stochastic evolution equations, *The Annals of Prob.*, 34(2)(2006), 638-662.

[11] M. McKibben, *Discovering evolution equations with applications: Vol. 1-Deterministic equations* (Chapman & Hall/CRC Applied Mathematics and Nonlinear Science) 1st Edition, 2010 London N.Y.

[12] M. O. Ogundiran, On the existence and uniqueness of solution of impulsive quantum stochastic differential equation, *Differential equations and control processes*, N 2(2013), 63-73.

[13] B. L. Rozovskii, *Stochastic differential equations: theory and applications, interdisciplinary mathematical sciences*, A Volume in Honor of Professor Boris L. Rozovskii, April 2007, pp. 420.

[14] M. C. Veraar, Non-autonomous stochastic evolution equations and applications to stochastic partial differential equations, *J. Evol. Equ.*, 10 (2010), 85-127.

[15] S. Zheng, *Nonlinear evolution equations, monograhps and surveys in pure and applied mathematics*, Chapman and Hall/CRC, 2004 pp. 304, London N.Y.

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FURTHER RESULTS ON THE EXISTENCE OF CONTINUOUS SELECTIONS OF SOLUTION SETS OF QUANTUM STOCHASTIC DIFFERENTIAL INCLUSIONS

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ABSTRACT. We prove that the map that associates to the initial value the set of solutions to the Lipschitzian Quantum Stochastic Differential Inclusion (QSDI) admits a selection which is continuous from the locally convex space of stochastic processes to the space of adapted and weakly absolutely continuous solutions. As a corollary, the reachable set multifunction admits a continuous selection. In the framework of the Hudson-Parthasarathy formulation of quantum stochastic calculus, these results are achieved subject to some compactness conditions on the set of initial values and on some coefficients of the inclusion.

Key Words: Continuous selections, Lipschitzian quantum stochastic differential inclusions, Reachable sets, solution sets

AMS (MOS) Subject Classification: 81S25, 60H10

1. INTRODUCTION

This work is concerned with further investigations of the existence and applications of continuous selections of solution sets of quantum stochastic differential inclusions (QSDI). In the context of classical differential inclusions defined in finite dimensional Euclidean spaces, such investigations have attracted considerable attention in the literature. Some well known results on continuous selections and their applications in the finite dimensional Euclidean settings can be found in [1, 2, 14, 15, 18, 20, 22]. As in [8, 18, 20, 22], selection results have been used among other things for the interpolation of a given finite set of trajectories of classical differential inclusions.

However, in the non commutative quantum setting, investigations of the existence of continuous selections and their applications have not received a comparable attention in the literature. In the framework of the Hudson and Parthasarathy [17, 19] formulations of quantum stochastic calculus, we established in our previous work [4], some continuous selections of solution sets of quantum stochastic differential inclusion (QSDI) defined on the set of the matrix elements of initial points with values

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in the set of matrix elements of solutions. However, on this occasion and in the same framework of quantum stochastic calculus, we establish the existence of a selection map continuous from a compact set of initial values contained in the space of quantum stochastic processes into the locally convex space of adapted weakly absolutely continuous quantum stochastic processes. In addition, as a corollary, we deduce that the reachable set multifunction admits a continuous selection. This work, therefore, complements our results in [4] where the set of the matrix elements of solutions and the reachable set respectively admit continuous selections and some continuous representations.

The proof of our main results here adapts the techniques employed in Cellina [1] in a way that is suitable for the analysis of QSDI where the solutions live in certain locally convex spaces. Our main tools in the construction of the selection are some suitable use of Liapunov's theorem on the range of vector measures (see [1, 15, 16]) and Ekhaguere's existence result [11] for the solutions of QSDI (2.3). The result is a generalization of Filippov's extension of Gronwall's inequalities to solutions of QSDI (2.3).

The plan for the rest of the paper is as follows: In section 2, we present some fundamental results, notations and assumptions. The main results of the paper are reported in Section 3.

2. PRELIMINARY RESULTS AND ASSUMPTIONS

In what follows, we adopt the notations, formulation and the frameworks as reported in [3, 4, 11, 12, 13]. Detailed definitions of various spaces that appear below can be found in [11]. In the sequel, γ is a fixed Hilbert space, $\mathbb{D}$ is an inner product space with $\mathcal{R}$ as its completion, and $\Gamma(L^2_\gamma(\mathbb{R}_+))$ is the Boson Fock Space determined by the function space $L^2_\gamma(\mathbb{R}_+)$. The set $\mathbb{E}$ is the subset of the Fock space generated by the exponential vectors. If $\mathcal{N}$ is a topological space, then we denote by $clos(\mathcal{N})$ (resp. $comp(\mathcal{N})$), the family of all nonempty closed subsets of $\mathcal{N}$ (resp. compact members of $clos(\mathcal{N})$).

In our formulations, quantum stochastic processes are $\tilde{\mathcal{A}}$-valued maps on $[t_0, T]$. The space $\tilde{\mathcal{A}}$ is the completion of the linear space

$$\mathcal{A} = L^+_W(\mathbb{D}\underline{\otimes}\mathbb{E}, \mathcal{R} \otimes \Gamma(L^2_\gamma(\mathbb{R}_+)))$$

endowed with the locally convex operator topology generated by the family of seminorms $\{x \to \|x\|_{\eta\xi} = |\langle \eta, x\xi\rangle|, \quad \eta, \xi \in \mathbb{D}\underline{\otimes}\mathbb{E}\}$. Here, $\mathcal{A}$ consists of linear operators from $\mathbb{D}\underline{\otimes}\mathbb{E}$ into $\mathcal{R} \otimes \Gamma(L^2_\gamma(\mathbb{R}_+))$ with the property that the domain of the adjoint operator contains $\mathbb{D}\underline{\otimes}\mathbb{E}$. We adopt the notation and the definitions of Hausdorff topology on $clos(\tilde{\mathcal{A}})$ as explained in [11]. The Hausdorff topology is determined by some family of pseudo-metrics. On the set $\mathbb{C}$ of complex numbers, we employ the metric topology on $clos(\mathbb{C})$ induced by the Hausdorff metric ρ. Thus for $A, B, \in clos(\mathbb{C})$, $\rho(A, B)$ is the Hausdorff distance between the sets and for arbitrary pair $\eta, \xi \in \mathbb{D}\underline{\otimes}\mathbb{E}$, $\mathcal{N}, \mathcal{M} \in clos(\tilde{\mathcal{A}})$, $\rho_{\eta\xi}(\mathcal{N}, \mathcal{M})$ denotes pseudo-metrics as in [11, 12, 13].

A quantum stochastic process $\Phi : [t_0, T] \to \tilde{\mathcal{A}}$ will be said to be weakly continuous on the interval $I = [t_0, T]$ if for each pair $\eta, \xi \in \mathbb{D}\underline{\otimes}\mathbb{E}$, the map $t \to \Phi_{\eta\xi}(t)$ is continuous. Here, $\Phi_{\eta\xi}(t) := \langle \eta, \Phi(t)\xi \rangle$. We shall denote by $C[I, \tilde{\mathcal{A}}]$ the set of all weakly continuous quantum stochastic processes on $[t_0, T]$ and for each $\Phi \in C[I, \tilde{\mathcal{A}}]$, we set

$$\|\Phi_{\eta\xi}\|_c := \sup_I |\Phi_{\eta\xi}(t)| = \sup_I \|\Phi(t)\|_{\eta\xi}. \tag{2.1}$$

By employing the symbol $Ad(\tilde{\mathcal{A}})_{wc}$ to denote the set of all adapted weakly continuous stochastic processes, then we have the following set inclusion

$$Ad(\tilde{\mathcal{A}})_{wac} \subseteq Ad(\tilde{\mathcal{A}})_{wc} \subseteq C[I, \tilde{\mathcal{A}}],$$

since all weakly absolutely continuous stochastic processes are weakly continuous.

As in [11], we denote by $wac(\tilde{\mathcal{A}})$, the completion of $Ad(\tilde{\mathcal{A}})_{wac}$ in the topology generated by the family of seminorms

$$|\Phi|_{\eta\xi} = \|\Phi(t_0)\|_{\eta\xi} + \int_{t_0}^{T} |\frac{d}{ds}\langle \eta, \Phi(s)\xi \rangle| ds \tag{2.2}$$

for each $\Phi \in Ad(\tilde{\mathcal{A}})_{wac}$ and arbitrary $\eta, \xi \in \mathbb{D}\underline{\otimes}\mathbb{E}$.

The existence of the continuous selections which we study in this paper concerns solution and the reachable sets of quantum stochastic differential inclusions in the integral form given by:

$$\begin{aligned} X(t) \in a + \int_0^t & (E(s, X(s))d\wedge_\pi(s) + F(s, X(s))dA_f(s) + G(s, X(s))dA_g^+(s) \\ & + H(s, X(s))ds), \; t \in [t_0, T], \end{aligned} \tag{2.3}$$

where the coefficients E, F, G, H are continuous and lie in the space $L^2_{loc}([t_0, T] \times \tilde{\mathcal{A}}))_{mvs}$, $f, g \in L^\infty_{\gamma,loc}(\mathbb{R}_+)$, $\pi \in L^\infty_{B(\gamma),loc}(\mathbb{R}_+)$. Here, $B(\gamma)$ is the space of bounded endomorphisms of γ and $(t_0, a) \in [t_0, T] \times \tilde{\mathcal{A}}$ is a fixed point.

For any pair of $\eta, \xi \in \mathbb{D}\underline{\otimes}\mathbb{E}$ such that $\eta = c \otimes e(\alpha)$, $\xi = d \otimes e(\beta)$, $\alpha, \beta \in L^2_\gamma(\mathbb{R}_+)$, $c, d \in \mathbb{D}$, as in our previous works in [3, 4, 5, 6, 7], we shall in what follows, employ the equivalent form of (2.3) as established in [11] given by the nonclassical ordinary differential inclusion:

$$\frac{d}{dt}\langle \eta, X(t)\xi \rangle \in P(t, X(t))(\eta, \xi), \qquad X(t_0) = a, \; t \in [t_0, T]. \tag{2.4}$$

The multivalued map P appearing in (2.4) is of the form

$$P(t, x)(\eta, \xi) = \langle \eta, P_{\alpha\beta}(t, x)\xi \rangle$$

where the map $P_{\alpha\beta} : [t_0, T] \times \tilde{\mathcal{A}} \to 2^{\tilde{\mathcal{A}}}$ is given by

$$P_{\alpha\beta}(t, x) = \mu_{\alpha\beta}(t)E(t, x) + \nu_\beta(t)F(t, x) + \sigma_\alpha(t)G(t, x) + H(t, x).$$

The complex valued functions $\mu_{\alpha\beta}, \nu_\beta, \sigma_\alpha : [t_0, T] \to \mathbb{C}$ are defined by

$$\mu_{\alpha\beta}(t) = \langle \alpha(t), \pi(t)\beta(t)\rangle_\gamma,\ \nu_\beta(t) = \langle f(t), \beta(t)\rangle_\gamma,$$

$$\sigma_\alpha(t) = \langle \alpha(t), g(t)\rangle_\gamma,\ t \in [t_0, T]$$

for all $(t, x) \in [0, T] \times \tilde{\mathcal{A}}$ and the coefficients E, F, G, H belong to the space $L^2_{loc}([t_0, T] \times \tilde{\mathcal{A}})_{mvs}$ of multivalued stochastic processes with closed values.

As explained in [11], the map P cannot in general be written in the form:

$$P(t, x)(\eta, \xi) = \tilde{P}(t, \langle \eta, x\xi\rangle)$$

for some complex valued multifunction $\tilde{P}$ defined on $[t_0, T] \times \mathbb{C}$, for $t \in [t_0, T],\ x \in \tilde{\mathcal{A}},\ \eta, \xi \in \mathbb{D}\underline{\otimes}\mathbb{E}$. Under the condition of compactness of the values of the map $(t, x) \to P(t, x)(\eta, \xi)$ for arbitrary $\eta, \xi \in \mathbb{D}\underline{\otimes}\mathbb{E}$, we prove that the map which associates to the initial point $a \in \tilde{\mathcal{A}}$, the set of solutions $S^{(T)}(a)$ to (2.4) admits a continuous selection from the space $\tilde{\mathcal{A}}$ to the completion (denoted by $wac(\tilde{\mathcal{A}})$) of the locally convex space of adapted weakly absolutely continuous stochastic processes indexed by elements of the interval $[t_0, T]$. In particular, we show that the map $a \to R^{(T)}(a)$ admits a continuous selection, where $R^{(T)}(a)$ is the reachable set at $t = T$ of the QSDI (2.3).

To establish our main results, we need the notion of partition of unity subordinate to any covering of a compact subset of $\tilde{\mathcal{A}}$ corresponding to an arbitrary pair of vectors in $\mathbb{E}$, the subspace of the Fock space generated by the exponential vectors. In what follows, unless otherwise indicated, we consider quantum stochastic processes defined on a simple Fock space. That is we shall take the initial space $\mathcal{R} = \mathbb{C}$ so that $\mathcal{R} \otimes \Gamma(L^2_\gamma(\mathbb{R}_+)) \equiv \Gamma(L^2_\gamma(\mathbb{R}_+))$ and $\mathbb{D}\underline{\otimes}\mathbb{E} \equiv \mathbb{E}$.

Definition 2.1. . Let A be a compact subset of the locally convex space $\tilde{\mathcal{A}}$ and let $\{\Omega_i\}_{i\in J}$ be an open covering for A with a finite sub covering $\{\Omega_i,\ \ i = 1, 2, \ldots, m\}$. A family of functions $\{\Pi_{\eta\xi,i}(\cdot)\},\ \ i = 1, 2, \ldots, m$ corresponding to an arbitrary pair of elements $\eta,\ \xi\ \in \mathbb{E}$ defined on A is called a Lipschitzian partition of unity subordinate to the finite subcovering if:

(1) The map $\Pi_{\eta\xi,i}(\cdot)$ is Lipschitzian for all $i = 1, 2, \ldots, m$. That is there exist constants $L_{\eta\xi} > 0$ such that for any pair $a,\ a'\ \in A$,

$$|\Pi_{\eta\xi,i}(a) - \Pi_{\eta\xi,i}(a')| \leq L_{\eta\xi}\|a - a'\|_{\eta\xi}.$$

(2) $\Pi_{\eta\xi,i}(a) > 0$ for $a \in \Omega_i \bigcap A$ and $\Pi_{\eta\xi,i}(a) = 0$ for $a \in A\backslash\Omega_i$.

(3) For each $a \in A$, $\sum_{i=1}^m \Pi_{\eta\xi,i}(a) = 1$.

Lemma 2.2. . *Let A be a compact subset of the space $\tilde{\mathcal{A}}$. Then, there exists a family of Lipschitzian partitions of unity subordinate to any finite subcovering of an open covering for the set A.*

Proof. We outline the proof as follows: Let $\{\Omega_i\}, i = 1, 2, \ldots, m$ be a finite open subcovering of an open covering $\{\Omega_i\}_{i\in J}$ of A. First we claim that the map $q_{\eta\xi} : \tilde{\mathcal{A}} \to \mathbb{R}_+$ defined by

$$q_{\eta\xi}(x) = \mathbf{d}_{\eta\xi}(x, Q), \quad Q \in clos(\tilde{\mathcal{A}}),$$

satisfies for any pair $x_1, x_2 \in \tilde{\mathcal{A}}$,

$$|q_{\eta\xi}(x_1) - q_{\eta\xi}(x_2)| \leq \|x_1 - x_2\|_{\eta\xi}. \tag{2.5}$$

Inequality (2.5) can be established as follows: Let $\epsilon > 0$ be given. Since $\mathbf{d}_{\eta\xi}(x, Q) = \inf_{y\in Q} \|x - y\|_{\eta\xi}$, then there exists $y_1 \in Q$ satisfying

$$\|x_1 - y_1\|_{\eta\xi} \leq \mathbf{d}_{\eta\xi}(x_1, Q) + \epsilon.$$

Hence,

$$\begin{aligned}
\mathbf{d}_{\eta\xi}(x_2, Q) &\leq \|x_2 - y_1\|_{\eta\xi} \\
&\leq \|x_2 - x_1\|_{\eta\xi} + \|x_1 - y_1\|_{\eta\xi} \\
&\leq \|x_2 - x_1\|_{\eta\xi} + \mathbf{d}_{\eta\xi}(x_1, Q) + \epsilon.
\end{aligned}$$

Interchanging x_1 and x_2, we have

$$|\mathbf{d}_{\eta\xi}(x_1, Q) - \mathbf{d}_{\eta\xi}(x_2, Q)| \leq \|x_1 - x_2\|_{\eta\xi} + \epsilon.$$

Inequality (2.5) follows since ϵ is arbitrary.

For $i = 1, 2, \ldots, m$, define the family of functions $q_{\eta\xi,i} : A \to \mathbb{R}_+$ by

$$q_{\eta\xi,i}(a) = \mathbf{d}_{\eta\xi}(a, A\backslash\Omega_i)$$

and functions $\Pi_{\eta\xi,i} : A \to \mathbb{R}_+$ defined by

$$\Pi_{\eta\xi,i}(a) = \frac{q_{\eta\xi,i}(a)}{\sum_{j=1}^{m} q_{\eta\xi,j}(a)} \tag{2.6}$$

For at least one $j \in \{1, 2, \ldots, m\}$, $a \in \Omega_j$. Hence, $\sum_{j=1}^{m} q_{\eta\xi,j}(a) > 0$. Also, by the definition of the seminorm $\|\cdot\|_{\eta\xi}$ and the properties of the exponential vectors $\eta, \xi \in \mathbb{E}$, the value $\|x\|_{\eta\xi}$ can never be zero when x is not a zero process. This follows from the fact that for any pair of exponential vectors $\eta, \xi \in \mathbb{E}$ such that $\eta = e(\alpha)$, $\xi = e(\beta)$, α, $\beta \in L^2_\gamma(\mathbb{R}_+)$, we have $\langle e(\alpha), e(\beta)\rangle = e^{\langle\alpha,\beta\rangle}$ (see [6] for some details). Consequently, (2.6) is well defined. The rest of the proof follows a similar argument as in the proof of Lemma 2.1 in [4]. This shows that $\{\Pi_{\eta\xi,i}(\cdot)\}_{i=1}^{m}$ is a family of Lipschitzian partition of unity subordinate to the covering. □

In the proof of our main results, we shall make use of the following maps that are associated with the family $\Pi_{\eta\xi,i}(\cdot))$ given by (2.6). Define the maps

$$\sigma_{\eta\xi}(i,a) = \sum_{1\leq j\leq i} \Pi_{\eta\xi,j}(a), \quad a \in A, \ i \in \{1,2,\dots,m\}. \tag{2.7}$$

Definition 2.3: Let $\epsilon > 0$ be fixed. Then the common modulus of continuity $\Theta_{\eta\xi}(\epsilon)$ depending on the pair $\eta, \xi \in \mathbb{E}$, of the map $a \to \sigma_{\eta\xi}(i,a)$ is defined by:

$$\Theta_{\eta\xi}(\epsilon) = \sup\{|\sigma_{\eta\xi}(i,a) - \sigma_{\eta\xi}(i,a')| : a, a' \in A, \ \|a - a'\|_{\eta\xi} \leq \epsilon, \ i = 1,2,\dots,m\}. \tag{2.8}$$

Remarks: As in the case of the modulus of continuity of real valued functions defined on the real line, (see [21, p. 2], for example), the modulus of continuity $\Theta_{\eta\xi}(\epsilon)$ defined by (2.8) satisfies the following inequalities as consequences of the definition. That is,

$$\Theta_{\eta\xi}(\epsilon) \leq \Theta_{\eta\xi}(\epsilon'), \quad \text{whenever} \ \ \epsilon \leq \epsilon'$$

and

$$\Theta_{\eta\xi}(\lambda\epsilon) \leq (1+\lambda)\Theta_{\eta\xi}(\epsilon), \quad \text{for any positive number } \lambda. \tag{2.9}$$

These follow directly from (2.8).

In what follows, we shall employ the space of complex valued sesquilinear forms on $(\mathbb{D}\underline{\otimes}\mathbb{E})^2$ denoted by $Sesq(\mathbb{D}\underline{\otimes}\mathbb{E})$ and assume that the multivalued map $(t,x) \to P(t,x)(\eta,\xi)$ appearing in Equation (2.4) satisfies the following conditions:

$\mathcal{S}$(a). $P : \Omega \subseteq [t_0,T] \times \tilde{\mathcal{A}} \to 2^{Sesq(\mathbb{D}\underline{\otimes}\mathbb{E})}$ defined on an open subset $\Omega \subseteq [t_0,T] \times \tilde{\mathcal{A}}$ bounded on Ω by constants $M_{\eta\xi}$ that depend on η, ξ, i.e

$$|P(t,x)(\eta,\xi)| \leq M_{\eta\xi}, \quad (t,x) \in \Omega, \ \eta,\xi \in \mathbb{D}\underline{\otimes}\mathbb{E}.$$

$\mathcal{S}$(b). The map $t \to P(t,x)(\eta,\xi)$ is measurable for fixed $x \in \tilde{\mathcal{A}}$ and for all $\eta,\xi \in \mathbb{D}\underline{\otimes}\mathbb{E}$.

$\mathcal{S}$(c). The map $(t,x) \to P(t,x)(\eta,\xi)$ is Lipschitzian with Lipschitz function $K_{\eta\xi}(t)$ lying in $L^1_{loc}([t_0,T])$, i.e. for $x, y \in \tilde{\mathcal{A}}$

$$\rho(P(t,x)(\eta,\xi), \ P(t,y)(\eta,\xi)) \leq K_{\eta\xi}(t)\|x-y\|_{\eta\xi}.$$

$\mathcal{S}$(d). The set $P(t,x)(\eta,\xi)$ is compact in $\mathbb{C}$, the field of complex numbers, for all $(t,x) \in \Omega, \ \eta,\xi \in \mathbb{D}\underline{\otimes}\mathbb{E}$.

$\mathcal{S}$(e). There exists a compact set $A \subseteq \tilde{\mathcal{A}}$ such that $\forall \eta,\xi \in \mathbb{D}\underline{\otimes}\mathbb{E}$, the set

$$\{(t, a + v(t-t_0) : a \in A, \ v \in \tilde{\mathcal{A}} \text{ such that } \ \|v\|_{\eta\xi} \leq M_{\eta\xi}, \ t \in [t_0,T]\} \subseteq \Omega.$$

Moreover, we set

$$Y_{\eta\xi}(t) = \int_{t_0}^{t} K_{\eta\xi}(s)ds. \tag{2.10}$$

We shall assume that the interval $I = [t_0,T]$ satisfies the following:

$$\Lambda_{\eta\xi} = 3(e^{Y_{\eta\xi} - Y_{\eta\xi}(s)} - 1) < 1; \quad \forall \ \eta,\xi \in \mathbb{D}\underline{\otimes}\mathbb{E}, \tag{2.11}$$

where

$$Y_{\eta\xi} = \int_{t_0}^{T} K_{\eta\xi}(s)ds.$$

In what follows, we set

$$\Gamma_{\eta\xi} = \int_{t_0}^{T} e^{Y_{\eta\xi} - Y_{\eta\xi}(s)}ds.$$

3. ESTABLISHMENT OF THE SELECTION MAP

By a solution of QSDI (2.3) we mean a quantum stochastic process $\Phi : [t_0, T] \to \tilde{\mathcal{A}}$ lying in $Ad(\tilde{\mathcal{A}})_{wac} \bigcap L^2_{loc}(\tilde{\mathcal{A}})$ satisfying QSDI (2.3). We denote by $S^{(T)}(a)$, the set of solutions of Lipschitzian QSDI (2.3). It has been established in [11] that under the conditions $\mathcal{S}(a) - \mathcal{S}(e)$, this set is not empty. Similar existence result under a general Lipschitz condition has recently been established in [3]. Our main result below shows that there exists a continuous map $\tilde{\Phi} : A \to wac(\tilde{\mathcal{A}})$ such that for each $a \in A$, $\tilde{\Phi}(a) \in S^{(T)}(a) \subseteq wac(\tilde{\mathcal{A}})$.

Theorem 3.1. *Suppose that the map $(t, x) \to P(t, x)(\eta, \xi)$ satisfies the assumptions* $\mathcal{S}$(a)–$\mathcal{S}$(e). *Then there exists a continuous map $\tilde{\Phi} : A \to wac(\tilde{\mathcal{A}})$ such that for every $a \in A$, $\tilde{\Phi}(a)$ is a solution to the QSDI* (2.4).

Proof. The proof shall be presented in six parts in what follows. The pair of elements $\eta, \xi \in \mathbb{E}$ are arbitrary unless otherwise indicated. We note here that it would be enough for us to establish the existence of the continuous selection by establishing appropriate estimates in the seminorms that generate the topology of the spaces $\tilde{\mathcal{A}}$ and $wac(\tilde{\mathcal{A}})$. A justification for this can be found in [23, p. 5].

Part A: We claim that there exists two sequences of adapted stochastic processes $\Phi^n(a),\ \Psi^n(a) : [t_0, T] \to \tilde{\mathcal{A}}$ such that

(i) $\Psi^n(a) \in S^{(T)}(a)$; $\Phi^n(a)$ is adapted weakly absolutely continuous such that $\Phi^n(a)(t_0) = a$. Setting $\Phi^n_{\eta\xi}(a)(t) := \langle \eta, (\Phi^n(a)(t))\xi\rangle$, then,

(ii) $\|\Phi^n_{\eta\xi}(a) - \Psi^n_{\eta\xi}(a)\|_c = \sup_I |\langle \eta, (\Phi^n(a)(t))\xi\rangle - \langle \eta, (\Psi^n(a)(t))\xi\rangle| \leq M_{\eta\xi}\Gamma_{\eta\xi}\Lambda^{n-1}_{\eta\xi}$.

(iii) For every $\epsilon > 0$, there exists $\delta(\epsilon) = \delta(\epsilon, n, \eta, \xi) > 0$ and a function $R^n_{\eta\xi}(a, \epsilon) : I \to \mathbb{R}_+$ satisfying

$$\int_I R^n_{\eta\xi}(a, \epsilon)(s)ds \leq 2M_{\eta\xi}\epsilon \tag{3.1}$$

such that

$$|\frac{d}{dt}\langle \eta, (\Phi^n(a)(t))\xi\rangle - \frac{d}{dt}\langle \eta, (\Phi^n(a')(t))\xi\rangle| \leq R^n_{\eta\xi}(a, \epsilon)(t)$$

whenever $\|a - a'\|_{\eta\xi} \leq \delta(\epsilon)$.

(iv) $|\frac{d}{dt}\langle \eta, (\Phi^n(a)(t))\xi\rangle - \frac{d}{dt}\langle \eta, (\Psi^n(a)(t))\xi\rangle| \leq 3M_{\eta\xi}\Gamma_{\eta\xi}\Lambda^{n-2}_{\eta\xi}K_{\eta\xi}(t)e^{Y_{\eta\xi}(t)},\ n \geq 2.$

(v) $|\Phi^n(a) - \Phi^{n-1}(a)|_{\eta\xi} \leq 3M_{\eta\xi}\Gamma_{\eta\xi}\Lambda^{n-1}_{\eta\xi},\ \ n \geq 3.$

Part B: We apply mathematical induction as follows: Set $\Phi^1(a) = a$. Then trivially, $\Phi^1(a)$ lies in $Ad(\tilde{\mathcal{A}})_{wac}$. Also by the boundedness of the map P,

$$d\left(\frac{d}{dt}\langle\eta,(\Phi^1(a)(t))\xi\rangle,\ P(t,\Phi^1(t))(\eta,\xi)\right) = d(0,P(t,a)(\eta,\xi)) \leq M_{\eta\xi}.$$

By the existence results of Ekhaguere [11], there exists $\Psi^1(a) \in S^{(T)}(a)$ such that $\forall t \in [t_0,T]$,

$$\|\Phi^1(a)(t) - \Psi^1(a)(t)\|_{\eta\xi} \leq \int_{t_0}^{t} e^{(Y_{\eta\xi}(t)-Y_{\eta\xi}(s))} M_{\eta\xi}ds \leq M_{\eta\xi}\Gamma_{\eta\xi}.$$

The above shows that $\Phi^1,\ \Psi^1$ satisfy items (i), (ii) in Part A, with n=1. Item (iii) also holds by putting $R^1_{\eta\xi}(a,\epsilon) = 0$ for n=1.
Assume that we have defined $\Phi^\nu(a)$ and $\Psi^\nu(a)$ satisfying items (i) – (iii), for $\nu = 1,2,\ldots,n-1$. We claim that we can define $\Phi^n(a)$ and $\Psi^n(a)$ satisfying items (i) – (iv) for $n \geq 2$.

Part C: For notational simplification, we will denote Φ^{n-1} by Φ and Ψ^{n-1} by Ψ. The map $\Phi_{\eta\xi} : A \to C[I,\mathbb{C}]$, $a \to \Phi_{\eta\xi}(a)$ is uniformly continuous on account of our assumption in Part A above. This can be shown as follows:
Let $r > 0$ be a real number satisfying $r \leq \delta(\Gamma_{\eta\xi}\Lambda^{n-1}_{\eta\xi})$, where δ is defined in Part A, item (iii) above. Then $a',\ a''$ lying in the set $B[a,r] = \{x \in \tilde{\mathcal{A}} : \|x-a\|_{\kappa\vartheta} \leq r,\ \forall\kappa,\vartheta \in \mathbb{D}\underline{\otimes}\mathbb{E}\}$ implies that $\|a-a'\|_{\eta\xi} \leq r \leq \delta$ and $\|a-a''\|_{\eta\xi} \leq r \leq \delta$.

By item (iii), Part A,

$$|\frac{d}{dt}\langle\eta,(\Phi(a)(t))\xi\rangle - \frac{d}{dt}\langle\eta,(\Phi(a'(t))\xi\rangle| \leq R^{n-1}_{\eta\xi}(a,\epsilon)(t)$$

and

$$|\frac{d}{dt}\langle\eta,(\Phi(a)(t))\xi\rangle - \frac{d}{dt}\langle\eta,(\Phi(a''(t))\xi\rangle| \leq R^{n-1}_{\eta\xi}(a,\epsilon)(t)$$

so that

$$|\frac{d}{dt}\langle\eta,(\Phi(a')(t))\xi\rangle - \frac{d}{dt}\langle\eta,(\Phi(a''(t))\xi\rangle| \leq 2R^{n-1}_{\eta\xi}(a,\epsilon)(t). \tag{3.2}$$

But by the absolute continuity of the map $t \to (\langle\eta,\Phi(a')(t)\xi\rangle - \langle\eta,\Phi(a'')(t)\xi\rangle)$, we have

$$\begin{aligned}&|\langle\eta,(\Phi(a')(t))\xi\rangle - \langle\eta,(\Phi(a''(t))\xi\rangle| \\ &\qquad = |\int_I \frac{d}{ds}\left(\langle\eta,(\Phi(a')(s))\xi\rangle - \langle\eta,(\Phi(a''(s))\xi\rangle\right)ds|\end{aligned} \tag{3.3}$$

Hence from (3.3) and using (3.1)

$$|\langle\eta,\Phi(a')(t)\xi\rangle - \langle\eta,\Phi(a'')(t)\xi\rangle| \leq 2\int_I R^{n-1}_{\eta\xi}(a,\epsilon)(s)ds \leq 4M_{\eta\xi}\epsilon.$$

If $r \leq \frac{1}{3}M_{\eta\xi}\Gamma_{\eta\xi}\Lambda^{n-2}_{\eta\xi}$, then $\|a'-a''\|_{\eta\xi} \leq 2r \leq \frac{2}{3}M_{\eta\xi}\Gamma_{\eta\xi}\Lambda^{n-2}_{\eta\xi}$, implies that

$$\|\Phi(a')(t) - \Phi(a'')(t)\|_{\eta\xi} \leq \frac{1}{3}M_{\eta\xi}\Gamma_{\eta\xi}\Lambda^{n-2}_{\eta\xi},$$

where ϵ is small enough so that

$$\epsilon \leq \frac{1}{12}\Gamma_{\eta\xi}\Lambda_{\eta\xi}^{n-1} \leq \frac{1}{12}\Gamma_{\eta\xi}\Lambda_{\eta\xi}^{n-2}.$$

Consequently, we have for $a',\ a'' \in B[a,r]$

$$\|\Phi_{\eta\xi}(a') - \Phi_{\eta\xi}(a'')\|_c \leq \frac{1}{3}M_{\eta\xi}\Gamma_{\eta\xi}\Lambda_{\eta\xi}^{n-2}.$$

Our claim of uniform continuity of the map $a \rightarrow \Phi_{\eta\xi}(a)$ follows.

Let $\{B(a_i,r), i = 1,2,\ldots,m\}$ be a finite open cover of the compact set A, $a_i \in A\ \forall i$ and $\Pi_{\eta\xi,i} : A \rightarrow \mathbb{R}_+$, a partition of unity subordinate to the cover. Here

$$B(a,r) = \{x \in \tilde{\mathcal{A}} : \|x - a\|_{\kappa\vartheta} < r, \forall \kappa, \vartheta \in \mathbb{D}\underline{\otimes}\mathbb{E}\}$$

and

$$\sum_{i=1}^{m} \Pi_{\eta\xi,i}(a) = 1,\ \ \Pi_{\eta\xi,i}(a) > 0,\ \ \forall a \in A\bigcap B(a_i,r).$$

The existence of such family of Lipschitzian partition of unity follows from Lemma 2.2.

Next, we define

$$\sigma_{\eta\xi}(j,a) = \sum_{1\leq i\leq j} \Pi_{\eta\xi,i}(a) \ \text{ and } \ \Psi_i(t) = \Psi(a_i)(t).$$

Let $\delta > 0$ be such that $\frac{T-t_0}{\delta} = m'$, an integer and $\delta < \frac{1}{12}\Gamma_{\eta\xi}\Lambda_{\eta\xi}^{n-2}$.

The subintervals

$$J(j) = [t_0 + (j-1)\delta, t_0 + j\delta),\ j = 1,2,\ldots,m'$$

form a partition of the interval $I = [t_0,T]$. Corresponding to an arbitrary pair of elements $\eta,\xi \in \mathbb{E}$, we consider the family of complex valued maps on $[t_0,T]$) defined by:

$$D_{\eta\xi,i,j}(t) = \frac{d}{dt}\langle\eta, \Psi_i(t)\xi\rangle I_{J(j)}(t), \quad i = 1,2\ldots,m;\ j = 1,2,\ldots,m', \tag{3.4}$$

where $I_{J(j)}$ is the characteristic function on the set $J(j)$. For $\alpha \in [0,1]$, let $\{B(\alpha)\}$ be a nested family of measurable subsets of the interval $[t_0,T]$ such that $B(0) = \emptyset,\ B(1) = [t_0,T]$ satisfying

$$\int_{B(\alpha)} D_{\eta\xi,i,j}(t)dt = \alpha\int_{t_0}^{T} D_{\eta\xi,i,j}(t)dt,\ \ \mu(B(\alpha)) = \alpha(T-t_0). \tag{3.5}$$

Such a family exists by a Corollary to Liapunov's theorem (see [1, 15]).

Since $\Psi_i \in S^{(T)}(a_i)$ then as shown in [11], there exists processes $V_i : I \rightarrow \tilde{\mathcal{A}}$ lying in $L^1_{loc}(\tilde{\mathcal{A}})$ such that $\Psi_i(t) = a_i + \int_{t_0}^{t} V_i(s)ds$ and

$$\frac{d}{dt}\langle\eta, \Psi_i(t)\xi\rangle = \langle\eta, V_i(t)\xi\rangle.$$

It follows from (3.4) that

$$D_{\eta\xi,i,j}(t) = \langle\eta, V_i(t)I_{J(j)}(t)\xi\rangle, \quad i = 1,2,\ldots,m;\ j = 1,2,\ldots,m'.$$

Hence by (3.5) and putting

$$V_{i,j}(t) = V_i(t)I_{J(j)}(t), \quad i = 1, 2, \ldots, m,\ j = 1, 2, \ldots, m',$$

we have

$$\int_{B(\alpha)} V_{i,j}(t)dt = \alpha \int_{t_0}^{T} V_{i,j}(t)dt. \tag{3.6}$$

Next we define the stochastic process $\Phi^n(a) : [t_0, T] \to \tilde{\mathcal{A}}$ by

$$\Phi^n(a)(t) = a + \sum_i \int_{t_0}^{t} V_i(s)I_{B(\sigma_{\eta\xi}(i,a))\backslash B(\sigma_{\eta\xi}(i-1,a))}(s)ds, \tag{3.7}$$

with its matrix element given by

$$\langle \eta, (\Phi^n(a)(t))\xi\rangle = \langle \eta, a\xi\rangle + \sum_i \int_{t_0}^{t} \langle \eta, (V_i(s))\xi\rangle I_{B(\sigma_{\eta\xi}(i,a))\backslash B(\sigma_{\eta\xi}(i-1,a))}(s)ds.$$

We remark that the process $\Phi^n(a)$ given by (3.7) lies in $wac(\tilde{\mathcal{A}})$ since each $V_i \in L^1_{loc}(\tilde{\mathcal{A}})$ and in addition, $\Phi^n(a)$ is an adapted and weakly absolutely continuous process.

To show that $\Phi^n(a)$ satisfies item (iii) of Part A, we note that as in the proof of the only Theorem in [1], $\frac{d}{dt}\langle \eta, \Phi^n(a)\xi\rangle$ and $\frac{d}{dt}\langle \eta, \Phi^n(a')\xi\rangle$ differ only on the subset $E' \subset [t_0, T]$ given by

$$E' = \bigcup_{i=1}^{m} \{(B(\sigma_{\eta\xi}(i,a))\backslash B(\sigma_{\eta\xi}(i-1,a)))\triangle(B(\sigma_{\eta\xi}(i,a'))\backslash B(\sigma_{\eta\xi}(i-1,a')))\}$$

and that

$$E' \subset \bigcup_{i=1}^{m} \{B(\sigma_{\eta\xi}(i,a))\triangle B(\sigma_{\eta\xi}(i,a'))\}, \tag{3.8}$$

where for any two subsets S, B of $[t_0, T]$, $S\triangle B := (S \cup B)\backslash(S \cap B)$.

As in [1], we fix $\epsilon > 0$ and let $\Theta_{\eta\xi} = \Theta_{\eta\xi}(\epsilon)$ be the common modulus of continuity of the map $a \to \sigma_{\eta\xi}(i, a)$, given by (2.8). Then, whenever $\|a - a'\|_{\eta\xi} < \Theta_{\eta\xi}(\frac{\epsilon}{2m})$, the superset in (3.8) is contained in the set

$$E''(a,\epsilon) = \bigcup_{i=1}^{m} \{B(\sigma_{\eta\xi}(i,a) + \frac{\epsilon}{2m})\backslash B(\sigma_{\eta\xi}(i,a) - \frac{\epsilon}{2m})\} \tag{3.9}$$

and the total measure of $E''(a, \epsilon)$ is bounded by ϵ or

$$\int_I I_{E''(a,\epsilon)} < \epsilon. \tag{3.10}$$

The foregoing assertion follows from the fact that if

$$\|a - a'\|_{\eta\xi} < \Theta_{\eta\xi}(\frac{\epsilon}{2m}),$$

then

$$|\sigma_{\eta\xi}(i,a) - \sigma_{\eta\xi}(i,a')| \leq \Theta_{\eta\xi}\left(\Theta_{\eta\xi}(\frac{\epsilon}{2m})\right) \leq \Theta_{\eta\xi}\left(\Theta_{\eta\xi}(\epsilon)\right).$$

Since $\Theta_{\eta\xi}(\epsilon)$ is positive and finite, we can write $\Theta_{\eta\xi}(\epsilon) = \lambda_{\eta\xi}\epsilon$ for some $\lambda_{\eta\xi} > 0$. Then, by (2.9),

$$|\sigma_{\eta\xi}(i,a) - \sigma_{\eta\xi}(i,a')| \leq (1+\lambda_{\eta\xi})\lambda_{\eta\xi}\epsilon = \frac{\epsilon}{2m}.$$

Thus,

$$|\sigma_{\eta\xi}(i,a) - \sigma_{\eta\xi}(i,a')| \leq \frac{\epsilon}{2m}, \tag{3.11}$$

for some positive number $\lambda_{\eta\xi}$ satisfying the algebraic equation

$$\lambda_{\eta\xi}^2 + \lambda_{\eta\xi} - \frac{1}{2m} = 0.$$

The claim follows by employing (3.11) and the property of the nested family of sets $\{B(\cdot)\}$.

Consequently we have

$$|\frac{d}{dt}\langle\eta, \Phi^n(a)(t)\xi\rangle - \frac{d}{dt}\langle\eta, \Phi^n(a')(t)\xi\rangle| \leq 2M_{\eta\xi}I_{E''(a,\epsilon)}(t)$$

so that item (iii) in Part A follows with

$$\delta(\epsilon) = \Theta_{\eta\xi}(\frac{\epsilon}{2m}) \text{ and } R^n_{\eta\xi}(a,\epsilon)(t) = 2M_{\eta\xi}I_{E''(a,\epsilon)}(t).$$

Part D: We estimate here the pseudo-distance of $\Phi^n(a)$ from the set of solution $S^{(T)}(a)$. To this end, let $t \in [t_0 + r\delta, t_0 + (r+1)\delta))$. At the point $t = t_0 + r\delta$, the integral in (3.7) can be written as

$$\begin{aligned}
& \sum_i \int_{t_0}^{t_0+r\delta} V_i(s) I_{B(\sigma_{\eta\xi}(i,a))\setminus B(\sigma_{\eta\xi}(i-1,a))} ds \\
= \;& \sum_i \sum_{l\leq r} \int V_i(s) I_{B(\sigma_{\eta\xi}(i,a))\setminus B(\sigma_{\eta\xi}(i-1,a))} I_{J(l)}(s) ds \\
= \;& \sum_i \sum_{l\leq r} \int V_{i,l}(s) I_{B(\sigma_{\eta\xi}(i,a))\setminus B(\sigma_{\eta\xi}(i-1,a))}(s) ds \\
= \;& \sum_i \sum_{l\leq r} \int_{B(\sigma_{\eta\xi}(i,a))\setminus B(\sigma_{\eta\xi}(i-1,a))} V_{i,l}(s) ds \\
= \;& \sum_{l\leq r} \sum_i \Pi_{\eta\xi,i}(a) \int_I V_{i,l}(s) ds, \\
= \;& \sum_i \sum_{l\leq r} \Pi_{\eta\xi,i}(a)\{\Psi_i(t_0 + l\delta) - \Psi_i(t_0 + (l-1)\delta)\} \\
= \;& \sum_i \Pi_{\eta\xi,i}(a)\{\Psi_i(t_0 + r\delta) - \Psi_i(t_0)\}.
\end{aligned}$$

This follows from (3.6) and the definition of $\sigma_{\eta\xi}(\cdot,\cdot)$.

Hence, we have

$$\Phi^n(a)(t_0 + r\delta) - a = \sum_i \Pi_{\eta\xi,i}(a)(\Psi_i(t_0 + r\delta) - a_i)$$

For any $j \in \{1, 2, \ldots, m\}$, we can write

$$\|\Phi^n(a)(t) - \Psi_j(t)\|_{\eta\xi} \leq \|\Phi^n(a)(t_0 + r\delta) - \Psi_j(t_0 + r\delta)\|_{\eta\xi}$$
$$+ \|\Phi^n(a)(t_0 + r\delta) - \Phi^n(a)(t)\|_{\eta\xi} + \|\Psi_j(t) - \Psi_j(t_0 + r\delta)\|_{\eta\xi} \tag{3.12}$$

Since

$$|\frac{d}{dt} < \eta, \Phi^n(a)(t)\xi > | \leq M_{\eta\xi}$$

and

$$|\frac{d}{dt} < \eta, \Psi_j(t)\xi > | \leq M_{\eta\xi},$$

by our choice of δ, the sum of the last two terms in (3.12) is bounded by $\frac{1}{3}M_{\eta\xi}\Gamma_{\eta\xi}\Lambda_{\eta\xi}^{n-2}$. Hence, from (3.12)

$$\|\Phi^n(a)(t) - \Psi_j(t)\|_{\eta\xi} \leq \|a - \sum_i \Pi_{\eta\xi,i}(a)a_i\|_{\eta\xi}$$
$$+ \| \sum_i \Pi_{\eta\xi,i}(a) \left(\Psi_i(t_0 + r\delta) - \Psi_j(t_0 + r\delta)\right) \|_{\eta\xi} + \frac{1}{3}M_{\eta\xi}\Gamma_{\eta\xi}\Lambda_{\eta\xi}^{n-2}. \tag{3.13}$$

By our choice of r in Part C, whenever $\Pi_{\eta\xi,i}(a) > 0$, then

$$\|a - a_i\|_{\eta\xi} \leq \frac{1}{3}M_{\eta\xi}\Gamma_{\eta\xi}\Lambda_{\eta\xi}^{n-2}.$$

This estimate also holds for the first term at the right hand side of (3.13). Furthermore,

$$\|\Psi_i(t_0 + r\delta) - \Psi_j(t_0 + r\delta)\|_{\eta\xi}$$
$$\leq \|\Psi_i(t_0 + r\delta) - \Phi(a_i)(t_0 + r\delta)\|_{\eta\xi} + \|\Phi(a_i)(t_0 + r\delta) - \Phi(a_j)(t_0 + r\delta)\|_{\eta\xi}$$
$$+ \|\Phi(a_j)(t_0 + r\delta) - \Psi_j(t_0 + r\delta)\|_{\eta\xi}. \tag{3.14}$$

When both $\Pi_{\eta\xi,i}(a) > 0$ and $\Pi_{\eta\xi,j}(a) > 0$ and by the choice of r, the second term on the right of (3.14) satisfies

$$\|\Phi(a_i)(t_0 + r\delta) - \Phi(a_j)(t_0 + r\delta)\|_{\eta\xi} \leq \frac{1}{3}M_{\eta\xi}\Gamma_{\eta\xi}\Lambda_{\eta\xi}^{n-2}, \tag{3.15}$$

so that by item (ii) in Part A and the recursive assumption, we finally have

$$\|\Phi^n(a)(t) - \Psi_j(t)\|_{\eta\xi} \leq 3M_{\eta\xi}\Gamma_{\eta\xi}\Lambda_{\eta\xi}^{n-2}, \tag{3.16}$$

Equation (3.16) holds for every j such that $\Pi_{\eta\xi,j}(a) > 0$. By the definition of $\Phi^n(a)(t)$ given by (3.7), at any point t except on a set of measure zero in I,

$$\frac{d}{dt}\langle\eta, \Phi^n(a)(t)\xi\rangle = \frac{d}{dt}\langle\eta, \Psi_j(t)\xi\rangle$$

for some j such that $\Pi_{\eta\xi,j}(a) > 0$.

Since $\Psi_j \in S^{(T)}(a_j)$, then

$$\frac{d}{dt}\langle\eta, \Psi_j(t)\xi\rangle \in P(t, \Psi_j(t))(\eta, \xi)$$

and therefore we have

$$\begin{aligned} d\left(\frac{d}{dt}\langle\eta,\Phi^n(a)(t)\xi\rangle,\ P(t,\Phi^n(a)(t))(\eta,\xi)\right) &\le \rho\left(P(t,\Psi_j(t))(\eta,\xi),\ P(t,\Phi^n(a)(t))(\eta,\xi)\right) \\ &\le K_{\eta\xi}(t)\|\Psi_j(t)-\Phi^n(a)(t)\|_{\eta\xi} \\ &\le 3M_{\eta\xi}\Gamma_{\eta\xi}\Lambda_{\eta\xi}^{n-2}K_{\eta\xi}(t) \end{aligned} \tag{3.17}$$

on account of (3.16) and the fact that the map $(t,x)\to P(t,x)(\eta,\xi)$ is Lipschitzian. We notice that estimate (3.17) is independent of j and therefore holds on $I=[t_0,T]$. Again by the existence result in [11], there exists a stochastic process $\Psi^n(a)\in S^{(T)}(a)$ such that

$$\|\Psi^n(a)(t)-\Phi^n(a)(t)\|_{\eta\xi}\le 3M_{\eta\xi}\Gamma_{\eta\xi}\Lambda_{\eta\xi}^{n-2}(e^{Y_{\eta\xi}(t)}-1)\le M_{\eta\xi}\Gamma_{\eta\xi}\Lambda_{\eta\xi}^{n-1}, \tag{3.18}$$

and

$$|\frac{d}{dt}\langle\eta,\Psi^n(a)(t)\xi\rangle-\frac{d}{dt}\langle\eta,\Phi^n(a)(t)\xi\rangle|\le 3M_{\eta\xi}\Gamma_{\eta\xi}\Lambda_{\eta\xi}^{n-2}K_{\eta\xi}(t)e^{Y_{\eta\xi}(t)}. \tag{3.19}$$

Inequalities (3.18) and (3.19) prove items (ii) and (iv) in Part A for all $n\ge 2$.

Part E: It is now left for us to show that if items (i)–(iv) hold up to $n-1$, then item (v) holds for n. We use the same notations as before to fix any t and let j be such that

$$\frac{d}{dt}\langle\eta,\Phi^n(a)(t)\xi\rangle=\frac{d}{dt}\langle\eta,\Psi_j(t)\xi\rangle$$

so that $\Pi_{\eta\xi,j}(a)>0$. Then we have

$$\begin{aligned} |\frac{d}{dt}\langle\eta,\Phi^n(a)(t)\xi\rangle-\frac{d}{dt}\langle\eta,\Phi^{n-1}(a)(t)\xi\rangle| &= |\frac{d}{dt}\langle\eta,\Psi_j(t)\xi\rangle-\frac{d}{dt}\langle\eta,\Phi(a)(t)\xi\rangle| \\ &\le |\frac{d}{dt}\langle\eta,\Psi_j(t)\xi\rangle-\frac{d}{dt}\langle\eta,\Phi(a_j)(t)\xi\rangle| \\ &\quad +|\frac{d}{dt}\langle\eta,\Phi(a_j)(t)\xi\rangle-\frac{d}{dt}\langle\eta,\Phi(a)(t)\xi\rangle| \end{aligned} \tag{3.20}$$

By item (iv), the first term in (3.20) is bounded by $3M_{\eta\xi}\Gamma_{\eta\xi}\Lambda_{\eta\xi}^{n-2}K_{\eta\xi}(t)e^{Y_{\eta\xi}(t)}$ while by the choice of r, and applying item (iii), the second term in (3.20) is bounded by the functions $R_{\eta\xi}^{n-1}(a,\Gamma_{\eta\xi}\Lambda_{\eta\xi}^{n-1}):I\to\mathbb{R}_+$ satisfying the conditions of item (iii). These bounds do not depend on j and so hold on the whole of interval I.

Since

$$\int_I R_{\eta\xi}^{n-1}(a,\epsilon)(t)dt<2M_{\eta\xi}\epsilon,$$

we have

$$\begin{aligned} |\Phi^n(a)-\Phi^{n-1}(a)|_{\eta\xi} &= \int_I|\frac{d}{dt}\langle\eta,(\Phi^n(a)(t)-\Phi^{n-1}(a)(t))\xi\rangle|dt \\ &\le 3\int_I M_{\eta\xi}\Gamma_{\eta\xi}\Lambda_{\eta\xi}^{n-2}K_{\eta\xi}(t)e^{Y_{\eta\xi}(t)}dt+\int_I R_{\eta\xi}^{n-1}(a,\Gamma_{\eta\xi}\Lambda_{\eta\xi}^{n-1})(t)dt \\ &\le M_{\eta\xi}\Gamma_{\eta\xi}\Lambda_{\eta\xi}^{n-1}+2M_{\eta\xi}\Gamma_{\eta\xi}\Lambda_{\eta\xi}^{n-1}, \end{aligned}$$

proving item (v).

Part F: By item (iii) in Part A, we have

$$|\Phi^n(a) - \Phi^n(a')|_{\eta\xi} = \|a - a'\|_{\eta\xi} + \int_{t_0}^{T} \frac{d}{dt}|\langle \eta, \Phi^n(a)(t)\xi\rangle - \langle \eta, \Phi^n(a')(t)\xi\rangle|dt$$
$$\leq \delta(\epsilon) + 2M_{\eta\xi}\epsilon.$$

This shows that each map $\Phi^n : A \to wac(\tilde{\mathcal{A}})$ is uniformly continuous. Since $\Lambda_{\eta\xi} < 1$ for arbitrary pair $\eta, \xi \in \mathbb{D}\underline{\otimes}\mathbb{E}$, item (v) shows that the sequence $\{\Phi^n(a)\}$ is Cauchy. Since $wac(\tilde{\mathcal{A}})$ is complete, the sequence converges to a continuous map $\tilde{\Phi} : A \to wac(\tilde{\mathcal{A}})$.

By construction, the sequence $\{\frac{d}{dt}\langle \eta, \Phi^n(a)(t)\xi\rangle\}$ converges in $L^1[I]$ to $\frac{d}{dt}\langle \eta, \tilde{\Phi}(a)(t)\xi\rangle$. Hence, a subsequence converges to $\frac{d}{dt}\langle \eta, \tilde{\Phi}(a)(t)\xi\rangle$ pointwise almost everywhere.

By item (iv),

$$d\left(\frac{d}{dt}\langle \eta, \Phi^n(a)(t)\xi\rangle,\ P(t, \Phi^n(a)(t))(\eta, \xi)\right) \to 0 \quad \text{as} \quad n \to \infty.$$

Since the images $P(t, x)(\eta, \xi)$ are compact in the field of complex numbers, and therefore closed and since the map $(t, x) \to P(t, x)(\eta, \xi)$ is continuous, then we have:

$$\frac{d}{dt}\langle \eta, \tilde{\Phi}(a)(t)\xi\rangle \in P(t, \tilde{\Phi}(a)(t))(\eta, \xi)$$

showing that

$$\tilde{\Phi}(a) \in S^{(T)}(a) \subseteq wac(\tilde{\mathcal{A}}).$$

□

The next result is a direct consequence of Theorem 3.1 concerning the reachable sets of QSDI (2.3) at the time $t = T$ defined by:

$$R^{(T)}(a) = \{\Psi(a)(T) : \Psi(a) \in S^{(T)}(a)\} \subseteq \tilde{\mathcal{A}}. \tag{3.21}$$

Corollary 3.2. The multivalued map $R^{(T)} : A \to 2^{\tilde{\mathcal{A}}}$ admits a continuous selection where $R^{(T)}(a)$ is given by (3.21).

Proof. We define a continuous map $h : wac(\tilde{\mathcal{A}}) \to \tilde{\mathcal{A}}$ by

$$h(\Phi(\cdot)) = \Phi(T),\ \Phi(\cdot) \in wac(\tilde{\mathcal{A}}).$$

Thus, by Theorem (3.1), the map $h(\tilde{\Phi}(a)(\cdot)) = \tilde{\Phi}(a)(T)$ is continuous for each $a \in A$ and $\tilde{\Phi}(a)(T) \in R^{(T)}(a)$.

The conclusion of the corollary follows. □

REFERENCES

[1] Arrigo Cellina: On the set of Solutions to Lipschitzian Differential Inclusions, *Differential and Integral Equations*, 1(4), (1988), 495–500.

[2] Aubin, J. P; Cellina, A. *Differential Inclusions*, Springer-Verlag, Berlin, 1984.

[3] Ayoola, E. O. Quantum stochastic differential inclusions satisfying a general Lipschitz condition, *Dynamic Systems and Applications*, Vol 17 (2008) 487–502.

[4] Ayoola, E. O. Continuous selections of solution sets of Lipschitzian quantum stochastic differential inclusions, *International Journal of Theoretical Physics,* Vol 43 (10),(2004), 2041–2059.

[5] Ayoola, E. O. Error estimates for discretized quantum stochastic differential inclusions, *Stochastic Analysis and Applications,* Vol 21 (6) (2003), 1215–1230.

[6] Ayoola, E. O. Exponential formula for the reachable sets of quantum stochastic differential inclusions, *Stochastic Analysis and Applications* Vol 21 (3), (2003) 515–543.

[7] Ayoola, E. O. Construction of approximate attainability sets for Lipschitzian quantum stochastic differential inclusions, *Stochastic Analysis and Applications,* Vol 19 (3), (2001) 461–471.

[8] Broucke, M; Arapostathis,A: Continuous interpolation of solutions of Lipsschitz inclusions. *J. Math. Analysis and Appl.*, Vol 258, (2001), 565–572.

[9] Dontchev, A. L; Farkhi, E. M: Error estimates for discretized differential inclusions. *Computing,* Vol 41, (1989) 349–358.

[10] Dontchev, A; Lempio, F.: Difference methods for differential inclusions: A survey. *SIAM Reviews,* Vol 34 (2), (1994) 263–294.

[11] Ekhaguere, G. O. S. Lipschitzian quantum stochastic differential inclusions, *International Journal of Theoretical Physics* Vol 31 (11) (1992), 2003–2034.

[12] Ekhaguere, G. O. S. Quantum stochastic differential inclusions of hypermaximal monotone type, *International Journal of Theoretical Physics,* Vol 34 (3), (1995), 323–353.

[13] Ekhaguere, G.O.S. Quantum stochastic evolutions, *International Journal of Theoretical Physics 35*(9) (1996), 1909–1946.

[14] Filippov, A. F. Classical solutions of differential equations with multivalued right hand side, *SIAM Journal on Control* Vol 5,(1967), 609–621.

[15] Fryszkowski, A.: Continuous selections for a class of nonconvex multivalued maps, *Studia Math.* Vol 76, (1983), 163–174.

[16] Hermes, H; LaSalle, J.P: *Functional Analysis and Time Optimal Control*, Academic Press, (1969), New York.

[17] Hudson, R. L; Parthasarathy, K. R. Quantum Ito's formulae and stochastic evolutions, *Communications in Mathematical Physics,* Vol 93, (1984), 301–324

[18] Michal Kisielewicz: *Differential Inclusions and Optimal Control*, Mathematics and its Applications (East European Series), Volume 44, Kluwer Academic Publishers, Dordrecht, Boston, London, 1991.

[19] Parthasarathy, K. R. *An introduction to quantum stochastic calculus*, Monographs in Mathematics, Vol. 85, Birkhause Verlag, 1992.

[20] Repovs, D; Semenov, P.V. : *Continuous selections of multivalued mappings*. Mathematics and its Applications. Kluwer, Dordrecht, 1998.

[21] Sendov, B.; Popov, V. A: *The Averaged Moduli of Smoothness, With Applications in Numerical Methods and Approximations.* A Wiley-Interscience Series in Pure and Applied Mathematics, John Wiley, Chichester, NY, 1988.

[22] Smirnov, G. V.: *Introduction to the Theory of Differential Inclusions*. Graduate Studies in Mathematics, AMS, Providence, RI Volume 41, 2002.

[23] Yamamuro, S.: *A Theory of Differentiation in Locally Convex Spaces*, Memoirs of the AMS, Vol. 17, No. 212, AMS, Providence, RI, January, 1979.

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LIPSCHITZIAN QUANTUM STOCHASTIC DIFFERENTIAL EQUATIONS AND THE ASSOCIATED KURZWEIL EQUATIONS

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ABSTRACT

Kurzweil or generalized differential equations associated with Lipschitzian quantum stochastic differential equations (QSDEs) are introduced and studied. This is accomplished within the framework of the Hudson-Parthasarathy formulations of quantum stochastic calculus. Results concerning the equivalence of these classes of equations satisfying the Caratheodory conditions are presented. It is further shown that the associated Kurzweil equation may be used to obtain a reasonably high accurate approximate solutions of the QSDEs. This generalize analogous results for classical initial value problems to the noncommutative quantum setting involving unbounded linear operators on a Hilbert space. Numerical examples are given.

Key Words: QSDE; Fock spaces; Exponential vectors; Kurzweil equations and integrals; Noncommutative stochastic processes.

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1. INTRODUCTION

The role of generalized ordinary differential or Kurzweil equations in applying topological dynamics to the study of ordinary differential equations as well as their semigroup properties outlined in Artstein [1] is an interesting motivation for studying this class of equations associated with the weak forms of the Lipschitzian quantum stochastic differential equations.

In the framework of the Hudson-Parthasarathy [13] formulations of quantum stochastic calculus, existence,uniqueness and the equivalent forms of Lipschitzian quantum stochastic differential equations have been established. In the formulations of Ekhaguere [8], the equivalent form is a first order initial value ordinary differential equation of a nonclassical type having a sesquilinear form - valued map as the right hand side (see [2,3,4]).

We remark that the equivalent form of QSDEs facilitates the introduction and study of the associated Kurzweil equations. This is accomplished in the framework of the Kurzweil integral calculus (called the generalized Perron integral calculus in the formulations of [18]). The results obtained here are generalizations of analogous results due to references [1,5,6,7,18] concerning classical initial value problems to the noncommutative quantum setting involving unbounded linear operators on a Hilbert space.

Consequently, the technique of topological dynamics can be applied to QSDEs as outlined in [1] by embedding the equivalent forms of these equations in the space of the associated Kurzweil equations when sufficient analytical properties of these equations have been developed. This question as well as the applications of this concept to quantum fields/systems will be addressed elsewhere.

Finally, since the construction of Kurzweil integrals is a simple extension of the Riemann theory of integration based on Riemann type integral sums,we use this fact to obtain discrete approximations of weak solutions of QSDEs using the associated Kurzweil equations.

Our numerical experiments show that the approximation methods developed in this paper are of a reasonably high level of accuracy than the Euler scheme and some multistep schemes considered in [4]. Moreover, the methods here are applicable to a wider class of equations than the considerations in [4] since we work with pure Caratheodory conditions. The rest of the paper is organised as follows: In section 2, we outline some of the concepts which feature in the subsquent analysis including the Kurzweil integral and some of its properties that are of interest in respect of noncummutative quantum stochastic processes.

The Kurzweil equations associated with quantum stochastic differential equation and some results on approximation of matrix elements of solution of the equation are established in section 3. Sections 4 and 5 contain the major results of this paper. In section 4, we derive a necessary and sufficient condition for a sesquilinear form-valued map to be Kurzweil integrable. We then show that the space of Kurzweil integrable sesquilinear form-valued maps contains sesquilinear form-valued maps that satisfy the Caratheodory conditions. In section 5, we employ our results in the previous section to prove that the weak form of quantum stochastic differential equation and its associated Kurzweil equation are equivalent. We then employ our approximation results of section 3 to generate approximate values of the weak solution of quantum stochastic differential equation formulated in Kurzweil form in section 6. We present some numerical examples.

In what follows, as in [2,3,4,8,9,10] we employ the locally convex topological state space $\tilde{\mathcal{A}}$ of noncommutative stochastic processes and we adopt the definitions and notations of spaces $Ad(\tilde{\mathcal{A}})$, $Ad(\tilde{\mathcal{A}})_{wac}$, $L^p_{loc}(\tilde{\mathcal{A}})$, $L^\infty_{\gamma,loc}(\mathbb{R}_+)$, and the integrator processes $\wedge_\pi$, A^+_g, A_f, for $f, g \in L^\infty_{\gamma,loc}(\mathbb{R}_+)$, $\pi \in L^\infty_{B(\gamma),loc}(\mathbb{R}_+)$. For E, F, G, H lying in $L^2_{loc}[\tilde{\mathcal{A}} \times I]$, we consider the quantum stochastic differential equation in integral form given by

$$\begin{aligned} X(t) = X_0 + \int_{t_0}^{t} (E(X(s), s)d \wedge_\pi (s) + F(X(s), s)dA_f(s) \\ + G(X(s), s)dA^+_g(s) + H(X(s), s)ds), \quad t \in I, \end{aligned} \tag{1.1}$$

where the integral in equation (1.1) is understood in the sense of Hudson and Parthasarathy [13]. However, Ekhaguere [8] has shown that equation (1.1) is equivalent to the following first order initial value nonclassical ordinary differential equation

$$\begin{aligned} \frac{d}{dt}\langle \eta, X(t)\xi \rangle &= P(X(t), t)(\eta, \xi) \\ X(t_0) &= X_0, \quad t \in [t_0, T] \end{aligned} \tag{1.2}$$

As explained in [2,3,4,8–10], the map P appearing in equation (1.2) has the form

$$\begin{aligned} P(x, t)(\eta, \xi) = (\mu E)(x, t)(\eta, \xi) + (\gamma F)(x, t)(\eta, \xi) + (\sigma G)(x, t)(\eta, \xi) \\ + H(x, t)(\eta, \xi) \end{aligned} \tag{1.3}$$

$\eta, \xi \in \mathbb{D}\underline{\otimes}\mathbb{E}$, $(x, t) \in \tilde{\mathcal{A}} \times I$ where $H(x, t)(\eta, \xi) := \langle \eta, H(x, t)\xi \rangle$.

The map P may sometimes be written in the form $P(x, t)(\eta, \xi) = \langle \eta, P_{\alpha\beta}(x, t)\xi \rangle$ where $P_{\alpha\beta} : \tilde{\mathcal{A}} \times I \longrightarrow \tilde{\mathcal{A}}$ is given by

$$P_{\alpha\beta}(x, t) = \mu_{\alpha\beta}(t)E(x, t) + \gamma_\beta(t)F(x, t) + \sigma_\alpha(t)G(x, t) + H(x, t)$$

for $(x, t) \in \tilde{\mathcal{A}} \times I$.

Equation (1.2) is known to have a unique weakly absolutely continuous adapted solution $\Phi : I \to \tilde{\mathcal{A}}$ for the Lipschitzian coefficients E, F, G, H.

2. KURZWEIL INTEGRALS ASSOCIATED WITH QUANTUM STOCHASTIC PROCESSES

Let $\eta, \xi \in \mathbb{D}\underline{\otimes}\mathbb{E}$ be arbitrary. Assume that $U : [t_0, T] \times [t_0, T] \to \tilde{\mathcal{A}}$ is an $\tilde{\mathcal{A}}$-valued map of two variables $\tau, t \in [t_0, T]$. We consider the family of complex valued functions: $U(\tau, t)(\eta, \xi) : = \langle \eta, U(\tau, t)\xi \rangle$ for arbitrary $\eta, \xi \in \mathbb{D}\underline{\otimes}\mathbb{E}$ associated with the map U. We shall use the notation $\int_{t_0}^{T} DU(\tau, t)(\eta, \xi)$ to denote the Kurzweil integral of $U(\tau, t)(\eta, \xi)$ in the sense and notations of Artstein [1] and using the formulations of Schwabik [18] and write

$$S(U, D)(\eta, \xi) = \sum_{j=1}^{k} [U(\tau_j, t_j)(\eta, \xi) - U(\tau_j, t_{j-1})(\eta, \xi)]$$

for the Riemann-Kurzweil sum corresponding to the function $U(\tau, t)(\eta, \xi)$ and partition $D : t_0 < \tau_1 < t_1 < \ldots . < t_k = T$ of $[t_0, T]$.

If $f : [t_0, T] \to \tilde{\mathcal{A}}$ is a stochastic process, then for arbitrary $\eta, \xi \in \mathbb{D}\underline{\otimes}\mathbb{E}$, we set $U(\tau, t)(\eta, \xi) = (\langle \eta, f(\tau)\xi \rangle)t \quad$ for $\tau, t \in [t_0, T]$ and therefore we have

$$\begin{aligned} S(U, D)(\eta, \xi) &= \sum_{j=1}^{k} [U(\tau_j, t_j)(\eta, \xi) - U(\tau_j, t_{j-1})(\eta, \xi)] \\ &= \sum_{j=1}^{k} [\langle \eta, f(\tau_j)\xi \rangle (t_j - t_{j-1})] \end{aligned}$$

representing the classical Riemann sum for the function $f_{\eta\xi}(t) := \langle \eta, f(t)\xi \rangle$ and a given partition D of $[t_0, T]$. In this case, we write

$$\int_{t_0}^{T} \langle \eta, f(s)\xi \rangle ds = \int_{t_0}^{T} D[f_{\eta\xi}(\tau), t]$$

provided that the Kurzweil integral $\int_{t_0}^{T} DU(\tau, t)(\eta, \xi)$ exists in this case. Hence

$$\int_{t_0}^{T} DU(\tau, t)(\eta, \xi) = \int_{t_0}^{T} D[f_{\eta\xi}(\tau), t] = \int_{t_0}^{T} f_{\eta\xi}(s) ds. \tag{2.1}$$

We remark that by Theorem (1.16) (Schwabik [18]) if $U : [t_0, T] \times [t_0, T] \to \mathbb{C}$ be such that U is Kurzweil integrable over $[t_0, T]$, then for $c \in [t_0, T]$, we have

$$\lim_{s \to c} \left[\int_{t_0}^{s} DU(\tau, t) - U(c, s) + U(c, c) \right] = \int_{t_0}^{c} DU(\tau, t) \tag{2.2}$$

For several properties enjoyed by Kurzweil integrals and the existence of at least one ∂-fine partition D of $[t_0, T]$ for a given gauge ∂, we refer to Chapter 1 and Lemma (1.4) in Schwabik [18].

We now introduce the Kurzweil equations associated with equation (1.2).

3. KURZWEIL EQUATIONS ASSOCIATED WITH QUANTUM STOCHASTIC DIFFERENTIAL EQUATIONS

(i) Let the map $P : \tilde{\mathcal{A}} \times [t_0, T] \to sesq(\mathbb{D}\underline{\otimes}\mathbb{E})$ be given by equation (1.3). Then we refer to the equation

$$\frac{d}{d\tau}\langle \eta, X(\tau)\xi\rangle = DP(X(\tau), t)(\eta, \xi) \tag{3.1}$$

as the Kurzweil equation associated with equation (1.2).

(ii) A map $\Phi : [t_0, T] \to \tilde{\mathcal{A}}$ is called a solution of equation (3.1) if

$$\langle \eta, \Phi(s_2)\xi\rangle - \langle \eta, \Phi(s_1)\xi\rangle = \int_{s_1}^{s_2} DP(\Phi(\tau), t)(\eta, \xi) \tag{3.2}$$

holds for every $s_1, s_2 \in [t_0, T]$ identically.

The integral on the right hand side of equation (3.2) is the Kurzweil integral introduced in section 2. Equation (3.1) is understood in integral form (3.2) via its solution.

We have the following results as immediate consequences of our definitions.

Proposition 3.1. *If a map $\Phi : [t_0, T] \to \tilde{\mathcal{A}}$ is a solution of the Kurzweil equation* (3.1) *on $[t_0, T]$, then for every $u \in [t_0, T]$, we have*

$$\langle \eta, \Phi(s)\xi\rangle = \langle \eta, \Phi(u)\xi\rangle + \int_u^s DP(\Phi(\tau), t)(\eta, \xi), \ \ s \in [t_0, T] \tag{3.3}$$

Conversely if a map $\Phi : [t_0, T] \to \tilde{\mathcal{A}}$ satisfies the integral equation (3.3) *for some $u \in [t_0, T]$ and all $s \in [t_0, T]$ then Φ is a solution of equation* (3.1).

Proof: The first statement follows directly from the definition of a solution of (3.1) when we put $s_1 = u$ and $s_2 = s$. Conversely, if $\Phi : [t_0, T] \to \tilde{\mathcal{A}}$ satisfies the integral equation (3.3) then by the additivity of the integral, equation (3.2) follows.

Proposition 3.2. *If $\Phi : [t_0, T] \to \tilde{\mathcal{A}}$ is a solution of equation* (3.1) *on $[t_0, T]$ then*

$$\begin{aligned}&\lim_{s\to\sigma}[\langle \eta, \Phi(s)\xi\rangle - P(\Phi(\sigma), s)(\eta, \xi) + P(\Phi(\sigma), \sigma)(\eta, \xi)]\\ &\quad = \langle \eta, \Phi(\sigma)\xi\rangle\end{aligned} \tag{3.4}$$

Proof: Let $\sigma \in [t_0, T]$ be fixed. Then by Proposition (3.1) we have

$$\langle \eta, \Phi(s)\xi\rangle - \int_\sigma^s DP(\Phi(\tau), t)(\eta, \xi) = \langle \eta, \Phi(\sigma)\xi\rangle$$

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therefore

$$\langle\eta,\Phi(s)\xi\rangle - P(\Phi(\sigma),s)(\eta,\xi) + P(\Phi(\sigma),\sigma)(\eta,\xi) - \int_\sigma^s DP(\Phi(\tau),t)(\eta,\xi) + P(\Phi(\sigma),s)(\eta,\xi) - P(\Phi(\sigma),\sigma)(\eta,\xi) - \langle\eta,\Phi(\sigma)\xi\rangle = 0 \tag{3.5}$$

for every $s \in [t_0, T]$.

By equation (2.2)

$$\lim_{s\to\sigma}\left[\int_\sigma^s DP(\Phi(\tau),t)(\eta,\xi) - P(\Phi(\sigma),s)(\eta,\xi) + P(\Phi(\sigma),\sigma)(\eta,\xi)\right] = 0 \tag{3.6}$$

Equation (3.6) and (3.5) yield the existence of the limit given by

$$\lim_{s\to\sigma}[\langle\eta,\Phi(s)\xi\rangle - P(\Phi(\sigma),s)(\eta,\xi) + P(\Phi(\sigma),\sigma)(\eta,\xi) - \langle\eta,\Phi(\sigma)\xi\rangle]$$

as well as the relation

$$\lim_{s\to\sigma}[\langle\eta,\Phi(s)\xi\rangle - P(\Phi(\sigma),s)(\eta,\xi) + P(\Phi(\sigma),\sigma)(\eta,\xi) - \langle\eta,\Phi(\sigma)\xi\rangle] = 0$$

which gives (3.4).

Remark 3.3. By virtue of Proposition (3.2), the following approximation holds:- If $\Phi : [t_0, T] \to \tilde{\mathcal{A}}$ is a solution of equation (3.1), then for every $\sigma \in [t_0, T]$ and for arbitrary $\eta, \xi \in I\!D \underline{\otimes} I\!E$, the matrix element

$$\langle\eta,\Phi(s)\xi\rangle \;\cong\; \langle\eta,\Phi(\sigma)\xi\rangle + P(\Phi(\sigma),s)(\eta,\xi) - P(\Phi(\sigma),\sigma)(\eta,\xi),$$

provided that s in $[t_0, T]$ is sufficiently close to σ.

We now introduce a class of sesquilinear form - valued maps $P : \tilde{\mathcal{A}} \times [t_0, T] \longrightarrow sesq(I\!D \underline{\otimes} I\!E)$, which are Kurzweil integrable.

4. A CLASS OF KURZWEIL INTEGRABLE SESQUILINEAR FORM - VALUED MAPS

In what follows, we adopt some notations and terminologies employed in [18, Chapter 1]. For each $\eta, \xi \in I\!D \underline{\otimes} I\!E$, let $h_{\eta\xi} : [t_0, T] \to I\!R$ be a family of nondecreasing functions defined on $[t_0, T]$ and $W : [0, \infty) \to I\!R$ be a continuous and increasing function such that $W(0) = 0$. Then we say that the map $P : \tilde{\mathcal{A}} \times [t_0, T] \longrightarrow sesq(I\!D\underline{\otimes}I\!E)$ belongs to the class $I\!F(\tilde{\mathcal{A}} \times [t_0, T], h_{\eta\xi}, W)$ for each $\eta, \xi \in I\!D\underline{\otimes}I\!E$ if for all $x, y \in \tilde{\mathcal{A}}, t_1, t_2 \in [t_0, T]$

(i) $|P(x, t_2)(\eta, \xi) - P(x, t_1)(\eta, \xi)| \le |h_{\eta\xi}(t_2) - h_{\eta\xi}(t_1)|$ (4.1.)

(ii) $|P(x, t_2)(\eta, \xi) - P(x, t_1)(\eta, \xi) - P(y, t_2)(\eta, \xi) + P(y, t_1)(\eta, \xi)|$
$\le W(\|x - y\|_{\eta\xi})|h_{\eta\xi}(t_2) - h_{\eta\xi}(t_1)|$ (4.2.)

We now present a number of results which are simple extensions of similar results in Schwabik [18] to the present noncommutative quantum setting. The next Theorem is an extension of the convergence results of Corollary 1.31 in [18]. The proof follows exact arguments as in [18].

Theorem 4.1. *Assume that the following conditions hold* :

(i) *the maps* $U, U_m : [t_0, T] \times [t_0, T] \to \tilde{\mathcal{A}}$ *are such that* $(\tau, t) \to U_m(\tau, t)$ (η, ξ) *are real valued and Kurzweil integrable over* $[t_0, T]$ *for each* $\eta, \xi \in I\!D\underline{\otimes}I\!E$, $\forall\, m = 1, 2, \ldots$

(ii) *there is a gauge* w *on* $[t_0, T]$ *such that for every* $\epsilon > 0$, *there exists a map* $p : [t_0, T] \to I\!N$ *and a family of positive superadditive interval functions* $\Phi_{\eta\xi}$ *on* $[t_0, T]$ *defined for closed intervals* $J \subset [t_0, T]$ *with* $\Phi_{\eta\xi}([t_0, T]) < \epsilon$ *such that for every* $\tau \in [t_0, T]$

$$|U_m(\tau, J)(\eta, \xi) - U(\tau, J)(\eta, \xi)| < \Phi_{\eta\xi}(J)$$

provided that $m > p(\tau)$, *and* (τ, J) *is an* w*-fine tagged interval with* $\tau \in J \subseteq [t_0, T]$.

(iii) *there exist real valued Kurzweil integrable functions*

$$V_{\eta\xi}, W_{\eta\xi} : [t_0, T] \times [t_0, T] \to I\!R$$

and a gauge θ *on* $[t_0, T]$ *such that for all* $m \in I\!N$, $\tau \in [t_0, T]$,

$$V_{\eta\xi}(\tau, J) \le U_m(\tau, J)(\eta, \xi) \le W_{\eta\xi}(\tau, J).$$

for any θ*-fine interval* (τ, J), $\forall\, \eta, \xi \in I\!D\underline{\otimes}I\!E$. *Then the map* $(\tau, t) \to U(\tau, t))(\eta, \xi)$ *is Kurzweil integrable over* $[t_0, T]$ *and that*

$$\lim_{m\to\infty} \int_{t_0}^{T} DU_m(\tau, t)(\tau, t)(\eta, \xi) = \int_{t_0}^{T} DU(\tau, t)(\eta, \xi).$$

The next Theorem concerns some fundamental properties of Kurzweil integrals in the framework of [18].

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Theorem 4.2.

(i) *Let* $U : [t_0, T] \times [t_0, T] \longrightarrow \tilde{\mathcal{A}}$ *be such that* $(\tau, t) \rightarrow U(\tau, t)(\eta, \xi)$ *is Kurzweil integrable over* $[t_0, c]$ *for* $c \in [t_0, T]$ *and that the limit*

$$\lim_{c \to T-} \left[\int_{t_0}^{c} DU(\tau, t)(\eta, \xi) - U(T, c)(\eta, \xi) + U(T, T)(\eta, \xi) \right] = I. \tag{4.3}$$

exists for all $\eta, \xi \in \mathbb{D}\underline{\otimes}\mathbb{E}$. *Then* $\int_{t_0}^{T} DU(\tau, t)(\eta, \xi)$ *exists and equals* I.

(ii) *Let* $U : [t_0, T] \times [t_0, T] \longrightarrow \tilde{\mathcal{A}}$ *be such that* $(\tau, t) \rightarrow U(\tau, t)(\eta, \xi)$ *is Kurzweil integrable over* $[c, T]$ *for every* $c \in (t_0, T]$ *and that the limit*

$$\lim_{c \to t_0+} \left[\int_{c}^{T} DU(\tau, t)(\eta, \xi) + U(t_0, c)(\eta, \xi) - U(t_0, t_0)(\eta, \xi) \right] = I \tag{4.4}$$

exists for all $\eta, \xi \in \mathbb{D}\underline{\otimes}\mathbb{E}$. *Then* $\int_{t_0}^{T} DU(\tau, t)(\eta, \xi)$ *exists and equals* I.

(iii) *Let* $U : [t_0, T] \times [t_0, T] \longrightarrow \tilde{\mathcal{A}}$ *be such that* $(\tau, t) \rightarrow U(\tau, t)(\eta, \xi)$ *is Kurzweil integrable over* $[t_0, T]$. *Then for* $c \in [t_0, T]$

$$\lim_{s \to c} \left[\int_{t_0}^{s} DU(\tau, t)(\eta, \xi) - U(c, s)(\eta, \xi) + U(c, c)(\eta, \xi) \right] = \int_{t_0}^{c} DU(\tau, t)(\eta, \xi) \tag{4.5}$$

for all $\eta, \xi \in \mathbb{D}\underline{\otimes}\mathbb{E}$.

Proof: The proofs are simple adaptation of arguements employed in Theorem 1.14, Remark 1.15 and Theorem 1.16 in [18] to the present noncommutative quantum setting.

Next, we present some results concerning the existence of the integral involved in the definition of the solution of the Kurzweil equation (3.1).

Theorem 4.3. *Assume that the map* $(x, t) \rightarrow P(x, t)(\eta, \xi)$ *belongs to* $\mathbb{F}.(\tilde{\mathcal{A}} \times [t_0, T], h_{\eta\xi}, W)$, *and* $X : [a, b] \rightarrow \tilde{\mathcal{A}}$, $[a, b] \subseteq [t_0, T]$ *is the limit of a sequence* $\{X_k\}_{k \in \mathbb{N}}$ *of processes* $X_k : [a, b] \rightarrow \tilde{\mathcal{A}}$ *such that* $\int_a^b DP(X_k(\tau), t)(\eta, \xi)$ *exists for every* $k \in \mathbb{N}$. *Then the integral* $\int_a^b DP(X(\tau), t)(\eta, \xi)$ *exists and*

$$\int_a^b DP(X(\tau), t)(\eta, \xi) = \lim_{k \to \infty} \int_a^b DP(X_k(\tau), t)(\eta, \xi)$$

Proof: Since the complex field $\mathbb{C} \cong \mathbb{R}^2$, we assume without any lost of generality that the map $P(X(\tau), t)(\eta, \xi)$ is real valued. Let $\epsilon > 0$ be given, then by (4.2), we have

$$\begin{aligned} &|P(X_k(\tau), t_2)(\eta, \xi) - P(X_k(\tau), t_1)(\eta, \xi) - P(X(\tau), t_2)(\eta, \xi) \\ &\quad + P(X(\tau), t_1)(\eta, \xi)| \leq W(\|X_k(\tau) - X(\tau)\|_{\eta\xi})|h_{\eta\xi}(t_2) - h_{\eta\xi}(t_1)| \end{aligned} \tag{4.6}$$

for every $\tau \in [a, b]$, $t_1 < \tau < t_2$, and $[t_1, t_2] \subset [a, b]$.
If we set

$$U_{\eta\xi}(t) = \frac{\epsilon}{h_{\eta\xi}(b) - h_{\eta\xi}(a) + 1} h_{\eta\xi}(t),$$

for $t \in [a, b]$ then the function $U_{\eta\xi} : [a, b] \longrightarrow \mathbb{R}$ is nondecreasing and

$$(U_{\eta\xi}(b) - U_{\eta\xi}(a)) < \epsilon.$$

Since $\lim_{k\to\infty} X_k(\tau) = X(\tau)$ in $\tilde{\mathcal{A}}$ for every $\tau \in [a, b]$ and the function W is continuous at 0, then there is a $p = p(\tau) \in \mathbb{N}$ such that for $k > p(\tau)$,

$$W(\|X_k(\tau) - X(\tau)\|_{\eta\xi}) \leq \frac{\epsilon}{h_{\eta\xi}(b) - h_{\eta\xi}(a) + 1}$$

i.e. for $k \geq p(\tau)$, the inequality (4.2) can be written as

$$\begin{aligned} &|P(X_k(\tau), t_2)(\eta, \xi) - P(X_k(\tau), t_1)(\eta, \xi) - P(X(\tau), t_2)(\eta, \xi) \\ &\qquad + P(X(\tau), t_1)(\eta, \xi) \leq U_{\eta\xi}(t_2) - U_{\eta\xi}(t_1) \end{aligned}$$

By inequality (4.1)

$$|P(X_k(\tau), t_2)(\eta, \xi) - P(X_k(\tau), t_1)(\eta, \xi)| \leq |h_{\eta\xi}(t_2) - h_{\eta\xi}(t_1)|$$

for every $t \in [a, b]$, $k \in \mathbb{N}$, $t_1 \leq \tau \leq t_2$ and $[t_1, t_2] \subseteq [a, b]$.
Hence the last inequality implies that

$$\begin{aligned} -h_{\eta\xi}(t_2) + h_{\eta\xi}(t_1) &\leq P(X_k(\tau), t_2)(\eta, \xi) - P(X_k(\tau), t_1)(\eta, \xi) \\ &\leq h_{\eta\xi}(t_2) - h_{\eta\xi}(t_1) \end{aligned}$$

but the integrals

$$\int_a^b D(h_{\eta\xi}(t)) = h_{\eta\xi}(b) - h_{\eta\xi}(a)$$

and

$$\int_a^b D(-h_{\eta\xi}(t)) = h_{\eta\xi}(a) - h_{\eta\xi}(b)$$

exist. We conclude that the integral $\int_a^b DP(X(\tau), t)(\eta, \xi)$ exists and the conclusion of the theorem holds by Theorem (4.1) above .

Theorem 4.4. *Assume that the map* $(x, t) \to P(x, t)(\eta, \xi)$ *belongs to* $\mathbb{F}(\tilde{\mathcal{A}} \times [t_0, T], h_{\eta\xi}, W)$ *and that* $X : [a, b] \longrightarrow \tilde{\mathcal{A}}$ *is the limit of a sequence of simple processes. Then the integral* $\int_a^b DP(X(\tau), t)(\eta, \xi)$ *exists for arbitrary* $\eta, \xi \in \mathbb{D}\underline{\otimes}\mathbb{E}$.

Proof: By Theorem (4.3), it is sufficient to prove that the integral $\int_a^b DP(\phi(\tau), t)(\eta, \xi)$ exists for every simple processes $\phi : [a, b] \to \tilde{\mathcal{A}}$. If ϕ is a simple process then there is a partition $a = s_0 < s_1 < s_2 < \cdots < s_k = b$ of $[a, b]$ such that $\phi(s) = c_j \in \tilde{\mathcal{A}}$ for $s \in (s_{j-1}, s_j)$, $j = 1, 2, \ldots, k$ where c_j, $j = 1, \ldots, k$ are finite number of elements of $\tilde{\mathcal{A}}$.

By the definition of the Kurzweil integral, if $s_{j-1} < \sigma_1 < \sigma_2 < \sigma_j$, then we have the existence of the integral

$$\int_{\sigma 1}^{\sigma_2} DP(\phi(\tau), t)(\eta, \xi) = P(c_j, \sigma_2)(\eta, \xi) - P(c_j, \sigma_1)(\eta, \xi)$$

Assume that $\sigma_0 \in (s_{j-1}, s_j)$ is given, we have

$$\begin{aligned}
\lim_{s \to s_{j-1}+} \Bigg[\int_s^{\sigma_0} & DP(\phi(\tau), t)(\eta, \xi) + P(\phi(s_{j-1}), s)(\eta, \xi) \\
& - P(\phi(s_{j-1}), s_{j-1})(\eta, \xi) \Bigg] = \lim_{s \to s_{j-1}+} [P(c_j, \sigma_0)(\eta, \xi) \\
& - P(c_j, s)(\eta, \xi) + P(\phi(s_{j-1}), s)(\eta, \xi) - P(\phi(s_{j-1}), s_{j-1})(\eta, \xi)] \\
&= P(c_j, \sigma_0)(\eta, \xi) - P(c_j, s_{j-1+})(\eta, \xi) + P(\phi(s_{j-1}), s_{j-1+})(\eta, \xi) \\
& - P(\phi(s_{j-1}), s_{j-1})(\eta, \xi) \qquad (4.7)
\end{aligned}$$

Hence by Theorem (4.2) (ii), the integral $\int_{s_{j-1}}^{\sigma_0} DP(\phi(\tau), t)(\eta, \xi)$ exists and equals the computed limit given by (4.7). Similarly, it can be shown that the integral $\int_{\sigma_0}^{s_j} DP(\phi(\tau), t)(\eta, \xi)$ exists and the following equality holds.

$$\begin{aligned}
\int_{\sigma_0}^{s_j} DP(\phi(\tau), t)(\eta, \xi) &= P(c_j, s_j-)(\eta, \xi) - P(c_j, \sigma_0)(\eta, \xi) \\
& - P(\phi(s_j), s_j-)(\eta, \xi) + P(\phi(s_j), s_j)(\eta, \xi) \qquad (4.8)
\end{aligned}$$

by Theorem (4.2)(i).
Hence by additivity of the integral, we obtain

$$\begin{aligned}
&\int_{s_{j-1}}^{s_j} DP(\phi(\tau), t)(\eta, \xi) \\
&= \int_{s_{j-1}}^{\sigma_0} DP(\phi(\tau), t)(\eta, \xi) + \int_{\sigma_0}^{s_j} DP(\phi(\tau), t)(\eta, \xi)
\end{aligned}$$

which equals the sum of expressions in (4.7) and (4.8) over the subinterval $[s_{j-1}, s_j]$ of the partition.

Repeating this argument for every interval $[s_{j-1}, s_j]$, $j = 1, 2, \ldots, k$ and using the additivity of the integral, we obtain the existence of the integral $\int_a^b DP(\phi(\tau), t)(\eta, \xi)$ and the identity

$$\begin{aligned}\int_a^b & DP(\phi(\tau), t)(\eta, \xi) \\ &= \sum_{j=1}^{k}[P(c_j, s_j-)(\eta, \xi) - P(c_j, s_{j-1+})] + \sum_{j=1}^{k}[P(\phi(s_{j-1+}), s_{j-1}+)(\eta, \xi) \\ &\quad - P(\phi(s_{j-1}), s_{j-1})(\eta, \xi) - P(\phi(s_j), s_{j-})(\eta, \xi) + P(\phi(s_j, s_j)(\eta, \xi)]\end{aligned} \tag{4.9}$$

Theorem 4.5. *Assume that the map $(x, t) \rightarrow P(x,t)(\eta,\xi)$ is of class $\mathbb{F}(\tilde{\mathcal{A}} \times [t_0, T], h_{\eta\xi}, W)$ and $X : [a, b] \rightarrow \tilde{\mathcal{A}}$, $[a, b] \subseteq [t_0, T]$ is of bounded variation, then the integral $\int_a^b DP(X(\tau), t)(\eta, \xi)$, exists.*

Proof: The result follows from Theorem (4.4) because every process $X : [a, b] \rightarrow \tilde{\mathcal{A}}$ in $L^2_{loc}(\tilde{\mathcal{A}})$ of bounded variation is the uniform limit of finite simple processes (cf [9,10,13]).

Next, we denote by $C(\tilde{\mathcal{A}} \times [t_0, T], W)$, the class of sesquilinear form-valued maps which are Lipschitzian and satisfy the Caratheodory conditions. We then give a result that connects this class with the class $\mathbb{F}(\tilde{\mathcal{A}} \times [t_0, T], h_{\eta\xi}, W)$.

Definition 4.6. A map $P : \tilde{\mathcal{A}} \times [t_0, T] \longrightarrow Sesq(\mathbb{D}\underline{\otimes}\mathbb{E})$ belongs to the class $C(\tilde{\mathcal{A}} \times [t_0, T], W)$ if for arbitrary $\eta, \xi \in \mathbb{D}\underline{\otimes}\mathbb{E}$,

(i) $P(x, \cdot)(\eta, \xi)$ is measurable for each $x \in \tilde{\mathcal{A}}$.

(ii) There exists a family of measurable functions $M_{\eta\xi} : [t_0, T] \rightarrow \mathbb{R}_+$ such that $\int_{t_0}^T M_{\eta\xi}(s)ds < \infty$ and

$$|P(x, s)(\eta, \xi)| \leq M_{\eta\xi}(s), \quad (x, s) \in \tilde{\mathcal{A}} \times [t_0, T] \tag{4.10}$$

(iii) There exists measurable functions $K^p_{\eta\xi} : [t_0, T] \rightarrow \mathbb{R}_+$ such that for each $t \in [t_0, T]$, $\int_{t_0}^t K_{\eta\xi}(s)ds < \infty$, and

$$|P(x, s)(\eta, \xi) - P(y, s)(\eta, \xi)| \leq K^p_{\eta\xi}(s)W(\|x - y\|_{\eta\xi}) \tag{4.11}$$

for $(x, s), (y, s) \in \tilde{\mathcal{A}} \times [t_0, T]$ and where in (i) - (iii) we take $W(t) = t$.

Definition 4.7. For $(x, t) \in \tilde{\mathcal{A}} \times [t_0, T]$ and P belonging to $C(\tilde{\mathcal{A}} \times [t_0, T], W)$, we define for arbitrary $\eta, \xi \in \mathbb{D}\underline{\otimes}\mathbb{E}$,

$$F(x,t)(\eta,\xi) = \int_{t_0}^{t} P(x,s)(\eta,\xi)ds \tag{4.12}$$

We have the following results that connect the two classes of maps.

Theorem 4.8. *Assume that for arbitrary $\eta, \xi \in \mathbb{D}\underline{\otimes}\mathbb{E}$, the map $P : \tilde{\mathcal{A}} \times [t_0, T] \to sesq(\mathbb{D}\underline{\otimes}\mathbb{E})$ is of class $C(\tilde{\mathcal{A}} \times [t_0, T], W)$. Then for every $x,\ y \in \tilde{\mathcal{A}}, t_1, t_2 \in [t_0, T]$, $F(x, t)(\eta, \xi)$ defined by (4.12) satisfies*

(i) $|F(x, t_2)(\eta, \xi) - F(x, t_1)(\eta, \xi)| \le \int_{t_1}^{t_2} M_{\eta\xi}(s)ds$

(ii) $|F(x, t_2)(\eta, \xi) - F(x, t_1)(\eta, \xi) - F(y, t_2)(\eta, \xi) + F(y, t_1)(\eta, \xi)| \le W(\|x - y\|_{\eta\xi}) \int_{t_1}^{t_2} K^p_{\eta\xi}(s)ds$

(iii) *The map $F(x, t)(\eta, \xi)$ belong to the class $\mathbb{F}(\tilde{\mathcal{A}} \times [t_0, T], h_{\eta\xi}, W)$ for each $\eta, \xi \in \mathbb{D}\underline{\otimes}\mathbb{E}$, where*

$$h_{\eta\xi}(t) = \int_{t_0}^{t} M_{\eta\xi}(s)ds + \int_{t_0}^{t} K^p_{\eta\xi}(s)ds$$

Proof: (i) Since (4.10) holds we have by (4.12) and for all $x \in \tilde{\mathcal{A}}$, $t_1, t_2 \in [t_0, T]$.

$$\begin{aligned}|F(x, t_2)(\eta, \xi) - F(x, t_1)(\eta, \xi)| &= \left|\int_{t_1}^{t_2} P(x,s)(\eta,\xi)ds\right| \\ &\le \int_{t_1}^{t_2} |P(x,s)(\eta,\xi)|ds \\ &\le \int_{t_1}^{t_2} M_{\eta,\xi}(s)ds\end{aligned}$$

(ii) Again by (4.12) and (4.11)

$$\begin{aligned}&|F(x, t_2)(\eta, \xi) - F(x, t_1)(\eta, \xi) - F(y, t_2)(\eta, \xi) + F(y, t_1)(\eta, \xi)| \\ &\quad = \left|\int_{t_1}^{t_2} [P(x,s)(\eta,\xi) - P(y,s)(\eta,\xi)]ds\right| \\ &\quad \le \int_{t_1}^{t_2} |P(x,s)(\eta,\xi) - P(y,s)(\eta,\xi)|ds \le W(\|x-y\|_{\eta\xi}) \int_{t_1}^{t_2} K^p_{\eta\xi}(s)ds\end{aligned}$$

for every $x, y \in \tilde{\mathcal{A}}$ and $t_1, t_2 \in [t_0, T]$.

By (i) above

$$|F(x,t_2)(\eta,\xi) - F(x,t_1)(\eta,\xi)| \leq \int_{t_1}^{t_2} M_{\eta\xi}(s)ds \leq |h_{\eta\xi}(t_2) - h_{\eta\xi}(t_1)|$$

$\forall x \in \tilde{\mathcal{A}}$, $t_1, t_2 \in [t_0, T]$ satisfying inequality (4.1).

Again by (ii) above

$$\begin{aligned} &|F(x,t_2)(\eta,\xi) - F(x,t_1)(\eta,\xi) - F(y,t_2)(\eta,\xi) + F(y,t_1)(\eta,\xi)| \\ &\quad \leq W(\|x-y\|_{\eta\xi}) \int_{t_1}^{t_2} K^p_{\eta\xi}(s)ds \\ &\quad \leq W(\|x-y\|_{\eta\xi})|h_{\eta\xi}(t_2) - h_{\eta\xi}(t_1)| \end{aligned}$$

for every $x, y \in \tilde{\mathcal{A}}$, $t_1, t_2 \in [t_0, T]$ satisfying inequality (4.2) .

In the next section, we prove that the Kurzweil integral of $F(x,t)(\eta,\xi)$ equals the Lebesque integral of $P(x,t)(\eta,\xi)$. This facilitates the proof of the equivalence of equation (1.2) and the associated Kurzweil equation.

5. EQUIVALENCE OF QUANTUM STOCHASTIC DIFFERENTIAL EQUATION AND THE ASSOCIATED KURZWEIL EQUATION

In connection with subsequent results, we assume that the map $P : \tilde{\mathcal{A}} \times [t_0, T] \to Sesq(I\!D \underline{\otimes} I\!E)$ given by equation (1.3) is of class $C(\tilde{\mathcal{A}} \times [t_0, T], W)$ and that $F(x,t)(\eta,\xi)$ is given by (4.12).

Theorem 5.1. *If $x : [a,b] \to \tilde{\mathcal{A}}$, $[a,b] \subseteq [t_0, T]$ is the limit of simple processes then*

$$\int_a^b DF(x(\tau),t)(\eta,\xi) = \int_a^b P(x(s),s)(\eta,\xi)ds$$

Proof: By Theorem (4.8)(iii) the map $(x,t) \to F(x,t)(\eta,\xi)$ belongs to $I\!F(\tilde{\mathcal{A}} \times [t_0, T], h_{\eta\xi}, W)$. Therefore the existence of the integral $\int_a^b DF(x(\tau),t)(\eta,\xi)$ is guaranteed by Theorem (4.4). Also by Theorem (4.4), for every simple process $\phi : [a,b] \to \tilde{\mathcal{A}}$ the integral $\int_a^b P(\phi(s),s)(\eta,\xi)ds$ exists and equals $\int_a^b DF(\phi(\tau),t)(\eta,\xi)$.

Assume now that $\phi_k : [a,b] \to \tilde{\mathcal{A}}$, $k \in I\!N$ is a sequence of simple processes such that

$$\lim_{k\to\infty} \phi_k(s) = x(s), \quad s \in [a,b]$$

Then by (4.11),

$$\lim_{k\to\infty} P(\phi_k(s),s)(\eta,\xi) = P(x(s),s)(\eta,\xi)$$

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and inequality (4.10) enables us to use the Lebesgue dominated convergence theorem for showing that $\int_a^b P(x(s), s)(\eta, \xi)ds$ exists and by Theorem (4.3)

$$\int_a^b DF(x(\tau), t)(\eta, \xi) = \lim_{k\to\infty} \int_a^b DF(\phi_k(\tau), t)(\eta, \xi)$$
$$= \lim_{k\to\infty} \int_a^b P(\phi_k(s), s)(\eta, \xi)ds = \int_a^b P(x(s), s)(\eta, \xi)ds$$

Remark 5.2.

(i) The results given above will be used for the representation of equation (1.2) within the framework of the Kurzweil integral calculus. This is accomplished based on the construction of the map $F(x, t)(\eta, \xi)$ for a given sesquilinear form - valued map $P : \tilde{\mathcal{A}} \times [t_0, T] \to Sesq(\mathbb{D}\underline{\otimes}\mathbb{E})$ of class $C(\tilde{\mathcal{A}} \times [t_0, T], W)$ and for arbitrary $\eta, \xi \in \mathbb{D}\underline{\otimes}\mathbb{E}$.

(ii) Let the quantum stochastic differential equation (1.2) be given. The Caratheodory concept of a solution of (1.2) is equivalent to the requirement that for every $s_1, s_2 \in [t_0, T]$ we have a weakly absolutely continuous map $X : [t_0, T] \to \tilde{\mathcal{A}}$ satisfying

$$\langle \eta, x(s_2)\xi\rangle - \langle \eta, x(s_1)\xi\rangle = \int_{s_1}^{s_2} P(x(s), s)(\eta, \xi)ds \tag{5.1}$$

(iii) The solution X of equation (1.2) lies in $L^2_{loc}(\tilde{\mathcal{A}})$ and is therefore the limit of simple processes in $Ad(\tilde{\mathcal{A}})_{wac}$, see [2,8,13]. Consequently the hypothesis of the last theorem remain true.

We now present our major result in this section.

Theorem 5.3. *A stochastic process $X : [t_0, T] \to \tilde{\mathcal{A}}$ is a solution of equation* (1.2) *if and only if X is a solution of the Kurzweil equation*

$$\frac{d}{d\tau}\langle \eta, X(\tau)\xi\rangle = DF(X(\tau), t)(\eta, \xi) \tag{5.2}$$

on $[t_0, T]$, $t \in [t_0, T]$, and for arbitrary $\eta, \xi \in \mathbb{D}\underline{\otimes}\mathbb{E}$.

Proof: Assume that $X : [t_0, T] \to \tilde{\mathcal{A}}$ is a solution of (1.2). By Theorem (4.8), the integral $\int_{s_1}^{s_2} DF(X(\tau), t)(\eta, \xi)$ exists and

$$\langle \eta, X(s_2)\xi\rangle - \langle \eta, X(s_1)\xi\rangle = \int_{s_1}^{s_2} P(X(s), s)(\eta, \xi)ds$$
$$= \int_{s_1}^{s_2} DF(X(\tau), t)(\eta, \xi)$$

for all $s_1, s_2 \in [t_0, t]$. Hence X is a solution of (5.2).

If conversely X is a solution of (5.2), then again Theorem (4.8) shows that X satisfies equation (5.1). Since $F(X,t)(\eta,\xi)$ belongs to $\mathbb{F}(\tilde{\mathcal{A}} \times [t_0, T], h_{\eta\xi}, W)$, we have

$$|\langle \eta, X(s_2)\xi\rangle - \langle \eta, X(s_1)\xi\rangle| = \left| \int_{s_1}^{s_2} DF(X(\tau), t)(\eta, \xi) \right|$$
$$\leq |h_{\eta\xi}(s_2) - h_{\eta\xi}(s_1)|.$$

Hence the map $t \to \langle \eta, X(t)\xi\rangle$ is absolutely continuous on $[t_0, T]$ since $h_{\eta\xi}(t)$ is absolutely continuous for each $\eta, \xi \in \mathbb{D}\underline{\otimes}\mathbb{E}$. Hence X is weakly absolutely continuous.

Remark 5.4. Owning to several properties of the sesquilinear form -valued map P given by equation (1.3) as outlined in Ekhaguere [8], it is enough for P to be Lipschitzian and for inequality (4.10) to be satisfied for all $(X, t) \in \tilde{\mathcal{A}} \times [t_0, T]$ for P to be of class $C(\tilde{\mathcal{A}} \times [t_o, T], W)$ where $W(t) = t$. Consequently, $F(X,t)(\eta,\xi)$ defined by equation (4.12) is of class $\mathbb{F}(\tilde{\mathcal{A}} \times [t_0, T], h_{\eta\xi}, W)$ and so by Theorem (5.1)

$$\int_{t_0}^{t} DF(X(\tau), t)(\eta, \xi) = \int_{t_0}^{t} P(X(s), s)(\eta, \xi)ds, \quad t \in [t_0, T]. \tag{5.3}$$

Again, Theorem (5.3) asserts that X satisfies equation (5.2) if and only if

$$\langle \eta, X(t)\xi\rangle - \langle \eta, X(t_0)\xi\rangle = \int_{t_0}^{t} DF(X(\tau), t)(\eta, \xi)$$
$$= \int_{t_0}^{t} P(X(s), s)(\eta, \xi)ds$$

by equation (5.3). This follows if and only if

$$\frac{d}{dt}\langle \eta, X(t)\xi\rangle = P(X(t), t)(\eta, \xi)$$
$$\langle \eta, X(t_0)\xi\rangle = \langle \eta, X_0\xi\rangle$$

Hence equations (5.2) and (1.2) are equivalent.

As a consequence of the above results, we now describe a procedure for obtaining approximate solutions of equation (1.2) as follows. We assume hypothesis of Theorems (4.8), (5.1) and (5.2).

The initial value problem (1.2) is equivalent to the integral equation

$$\langle \eta, X(s)\xi\rangle = \langle \eta, X(t_0)\xi\rangle + \int_{t_0}^{s} P(X(u), u)(\eta, \xi)du \tag{5.4}$$

with the Lebesgue integral on the right hand side. If X is a solution of (1.2) on $[t_0, T]$, then by the existence and uniqueness results, X is adapted and weakly absolutely continuous and lie in $L^2_{loc}(\tilde{\mathcal{A}})$. Consequently the matrix elements of the solution can be approximated by matrix elements $\langle \eta, X_l(t)\xi\rangle$ of a simple process $X_l(t) \in \mathrm{Ad}(\tilde{\mathcal{A}})_{wac}$ which is constant on intervals of the form (t_{j-1}, t_j) where $t_0 < t_1 < \cdots < t_{kl} = t$ and which on (t_{j-1}, t_j) assumes the value $\langle \eta, X(\tau_j)\xi\rangle$ where $t_{j-1} \le \tau_j \le t_j, \quad j = 1, 2. \ldots, k_l$ such that

$$\lim_{l\to\infty} \langle \eta, X_l(s)\xi\rangle = \langle \eta, X(s)\xi\rangle \tag{5.5}$$

i.e $\lim_{l\to\infty} X_l(s) = X(s)$ uniformly on $[t_0, T]$.
Since

$$P(X, t)(\eta, \xi) \text{ is of class } C(\tilde{\mathcal{A}} \times [t_0, T], h_{\eta\xi}, W)$$

then

$$\lim_{l\to\infty} P(X_l(s), s)(\eta, \xi) = P(X(s), s)(\eta, \xi)$$

on $[t_0, T]$ by inequality (4.11).
Assuming that the sequence

$$P(X_l(s), s)(\eta, \xi), \quad l = 1, 2, \ldots$$

satisfies (4.10) then by the Lebesgue dominated convergence theorem it can be concluded that

$$\lim_{l\to\infty} \int_{t_0}^{t} P(X_l(s), s)(\eta, \xi)ds = \int_{t_0}^{t} P(X(s), s)(\eta, \xi)ds. \tag{5.6}$$

However, for a fixed $l \in \mathbb{N}$, we have

$$\begin{aligned}\int_{t_0}^{t} P(X_l(s), s)(\eta, \xi)ds &= \sum_{j=1}^{k_l} \int_{t_{j-1}}^{t_j} P(X(\tau_j), s)(\eta, \xi)ds \\ &= \sum_{j=1}^{k_l} [F(X(\tau_j), t_j)(\eta, \xi) - F(X(\tau_j), t_{j-1})(\eta, \xi)],\end{aligned}$$

which shows that the integral $\int_{t_0}^{t} P(X(s), s)(\eta, \xi)ds$ appearing in (5.6) can be approximated by the Kurzweil integral sums of the form

$$\sum_{j=1}^{k_l} [F(X(\tau_j), t_j)(\eta, \xi) - F(X(\tau_j), t_{j-1})(\eta, \xi)].$$

Finally, using (5.4) the matrix element $\langle \eta, X(t)\xi\rangle$ of the solution X can be approximated by the sum

$$\begin{aligned}\langle \eta, X(t)\xi\rangle \cong \langle \eta, X_0\xi\rangle + \sum_{j=1}^{k_l} [F(X(\tau_j), t_j)(\eta, \xi) \\ - F(X(\tau_j), t_{j-1})(\eta, \xi)]\end{aligned} \tag{5.7}$$

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provided that a sufficiently fine division $t_0 < t_1 < t_2 < \cdots < t_k = t$ is constructed and the choice of $\tau_j \in [t_{j-1}, t_j]$, $j = 1, 2, \ldots, k$ is fixed in order to obtain the uniform convergence (5.5).

6. NUMERICAL EXAMPLES

In the notation of section 1, we consider the simple Fock space $\Gamma(L^2_\gamma(\mathbb{R}_+))$ where $\gamma = \mathcal{R} = \mathbb{C}$, $f = g \equiv 1$, and its $L^2(\Omega, \mathcal{F}, W)$ realization where $(\Omega, \mathcal{F}, W)$ is a Wiener space. Each random variable X is identified with the operator of multiplication by X so that $Q(t) = A(t) + A^+(t) = w(t)$ is the evaluation of the Brownian path w at time t. In this case, it has been shown that quantum stochastic integrals of adapted Brownian functional F such that $\int_{t_0}^t E[F(s, \cdot)^2]ds < \infty$ exists (see [2,4]). Here E is the expected value function.

For exponential vectors $\eta = e(\alpha)$ and $\xi = e(\beta)$ where α, β are purely imaginary- valued functions in $L^2_{\mathbb{C}}(\mathbb{R}_+)$, the equivalent form (1.2) of the quantum analogue of the classical Ito stochastic differential equation

$$dX(t, w) = -\frac{1}{2}X(t, w)dt - \sqrt{1 - X^2(t, w)}dW(t)$$
$$X(t_0) = X_0, \quad t \in [0, T] \tag{6.1}$$

is given by

$$\frac{d}{dt}E(X(t, w)z(w)) = E(-\beta(t)\sqrt{1 - X^2(t, w)}z(w))$$
$$+ E(-\bar{\alpha}(t)\sqrt{1 - X^2(t, w)}z(w)) + E\left(-\frac{1}{2}X(t, w)z(w)\right)$$
$$X(t_0) = X_0, \quad t \in [t_0, T] \tag{6.2}$$

where

$$z(w) = \exp\left\{\int_0^\infty(-\alpha(s) + \beta(s))dw(s) - \frac{1}{2}\int_0^\infty(\alpha^2(s) + \beta^2(s))ds\right\} \tag{6.3}$$

(see [2,4] for details).

With $X_0(w) = 1$, $\alpha(t) = \beta(t) = i$, and the interval $[0, T] = [0, 1]$, then we have by equation (6.3), $z(w) = e$ and $E(X_0(w)z(w)) = E(z(w)) = e$.
The exact solution of equation (6.2) is then given by

$$E(X(t)z(w)) = e^{1-\frac{1}{2}t} \tag{6.4}$$

We now apply our approximation procedures to discretize equation (6.2)

For $t \in [0, 1]$, $X \in \tilde{\mathcal{A}}$ the map $P_{\alpha\beta}$ defined in section 1 is

$$P_{\alpha\beta}(t, X) = -\beta(t)\sqrt{1 - X^2(t)} - \bar{\alpha}(t)\sqrt{1 - X^2(t)} - \frac{1}{2}X(t) = -\frac{1}{2}X(t)$$

and

$$P(X, t)(\eta, \xi) = \left\langle \eta, \left(-\frac{1}{2}X(t)\right)\xi \right\rangle$$

for arbitrary $\eta, \xi \in \mathbb{D}\underline{\otimes}\mathbb{E}$. By equation (4.12)

$$F(X(\tau), t)(\eta, \xi) = \left(\left\langle \eta, \left(-\frac{1}{2}X(\tau)\right)\xi \right\rangle\right)t = -\frac{1}{2}tE(X(\tau)z(w))$$

Equation (6.2) is equivalent to (5.2) by Theorem (5.3). Thus we use Proposition (3.2) which leads to the approximation

$$\langle \eta, X(s)\xi \rangle \cong \langle \eta, X(\sigma)\xi \rangle + F(X(\sigma), s)(\eta, \xi) - F(X(\sigma), \sigma)(\eta, \xi) \tag{6.5}$$

for every $\sigma \in [0, 1]$ provided that $s \in [0, 1]$ is sufficiently close to σ. Thus from equation (6.5),

$$\begin{aligned} E(X(s)z(w)) &\cong E(X(\sigma)z(w)) - \frac{1}{2}sE(X(\sigma)z(w)) + \frac{1}{2}\sigma E(X(\sigma)z(w)) \\ &= \left(1 - \frac{1}{2}(s - \sigma)\right)E(X(\sigma)z(w)) \end{aligned} \tag{6.6}$$

Again by equation (5.7), we have

$$\langle \eta, X(t)\xi \rangle \cong \langle \eta, X_0\xi \rangle + \sum_{j=1}^{k_l} \int_{t_{j-1}}^{t_j} P(X(\tau_j), s)(\eta, \xi)ds$$

i.e

$$E(X(t)z(w)) \cong E(X_0 z(w)) - \frac{1}{2}\sum_{j=1}^{k_l} E(X(\tau_j)z(w))(t_j - t_{j-1}) \tag{6.7}$$

where $t_o < t_1 < t_2 < \cdots < t_{k_l} = t$ and $\tau_j \in [t_{j-1}, t_j]$.

If we fix $\tau_j = t_{j-1}$ for each $j = 1, 2..$ and a constant steplength h, then we have from equation (6.7)

$$E(X(t_j)z(w)) = \left(1 - \frac{1}{2}h\right)E(X(t_{j-1})z(w)), \quad j = 1, 2..N \tag{6.8}$$

Again, fixing $\tau_j = \frac{1}{2}(t_j + t_{j-1})$, $j = 1, 2..$ then equation (6.7) gives

$$E(X(t_j)z(w)) = E(X(t_{j-1})z(w)) - \frac{1}{2}hE(X(\tau_j)z(w)) \tag{6.9}$$

Table 1. Numerical Values with $\tau_j = t_{j-1}$ and $\alpha(t) = \beta(t) = i$

N	h	Approximate Values $E(X(t_N)z(w))$	Exact Values $e^{1-\frac{1}{2}t_N}$	Absolute Errors $\lvert E(X(t_N)z(w)) - e^{1-\frac{1}{2}t_N}\rvert$
8	2^{-3}	1.622051700	1.648721300	0.02666960000
16	2^{-4}	1.6356182000	1.6487212710	0.01310307000

where the intermediate values $E(X(\tau_j)z(w))$ are calculated by setting $\sigma = t_{j-1}$ and $s = \tau_j$ in (6.6) to give

$$E(X(\tau_j)(z(w)) = \left[1 - \frac{1}{2}(\tau_j - t_{j-1})\right]E(X(t_{j-1})z(w)) \tag{6.10}$$

By putting $h = 2^{-3}$, $h = 2^{-4}$ and fixing τ_j as above ,we generate the following tables of values for the case $\alpha = \beta = i$. Equations (6.8),(6.9),(6.10) are used to generate the following values at the final time $t = 1$ in Tables 1 and 2 below.

In order to compare the accuracy of the method of this paper ,we now apply the method to generate approximate values for the equivalent form of Ito equation

$$\begin{aligned} dX(t) &= \frac{3}{2}X(t)dt + X(t)dW(t) \\ X(t_0) &= 1, \quad t \in [0, 1] \end{aligned} \tag{6.11}$$

given by

$$\frac{d}{dt}E(X(t)z(w)) = \frac{3}{2}E(X(t, w)) \tag{6.12}$$

where $z(w) = e, t \in [0, 1]$, for $\alpha(t) = \beta(t) = i$ with exact solution $E(X)t, w)z(w) = e^{1+\frac{3}{2}t}$

Equation (6.12) had been discretized in [4] using the Euler and a 2-step scheme. We compare the results with those of the present scheme.

Table 2. Numerical Values with $\tau_j = \frac{1}{2}(t_j + t_{j-1})$ and $\alpha(t) = \beta(t) = i$

N	h	τ_N	$E(X(\tau_N)z(w))$	$E(X(t_N)z(w))$	Exact value $= e^{1-\frac{1}{2}t_N}$	Absolute Error $\lvert E(X(t_N)z(w)) - e^{1-\frac{1}{2}t_N}\rvert$
8	2^{-3}	0.9375	1.700716800	1.649283900	1.648721300	0.000562600
16	2^{-4}	0.96875	1.67460900	1.648858600	1.648721271	0.000137329

Table 3. Numerical values with $\tau_j = t_{j-1}$ and $\alpha(t) = \beta(t) = i$

N	h	Approximate Values $E(X(t_N)z(w))$	Exact values $e^{1-\frac{1}{2}t_N}$	Absolute Errors $\lvert E(X(t_N)z(w)) - e^{1-\frac{1}{2}t_N}\rvert$
8	2^{-3}	10.74888529	12.18249396	1.433608668
16	2^{-4}	11.402065680	12.18249396	0.78042828

For equation (6.11),we have the followings. From equation (6.12) and using (4.12),

$$F(X(\tau), t) = \frac{3}{2} t E(X(\tau)z(w))$$

and from (6.5)

$$E(X(s)z(w)) \cong \left(1 + \frac{3}{2}(s - \sigma)\right) E(X(\sigma)z(w)). \tag{6.13}$$

From (5.7),

$$E(X(t)z(w)) = E(X_0 z(w)) + \frac{3}{2} \sum_{j=1}^{k_l} E(X(\tau_j)z(w))(t_j - t_{j-1}). \tag{6.14}$$

Fixing $\tau_j = t_{j-1}$ for each $j = 1, 2 \cdots$ and $h = (t_j - t_{j-1})$, then we have from (6.14)

$$E(X(t_j)z(w)) = \left(1 + \frac{3}{2}h\right) E(X(t_{j-1})z(w)) \tag{6.15}$$

Again fixing $\tau_j = \frac{1}{2}(t_j + t_{j-1})$, we have from (6.14)

$$E(X(t_j)z(w) = E(X(t_{j-1})z(w)) + \frac{3}{2} h E(X(\tau_j)z(w)), \tag{6.16}$$

with intermediate values

$$E(X(\tau_j)z(w)) = \left[1 + \frac{3}{2}(\tau_j - t_{j-1})\right] E(X(t_{j-1})z(w)). \tag{6.17}$$

Our numerical experiments yield the following results at the final time $t = 1$. We use equation (6.15) for Table 3.

Equations (6.16) and (6.17) are used to generate Table 4.

6.1 Conclusion

(i) It is discovered that the schemes (6.8) and (6.15) when τ_j is fixed at the starting point of each subinterval of the partition points generate exactly the same values as Euler scheme considered in [4]. This is confirmed

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Table 4. Numerical values with $\tau_j = \frac{1}{2}(t_j + t_{j-1})$ and $\alpha(t) = \beta(t) = i$

N	h	τ_N	$E(X(\tau_N)z(w))$	$E(X(t_N)z(w))$	Exact value $= e^{1-\frac{1}{2}t_N}$	Absolute Error $\lvert E(X(t_N)z(w)) - e^{1-\frac{1}{2}t_N}\rvert$
8	2^{-3}	0.9375	10.97032998	12.08924593	12.18249396	0.093248032
16	2^{-4}	0.96875	11.58995788	12.15756309	12.18249396	0.024930868

by Tables 1 and 3 . The Tables also show that the approximate schemes produce better results with finer gridpoints.

(ii) However, Tables 2 and 4 show a more superior convergence rate when τ_j is fixed at the midpoint of each subintervals of partition. In Table 2, with constant steplengths $h = 2^{-3}$ and $h = 2^{-4}$,we have convergence to at least three decimal places at each of the gridpoint with cummulative absolute errors at the end point $t = 1$ being 0.000562600 and 0.000137329 respectively. This experiment shows that the approximate scheme (5.7) has a superior convergence rate when τ_j is taken as midpoints of each subinterval and expression (6.5) is used to compute intermediate values. This level of accuracy is comparable to that of a 2- stage Runge-Kutta scheme reported in [2] applied to problem (6.12).

In comparison with the Euler and the 2-step method applied in [4] to problem (6.11), Table 4 shows that the method of this paper is more accurate than those two schemes when τ_j is taken as the midpoint of each of the partition subinterval. In particular, for a steplength of $h = 2^{-3}$, the global accumulated error at the final time $t = 1$ is 0.093248032 compared with the global errors of 0.39983038 and 0.11070989 with $h = 2^{-3}$ and 2^{-4} respectively for the 2-step scheme (see [4]). We remark that equation (5.7) permits a change of steplenghts at any point during computation and that this method is suitable for equation (1.2) where the map $(t, x) \mapsto P(t, x)(\eta, \xi)$ is not necessarily continuous jointly in t and x and the matrix elements are not necessarily differentiable more than one time.

In particular, the methods developed in this paper provide a simple approach for computations of expectations of functionals of Ito processes when the quantum equations are considered only in the simple Fock spaces. Applications of the methods to problems in quantum fields/systems will be considered elsewhere.

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REFERENCES

1. Z. Artstein, Topological Dynamics of Ordinary Differential Equations And Kurzweil Equations, *Journal of Differential Equations* 23, (1977), 224–243.
2. E.O. Ayoola, On Numerical Procedures For Solving Lipschitzian Quantum Stochastic Differential Equations. *Ph.D Thesis (*1998*), University of Ibadan, Nigeria.*
3. E.O. Ayoola, Solutions of Lipschizian Quantum Stochastic Differential Equations In A Locally Convex Topological Space. *To Appear In Journal Of Science Research,* Vol. 4, no. 1 (1999).
4. E.O. Ayoola, Converging Multistep Methods For Weak Solutions Of Quantum Stochastic Differential Equations, *Stochastic Analysis And Applications*, Vol.18, no. 4 (2000) pp. 525–554.
5. P.S. Bullen, A survey on integration by parts for Perron integrals, *J. Austral. Math. Soc. (Series A)*, 40, (1986), 343–363.
6. T.S. Chew, On Kurzweil generalized ordinary differential equations, *J. Differential Equations*, 76, no 2 (1988), 286–293.
7. T.S. Chew, F. Flordeliza, On $X' = f(t, X)$ and Henstock - Kurzweil integrals, *Differential and Integral Equations*, 4 (1991) 861–868.
8. G.O.S. Ekhaguere, Lipschitzian quantum stochastic differential inclusions, *Internat. J. Theoret. Phys.* Vol. 31, No. 11, (1992) 2003–2034.
9. G.O.S. Ekhaguere, Quantum stochastic differential inclusions of hypermaximal monotone type, *Internat. J. Theoret. Phys.* Vol. 34, no. 3, (1995), 323–353.
10. G.O.S. Ekhaguere, Quantum stochastic evolutions. *Internat. J. Theoret. Phys.* Vol. 35, no. 9, (1996), 1909–1946.
11. R.L. Hudson, *Quantum Stochastic Calculus and Some of its Applications.* Contemporary Stochastic Analysis 31–70, edited by G.O.S. Ekhaguere, World Scientific (1991).
12. R.L. Hudson, and K.R. Parthasarathy, *Construction of quantum diffusion.* Lecture Notes in Math. 1055, 173–198. Springer (1982).
13. R.L Hudson, and K.R. Parthasarathy, Quantum Ito's formulae and stochastic evolutions, *Commun. Math. Phys. 93,* (1984) 301–324
14. P.E. Kloeden, and E. Platen, *Numerical solutions of stochastic differential equations.* Springer-Verlag (1992).
15. P. Meyer, Quantum Probability for Probabilists. *Lecture Notes in Math. 1538* Springer-Verlag, (1993).

16. K.R. Parthasarathy, *An Introduction to Quantum Stochastic Calculus,* Monographs in Mathematics, Vol. 85, Birkhauser Verlag (1992).
17. K.R. Parthasarathy, Boson Stochastic Calculus, *Pramana J. Phys.* 25 (1985) 457–465.
18. S. Schwabik, *Generalized Ordinary Differential Equations.* World Scientific, (1992).

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Lower Semicontinuous Quantum Stochastic Differential Inclusions

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Abstract. In this paper we established the existence of solutions of Lower semicontinuous quantum stochastic differential inclusions(QSDI). The existence of a continuous selection of a predefined integral operator was established. This selection which is an adapted stochastic process is a solution of the Lower semicontinuous quantum stochastic differential inclusions.

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1. Introduction

The theory of quantum stochastic differential inclusions is a multivalued analogue of quantum stochastic calculus of Hudson and Parthasarathy formulation [9]. The theory of differential inclusions has vast applications and one of its motivations is the application in the study of control theory. In [5] the existence of solutions of quantum stochastic differential inclusions with Lipschitzian coefficients lying in certain locally convex spaces was established. A further study of this quantum stochastic differential inclusions was done in [6] with hypermaximal monotone type and in [7] for evolution type. The topological properties of solution sets and existence of continuous selections of the solution sets for the Lipschitzian quantum stochastic differential inclusions were established in [2] and [3].
For a classical differential inclusion the existence of solutions of discontinuous cases, upper and lower semicontinuous differential inclusions were established in [1] and [4]. These weaker forms of regularity of the coefficients are also applicable in the study of optimal quantum stochastic control theory [10]. The aim of this work is to establish the existence of solution of Lower semicontinuous quantum stochastic differential inclusions. We first define

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an integral operator which is a mapping consisting of adapted stochastic processes and established the existence of a continuous map which is a selection of the mapping. Hence we established the existence of at least a solution of the Lower semicontinuous quantum stochastic differential inclusions. This is a generalization of the result in [1] to our non commutative setting. This will lead in a later work to further applications of quantum stochastic calculus to quantum stochastic differential equations with discontinuous coefficients, solutions of pertinent quantum stochastic control problems and quantum optics. In sequel the work shall be arranged as follows: section 2 shall be for preliminaries on notations and definitions while section 3 shall be for our main results.

2. Preliminaries

In this section we state the definitions and notations which shall be employed in the sequel.

2.1. Notations

In what follows, if U is a topological space, we denote by clos(U), the collection of all non-empty closed subsets of U.
To each pair (D,H) consisting of a pre-Hilbert space D and its completion H, we associate the set $L_w^+(D,H)$ of all linear maps x from D into H, with the property that the domain of the operator adjoint contains D. The members of $L_w^+(D,H)$ are densely-defined linear operators on H which do not necessarily leave D invariant and $L_w^+(D,H)$ is a linear space when equipped with the usual notions of addition and scalar multiplication.
To H corresponds a Hilbert space $\Gamma(H)$ called the boson Fock space determined by H. A natural dense subset of $\Gamma(H)$ consists of linear space generated by the set of exponential vectors(Guichardet, [8]) in $\Gamma(H)$ of the form

$$e(f) = \bigoplus_{n=0}^{\infty}(n!)^{-\frac{1}{2}}\bigotimes^{n} f, \quad f \in H,$$

where $\bigotimes^0 f = 1$ and $\bigotimes^n f$ is the n-fold tensor product of f with itself for $n \geq 1$.
In what follows, $\mathbb{D}$ is some pre-Hilbert space whose completion is $\mathscr{R}$ and γ is a fixed Hilbert space.
$L_\gamma^2(\mathbb{R}_+)$(resp. $L_\gamma^2([0,t))$, resp. $L_\gamma^2([t,\infty))$ $t \in \mathbb{R}_+$) is the space of square integrable γ-valued maps on $\mathbb{R}_+$ (resp .$[0,t)$, resp. $[t,\infty)$).
The inner product of the Hilbert space $\mathscr{R} \otimes \Gamma(L_\gamma^2(\mathbb{R}_+))$ will be denoted by $\langle .,. \rangle$ and $\| . \|$ the norm induced by $\langle .,. \rangle$.
Let $\mathbb{E}, \mathbb{E}_t$ and $\mathbb{E}^t$, $t > 0$ be linear spaces generated by the exponential vectors in Fock spaces $\Gamma(L_\gamma^2(\mathbb{R}_+)), \Gamma(L_\gamma^2([0,t)))$ and $\Gamma(L_\gamma^2([t,\infty)))$ respectively ;

$$\begin{aligned}
\mathscr{A} &\equiv L_w^+(\mathbb{D}\underline{\otimes}\mathbb{E}, \mathscr{R} \otimes \Gamma(L_\gamma^2(\mathbb{R}_+))) \\
\mathscr{A}_t &\equiv L_w^+(\mathbb{D}\underline{\otimes}\mathbb{E}_t, \mathscr{R} \otimes \Gamma(L_\gamma^2([0,t)))) \otimes \mathbb{I}^t \\
\mathscr{A}^t &\equiv \mathbb{I}_t \otimes L_w^+(\mathbb{E}^t, \Gamma(L_\gamma^2([t,\infty)))),\ t > 0
\end{aligned}$$

where $\underline{\otimes}$ denotes algebraic tensor product and $\mathbb{I}_t$(resp.$\mathbb{I}^t$) denotes the identity map on $\mathcal{R} \otimes \Gamma(L^2_\gamma([0,t))))$(resp .$\Gamma(L^2_\gamma([t,\infty))))$, $t > 0$ For every $\eta, \xi \in \mathbb{D}\underline{\otimes}\mathbb{E}$ define

$$\| x \|_{\eta\xi} = | \langle \eta, x\xi \rangle |, \quad x \in \mathcal{A}$$

then the family of seminorms

$$\{\| . \|_{\eta\xi} : \eta, \xi \in \mathbb{D}\underline{\otimes}\mathbb{E}\}$$

generates a topology τ_w , weak topology .
The completion of the locally convex spaces $(\mathcal{A}, \tau_w)$, $(\mathcal{A}_t, \tau_w)$ and $(\mathcal{A}^t, \tau_w)$ are respectively denoted by $\widetilde{\mathcal{A}}$, $\widetilde{\mathcal{A}_t}$ and $\widetilde{\mathcal{A}^t}$.
We define the Hausdorff topology on clos$(\widetilde{\mathcal{A}})$ as follows:
For $x \in \widetilde{\mathcal{A}}$, $\mathcal{M}, \mathcal{N} \in$ clos$(\widetilde{\mathcal{A}})$ and $\eta, \xi \in \mathbb{D}\underline{\otimes}\mathbb{E}$, define

$$\rho_{\eta\xi}(\mathcal{M}, \mathcal{N}) \equiv \max(\delta_{\eta\xi}(\mathcal{M}, \mathcal{N}), \delta_{\eta\xi}(\mathcal{N}, \mathcal{M})),$$

where

$$\delta_{\eta\xi}(\mathcal{M}, \mathcal{N}) \equiv \sup_{x \in \mathcal{M}} \mathbf{d}_{\eta\xi}(x, \mathcal{N}),$$

and

$$\mathbf{d}_{\eta\xi}(x, \mathcal{N}) \equiv \inf_{y \in \mathcal{N}} \| x - y \|_{\eta\xi} .$$

The Hausdorff topology which shall be employed in what follows, denoted by, τ_H , is generated by the family of pseudometrics $\{\rho_{\eta\xi}(.) : \eta, \xi \in \mathbb{D}\underline{\otimes}\mathbb{E}\}$. Moreover, if $\mathcal{M} \in clos(\widetilde{\mathcal{A}})$, then $\| \mathcal{M} \|_{\eta\xi}$ is defined by

$$\| \mathcal{M} \|_{\eta\xi} \equiv \rho_{\eta\xi}(\mathcal{M}, \{0\});$$

for arbitrary $\eta, \xi \in \mathbb{D}\underline{\otimes}\mathbb{E}$.
For $A, B \in clos(\mathbb{C})$ and $x \in \mathbb{C}$, a complex number, define

$$\mathbf{d}(x, B) \equiv \inf_{y \in B} | x - y |,$$

$$\delta(A, B) \equiv \sup_{x \in A} \mathbf{d}(x, B),$$

and

$$\rho(A, B) \equiv \max(\delta(A, B), \delta(B, A)).$$

Then ρ is a metric on $clos(\mathbb{C})$ and induces a metric topology on the space. We also define:

$$d_{\eta\xi}((t, x), (t_0, x_0)) = \max\{| t - t_0 |, \| x - x_0 \|_{\eta\xi}\}.$$

Let $I \subseteq \mathbb{R}_+$. A *stochastic process* indexed by I is an $\widetilde{\mathcal{A}}$-valued measurable map on I.
A stochastic process X is called *adapted* if $X(t) \in \widetilde{\mathcal{A}_t}$ for each $t \in I$.
We write Ad$(\widetilde{\mathcal{A}})$ for the set of all adapted stochastic processes indexed by I.

Definition 1. *A member X of $Ad(\widetilde{\mathscr{A}})$ is called*

(i) weakly absolutely continuous if the map $t \mapsto \langle \eta, X(t)\xi \rangle$, $t \in I$ is absolutely continuous for arbitrary $\eta, \xi \in \mathbb{D}\underline{\otimes}\mathbb{E}$,

(ii) locally absolutely p-integrable if $\| X(.) \|^p_{\eta\xi}$ is Lebesgue -measurable and integrable on $[0,t) \subseteq I$ for each $t \in I$ and arbitrary $\eta, \xi \in \mathbb{D}\underline{\otimes}\mathbb{E}$.

We denote by $\mathrm{Ad}(\widetilde{\mathscr{A}})_{wac}$(resp.$L^p_{loc}(\widetilde{\mathscr{A}})$) the set of all weakly, absolutely continuous(resp. locally absolutely p-integrable) members of $\mathrm{Ad}(\widetilde{\mathscr{A}})$.
Stochastic integrators: Let $L^{\infty}_{\gamma,loc}(\mathbb{R}_+)$ [resp.$L^{\infty}_{B(\gamma),loc}(\mathbb{R}_+)$] be the linear space of all measurable , locally bounded functions from $\mathbb{R}_+$ to γ [resp. to B(γ) , the Banach space of bounded endomorphisms of γ]. If $f \in L^{\infty}_{\gamma,loc}(\mathbb{R}_+)$ and $\pi \in L^{\infty}_{B(\gamma),loc}(\mathbb{R}_+)$, then πf is the member of $L^{\infty}_{\gamma,loc}(\mathbb{R}_+)$ given by $(\pi f)(t) = \pi(t)f(t)$, $t \in \mathbb{R}_+$.
For $f \in L^2_{\gamma}(\mathbb{R})_+$ and $\pi \in L^{\infty}_{B(\gamma),loc}(\mathbb{R}_+)$; the annihilation , creation and gauge operators, $a(f), a^+(f)$ and $\lambda(\pi)$ in $L^+_w(\mathbb{D}, \Gamma(L^2_{\gamma}(\mathbb{R})_+))$ respectively, are defined as:

$$a(f)\mathbf{e}(g) = \langle f, g \rangle_{L^2_{\gamma}(\mathbb{R}_+)}\mathbf{e}(g),$$
$$a^+(f)\mathbf{e}(g) = \frac{d}{d\sigma}\mathbf{e}(g + \sigma f) \mid_{\sigma=0},$$
$$\lambda(\pi)\mathbf{e}(g) = \frac{d}{d\sigma}\mathbf{e}(e^{\sigma\pi}f) \mid_{\sigma=0}$$

for all $g \in L^2_{\gamma}(\mathbb{R}_+)$.
For arbitrary $f \in L^{\infty}_{\gamma,loc}(\mathbb{R}_+)$ and $\pi \in L^{\infty}_{B(\gamma),loc}(\mathbb{R}_+)$, they give rise to the operator-valued maps A_f, A^+_f and Λ_π defined by:

$$A_f(t) \equiv a(f\chi_{[0,t)}),$$
$$A^+_f(t) \equiv a^+(f\chi_{[0,t)}),$$
$$\Lambda_\pi(t) \equiv \lambda(\pi\chi_{[0,t)})$$

for all $t \in \mathbb{R}_+$, where χ_I denotes the indicator function of the Borel set $I \subseteq \mathbb{R}_+$. The maps A_f, A^+_f and Λ_π are stochastic processes , called annihilation, creation and gauge processes, respectively, when their values are identified with their ampliations on $\mathscr{R} \otimes \Gamma(L^2_{\gamma}(\mathbb{R}_+))$. These are the stochastic integrators in Hudson and Parthasarathy[9] formulation of boson quantum stochastic integration.
For processes $p, q, u, v \in L^2_{loc}(\widetilde{\mathscr{A}})$, the quantum stochastic integral:

$$\int_{t_0}^{t} \big(p(s)d\Lambda_\pi(s) + q(s)dA_f(s) + u(s)dA^+_g(s) + v(s)ds\big), \quad t_0, t \in \mathbb{R}_+$$

is interpreted in the sense of Hudson-Parthasarathy[9].

2.2. Quantum Stochastic Differential Inclusions

Definition 2. *(a) By a multivalued stochastic process indexed by $I \subseteq \mathbb{R}_+$ we mean a multifunction on I with values in $clos(\widetilde{\mathscr{A}})$.*

(b) If Φ is a multivalued stochastic process indexed by $I \subseteq \mathbb{R}_+$, then a selection of Φ is a stochastic process $X : I \to \widetilde{\mathscr{A}}$ with the property that $X(t) \in \Phi(t)$ for almost all $t \in I$.

(c) A multivalued stochastic process Φ will be called (i) adapted if $\Phi(t) \subseteq \widetilde{\mathscr{A}_t}$ for each $t \in \mathbb{R}_+$; (ii) measurable if $t \mapsto \mathbf{d}_{\eta\xi}(x, \Phi(t))$ is measurable for arbitrary $x \in \widetilde{\mathscr{A}}, \eta, \xi \in (\mathbb{D}\underline{\otimes}\mathbb{E})$

(d) locally absolutely p-integrable if $t \mapsto \| \Phi(t) \|_{\eta\xi}$, $t \in \mathbb{R}_+$ lie in $L^p_{loc}(I)$ for arbitrary $\eta, \xi \in (\mathbb{D}\underline{\otimes}\mathbb{E})$.

For $p \in (0, \infty)$ and $I \subseteq \mathbb{R}_+$, the set of all locally absolutely p-integrable multivalued stochastic processes will be denoted by $L^p_{loc}(\widetilde{\mathscr{A}})_{mvs}$. Denote by $L^p_{loc}(I \times \widetilde{\mathscr{A}})_{mvs}$ the set of maps $\Phi : I \times \widetilde{\mathscr{A}} \to clos(\widetilde{\mathscr{A}})$ such that $t \mapsto \Phi(t, X(t))$, $t \in I$, lies in $L^p_{loc}(\widetilde{\mathscr{A}})_{mvs}$ for every $X \in L^p_{loc}(\widetilde{\mathscr{A}})$.
Moreover, if $\Phi \in L^p_{loc}(I \times \widetilde{\mathscr{A}})_{mvs}$, then we denote by

$$L_p(\Phi) \equiv \{\phi \in L^p(\widetilde{\mathscr{A}}) : \phi \text{ is a selection of } \Phi\}.$$

Let $f, g \in L^2_\gamma(\mathbb{R}_+)$, $\pi \in L^\infty_{B(\gamma),loc}(\mathbb{R}_+)$, $\mathbb{I}$, the identity map on $\mathscr{R} \otimes \Gamma(L^2_\gamma(\mathbb{R}_+))$, and M is any of the stochastic processes A_f, A^+_g, Λ_π and $s \mapsto s\mathbb{I}$, $s \in \mathbb{R}_+$.
We introduce the *stochastic integral*{resp. *differential*} expressions as follows:
If $\Phi \in L^2_{loc}(I \times \widetilde{\mathscr{A}})_{mvs}$ and $(t, X) \in I \times L^2_{loc}(\widetilde{\mathscr{A}})$, then

$$\int_{t_0}^t \Phi(s, X(s))dM(s) \equiv \left\{ \int_{t_0}^t \phi(s)dM(s) : \phi \in L_2(\Phi) \right\}.$$

This leads to the following definition:

Definition 3. *Let $E, F, G, H \in L^2_{loc}(I \times \widetilde{\mathscr{A}})$ and (t_0, x_0) be a fixed point of $I \times \widetilde{\mathscr{A}}$. Then a relation of the form*

$$\begin{aligned} dX(t) \in\ & E(t, X(t))d\Lambda_\pi(t) + F(t, X(t))dA_f(t) \\ & + G(t, X(t))dA^+_g(t) + H(t, X(t))dt \text{ almost all } t \in I, \\ X(t_0) =\ & x_0 \end{aligned} \tag{1}$$

is called Quantum stochastic differential inclusions(QSDI) with coefficients E, F, G, H and initial data (t_0, x_0).
Equation(1) is understood in the integral form:

$$\begin{aligned} X(t) \in x_0 + \int_{t_0}^t \big(& E(s, X(s))d\Lambda_\pi(s) + F(s, X(s))dA_f(s) \\ & + G(s, X(s))dA^+_g(s) + H(s, X(s))ds \big), \text{ almost all } t \in I \end{aligned}$$

called a stochastic integral inclusion with coefficients E, F, G, H and initial data (t_0, x_0)

An equivalent form of (1) has been established in [5], Theorem 6.2 as follows:
For $\eta, \xi \in \mathbb{D}\underline{\otimes}\mathbb{E}$, $\alpha, \beta \in L^2_\gamma(\mathbb{R}_+)$ with $\eta = c \otimes e(\alpha)$, $\xi = d \otimes e(\beta)$, define the following complex-valued functions:

$$\mu_{\alpha\beta},\ \nu_\beta,\ \sigma_\alpha : I \to \mathbb{C}, \quad I \subset \mathbb{R}_+$$

by

$$\begin{aligned} \mu_{\alpha\beta}(t) &= \langle \alpha(t), \pi(t)\beta(t)\rangle_\gamma, \\ \nu_\beta(t) &= \langle f(t), \beta(t)\rangle_\gamma, \\ \sigma_\alpha(t) &= \langle \alpha(t), g(t)\rangle_\gamma, \end{aligned}$$

$t \in I$, $f, g \in L^2_{\gamma,loc}(\mathbb{R}_+)$, $\pi \in L^\infty_{B(\gamma),loc}$. To these functions we associate the maps μE, νF, σG, $\mathbb{P}$ from $I \times \widetilde{\mathscr{A}}$ into the set of sesquilinear forms on $\mathbb{D}\underline{\otimes}\mathbb{E}$ defined by :

$$\begin{aligned} (\mu E)(t,x)(\eta,\xi) &= \{\langle \eta, \mu_{\alpha\beta}(t)p(t,x)\xi\rangle : p(t,x) \in E(t,x)\}, \\ (\nu F)(t,x)(\eta,\xi) &= \{\langle \eta, \nu_\beta(t)q(t,x)\xi\rangle : q(t,x) \in F(t,x)\}, \\ (\sigma G)(t,x)(\eta,\xi) &= \{\langle \eta, \sigma_\alpha(t)u(t,x)\xi\rangle : u(t,x) \in G(t,x)\}, \\ \mathbb{P}(t,x)(\eta,\xi) &= (\mu E)(t,x)(\eta,\xi) + (\nu F)(t,x)(\eta,\xi) \\ &\quad + (\sigma G)(t,x)(\eta,\xi) + H(t,x)(\eta,\xi), \\ H(t,x)(\eta,\xi) &= \{v(t,x)(\eta,\xi) : v(.,X(.)) \end{aligned} \tag{2}$$

is a selection of

$$H(.,X(.))\forall X \in L^2_{loc}(\widetilde{\mathscr{A}})\}. \tag{3}$$

Then, Problem (1) is equivalent to

$$\begin{aligned} \frac{d}{dt}\langle \eta, X(t)\xi\rangle &\in \mathbb{P}(t,X(t))(\eta,\xi), \\ X(t_0) &= x_0 \end{aligned} \tag{4}$$

for arbitrary $\eta, \xi \in \mathbb{D}\underline{\otimes}\mathbb{E}$, almost all $t \in I$.
The notion of solution of (1) or equivalently (3) is defined as follows:

Definition 4. *By a solution of (1) or equivalently* (3)*, we mean a stochastic process* $\varphi \in Ad(\widetilde{\mathscr{A}})_{wac} \cap L^2_{loc}(\widetilde{\mathscr{A}})$ *such that*

$$\begin{aligned} d\varphi(t) &\in E(t,\varphi(t))d\Lambda_\pi(t) + F(t,\varphi(t))dA_f(t) \\ &\quad + G(t,\varphi(t))dA^+_g(t) + H(t,\varphi(t))dt \text{ almost all } t \in I, \\ \varphi(t_0) &= \varphi_0 \end{aligned}$$

or equivalently

$$\begin{aligned} \frac{d}{dt}\langle \eta, \varphi(t)\xi\rangle &\in \mathbb{P}(t,\varphi(t))(\eta,\xi), \\ \varphi(t_0) &= \varphi_0 \end{aligned}$$

for arbitrary $\eta, \xi \in \mathbb{D}\underline{\otimes}\mathbb{E}$ *, almost all* $t \in I$*.*

The existence of solution of (1) implies the existence of solution of (3) and vice-versa. As explained in [5], for the map $\mathbb{P}$:

$$\mathbb{P}(t,x)(\eta,\xi) \neq \widetilde{\mathbb{P}}(t,\langle\eta,x\xi\rangle)$$

for some complex-valued multifunction $\widetilde{\mathbb{P}}$ defined on $I\times\mathbb{C}$ for $t\in I$, $x\in\widetilde{\mathscr{A}}$, $\eta,\xi\in\mathbb{D}\underline{\otimes}\mathbb{E}$.

2.3. Lower Semicontinuous Multivalued Maps

Definition 5. *(a) Let $\mathscr{N}\in clos(\widetilde{\mathscr{A}})$ be non-empty and $I\subseteq\mathbb{R}_+$.*
A multifunction $\Phi: I\times\mathscr{N}\to clos(\widetilde{\mathscr{A}})$ will be said to be lower semicontinuous at a point $(t_0,x_0)\in I\times\mathscr{N}$, if for every $\epsilon>0,\eta,\xi\in\mathbb{D}\underline{\otimes}\mathbb{E}$ there exists $\delta_{\eta\xi}=\delta_{\eta\xi}((t_0,x_0),\epsilon)>0$ such that $\forall x\in\mathscr{N}$, $t\in I$ if

$$d_{\eta\xi}\big((t,x),(t_0,x_0)\big)<\delta_{\eta\xi} \text{ then } \Phi(t_0,x_0)\subset\Phi(t,x)+B_{\eta\xi,\epsilon}(0).$$

If Φ is lower semicontinuous(lsc) at every point $(t_0,x_0)\in I\times\mathscr{N}$, then it will be said to be lower semicontinuous on $I\times\mathscr{N}$.

(b) Analogously if Φ is a sesquilinear form valued multifunction, then the map $\Phi: I\times\mathscr{N}\to 2^{sesq(\mathbb{D}\underline{\otimes}\mathbb{E})^2}$ will be said to be lower semicontinuous at a point $(t_0,x_0)\in I\times\mathscr{N}$, if for every $\eta,\xi\in\mathbb{D}\underline{\otimes}\mathbb{E},\epsilon>0$ there exists $\delta_{\eta\xi}=\delta_{\eta\xi}((t_0,x_0),\epsilon)>0$ such that $\forall x\in\mathscr{N}$, $t\in I$, if

$$d_{\eta\xi}\big((t,x),(t_0,x_0)\big)<\delta_{\eta\xi} \text{ then } \Phi(t_0,x_0)(\eta,\xi)\subset\Phi(t,x)(\eta,\xi)+B_\epsilon(0).$$

In what follows, a map shall be called lower semicontinuous on a domain if it is so at every point of the domain.

The next result shows that, if μE, νF, σG, H are lower semicontinuous then $(t,x)\to\mathbb{P}(t,x)(\eta,\xi)$ is lower semicontinuous.

Proposition 1. *Assume that the following holds:*

(i) The coefficients E,F,G,H appearing in (1) belongs to the space $L^2_{loc}(I\times\widetilde{\mathscr{A}})_{mvs}$.

(ii) For an arbitrary elements $\eta,\xi\in\mathbb{D}\underline{\otimes}\mathbb{E}$, the maps μE, νF, σG, H defined by equation (2) are lower semicontinuous on $I\times\widetilde{\mathscr{A}}$.

Then, the map $(t,x)\to\mathbb{P}(t,x)(\eta,\xi)$ is lower semicontinuous on $I\times\widetilde{\mathscr{A}}$.

Proof. For an arbitrary $\eta,\xi\in\mathbb{D}\underline{\otimes}\mathbb{E}$, since $\mu E,\nu F,\sigma G,H$ are lower semicontinuous $I\times\widetilde{\mathscr{A}}$. Then for any point $(t_0,x_0)\in I\times\widetilde{\mathscr{A}}$, given $\epsilon>0$, there exist $\delta_{\eta\xi,E},\delta_{\eta\xi,F},\delta_{\eta\xi,G},\delta_{\eta\xi,H}>0$, such that for each $M\in\{\mu E,\nu F,\sigma G,H\}$,

$$M(t_0,x_0)(\eta,\xi)\subset M(t,x)(\eta,\xi)+B_\epsilon(0)\ \ \forall x\in\mathscr{N}, \text{ almost all } t\in I \text{ and } d_{\eta\xi}\big((t,x),(t_0,x_0)\big)<\delta_{\eta\xi,M}.$$

Hence the proposition follows from the relation:

$$\begin{aligned}\mathbb{P}(t_0,x_0)(\eta,\xi) &= (\mu E)(t_0,x_0)(\eta,\xi) + (\nu F)(t_0,x_0)(\eta,\xi)\\ &\quad + (\sigma G)(t_0,x_0)(\eta,\xi) + H(t_0,x_0)(\eta,\xi) + B_{\epsilon}(0)\\ &\subset \mathbb{P}(t,x)(\eta,\xi) + B_{5\epsilon}(0).\end{aligned}$$

3. Main Results

In this subsection under some assumptions, we prove an existence theorem for lower semi-continuous quantum stochastic differential inclusions by using a predefined integral operator.

Definition 6. *Let $C(I)$ be the space of continuous maps from I to $sesq(\mathbb{D}\underline{\otimes}\mathbb{E})$. For all $\eta,\xi \in \mathbb{D}\underline{\otimes}\mathbb{E}$; $X,Z \in Ad(\widetilde{\mathscr{A}})_{wac} \cap L^2_{loc}(\widetilde{\mathscr{A}})$, we define the set:*

$$\mathscr{K}_{\eta\xi} = \{\langle \eta, X(t)\xi\rangle \in C(I) : \exists \lambda \in \mathbb{R}_+; |\langle \eta,(X(t)-X(s))\xi\rangle| < \lambda\, |t-s|, t,s \in I \text{ and } X(t_0) = x_0\}.$$

Moreover, the integral operator $\mathscr{F}_{\eta\xi}$ is defined as

$$\mathscr{F}_{\eta\xi}(X) = \{\langle \eta, Z(t)\xi\rangle \in \mathscr{K}_{\eta\xi} : \frac{d}{dt}\langle \eta, Z(t)\xi\rangle \in \mathbb{P}(t,X(t))(\eta,\xi) \text{ a.e. } t \in I\}.$$

We also define the following sets as applicable in the subsequent result.
For any $(t,x),(t_0,x_0) \in I \times \widetilde{\mathscr{A}}, \lambda_{\eta\xi} > 0$, a real number; $\eta,\xi \in \mathbb{D}\underline{\otimes}\mathbb{E}$.

$$Q_{(t_0,x_0),\lambda_{\eta\xi}} = \{(t,x) \in I \times \widetilde{\mathscr{A}} : d_{\eta\xi}((t,x),(t_0,x_0)) \le \lambda_{\eta\xi}\},$$

where

$$\begin{aligned}d_{\eta\xi}((t,x),(t_0,x_0)) &= \max\{|t-t_0|, \| x-x_0 \|_{\eta\xi}\},\\ Q_{x_0,\lambda_{\eta\xi}} &= \{x \in \widetilde{\mathscr{A}} : \| x-x_0 \|_{\eta\xi} < \lambda_{\eta\xi}\},\\ Q_{\lambda_{\eta\xi}} &= \{x \in \widetilde{\mathscr{A}} : \| x \|_{\eta\xi} < \lambda_{\eta\xi}\},\end{aligned}$$

and set

$$Q_{\epsilon}(\eta,\xi) = \{\langle \eta, x\xi\rangle : x \in Q_{\epsilon}\}.$$

In what follows, we make the following assumptions:
$I = [t_0, T]$, $\lambda_{\eta\xi} > 0$ and $\Omega \subset I \times \widetilde{\mathscr{A}}$, open, such that:

(i) $I \times Q_{x_0,\frac{T}{2}\lambda_{\eta\xi}} \subseteq \Omega$,

(ii) $\exists\, \lambda_{M,\eta\xi} > 0\ \forall\, M \in \{E,F,G,H\}$ with $\max_M \lambda_{M,\eta\xi} < \lambda_{\eta\xi}$ and

(iii) $\| M(t,x) \|_{\eta\xi} \le \lambda_{M,\eta\xi}$ for each M on $I \times Q_{x_0,\frac{T}{2}\lambda_{\eta\xi}}$.

Lemma 1. *Suppose that $K \subseteq I \times \widetilde{\mathscr{A}}$ is compact.*
For arbitrary pair $\eta, \xi \in \mathbb{D}\underline{\otimes}\mathbb{E}$, suppose that the multivalued map

$$(t,x) \to M(t,x)(\eta,\xi)$$

is lower semicontinuous for each $M \in \{\mu E, \nu F, \sigma G, H\}$.
For $\epsilon > 0$, set

$$\omega_{\eta\xi,\epsilon}(t,x) = \sup\{\omega_{\eta\xi} : \bigcap_{(\tau,\zeta)\in Q_{(t,x),\omega_{\eta\xi}}} M(\tau,\zeta)(\eta,\xi) + Q_\epsilon(\eta,\xi) \neq \emptyset\}. \tag{5}$$

Then

(a) *for some $\omega_\epsilon > 0$ we have*

$$\omega_{\eta\xi,\epsilon}(t,x) \geq \omega_\epsilon \text{ for all } (t,x) \in I \times \widetilde{\mathscr{A}}, \eta, \xi \in \mathbb{D}\underline{\otimes}\mathbb{E},$$

(b) *for every continuous u , $(t,u(t)) \in K$, there exists a measurable map $t \to v(t)(\eta,\xi)$, such that*

$$d_{\eta\xi}((t,x),(t,u(t))) < \omega_\epsilon,$$

implies

$$d(v(t)(\eta,\xi), M(t,x)(\eta,\xi)) \leq \epsilon.$$

Proof. (a) The definition of lower semicontinuity implies that the set inside brackets in (4) is non-empty , so that $\omega_{\eta\xi,\epsilon}(t,x)$ is positive.
We claim that it is a continuous function.
Fix $\sigma > 0$ arbitrarily, and remark that whenever $d_{\eta\xi}((\tau_1,\zeta_1),(\tau_2,\zeta_2)) < \frac{\sigma}{3}$,

$$Q^1 = Q_{(\tau_1,\zeta_1),\omega_\epsilon((\tau_2,\zeta_2)-\frac{2\sigma}{3})} \subset Q_{(\tau_2,\zeta_2),\omega_\epsilon((\tau_2,\zeta_2)-\frac{\sigma}{3})} = Q^2$$

that is,

$$\bigcap_{(\tau,\zeta)\in Q^2} M(\tau,\zeta)(\eta,\xi) + Q_\epsilon(\eta,\xi) \neq \emptyset \Rightarrow \bigcap_{(\tau,\zeta)\in Q^1} M(\tau,\zeta)(\eta,\xi) + Q_\epsilon(\eta,\xi) \neq \emptyset.$$

Whenever $d_{\eta,\xi}((t,x),(t^*,x^*)) < \frac{\sigma}{3}$, setting $(t,x) = (\tau_1,\zeta_1)$, $(t^*,x^*) = (\tau_2,\zeta_2)$ we obtain

$$\omega_{\eta\xi,\epsilon}(t,x) \geq \omega_{\eta\xi,\epsilon}(t^*,x^*) - \frac{2\sigma}{3}$$

while interchanging (t,x) and (t^*,x^*), we have

$$\omega_{\eta\xi,\epsilon}(t,x) \geq \omega_{\eta\xi,\epsilon}(t^*,x^*) - \frac{2\sigma}{3}.$$

Hence $(t,x) \to \omega_{\eta\xi,\epsilon}(t,x)$ is a continuous and positive map defined on a compact set.
(b) We define the map Φ, given by

$$(t,x) \to \Phi(t,x)(\eta,\xi) = \bigcap_{(\tau,\zeta)\in Q_{(t,x),\omega_\epsilon}} M(\tau,\zeta)(\eta,\xi) + Q_\epsilon(\eta,\xi). \tag{6}$$

Then Φ is lower semicontinuous . In fact let y^* be in $\Phi(t^*,x^*)(\eta,\xi)$, so that for every (τ,ζ) in $Q_{(t^*,x^*),\omega_\epsilon}$,

$$d(y^*, M(\tau,\zeta)(\eta,\xi)) = \epsilon - \omega_{\eta\xi,\epsilon}(\tau,\zeta), \;\; \omega_{\eta\xi,\epsilon}(\tau,\zeta) > 0$$

or equivalently , there exists $y_{\eta,\xi}(\tau,\zeta)$ in $M(\tau,\zeta)(\eta,\xi)$ so that $| y^* - y_{\eta\xi}(\tau,\zeta) | \leq \epsilon - \frac{\sigma}{2}$.
By the lower semicontinuity of M, there exists $\delta = \delta(\tau,\zeta)$ so that (τ',ζ') in $Q_{(\tau,\zeta),\delta}$ implies $d(y_{\eta\xi}(\tau,\zeta), M(\tau',\zeta')(\eta,\xi)) < \frac{\sigma}{2}$, hence, in particular

$$d(y^*, M(\tau',\zeta')(\eta,\xi)) < \epsilon.$$

The open set

$$\mathscr{U} = \bigcup_{(\tau,\zeta)\in Q_{(t^*,x^*),\omega_\epsilon}} Q_{(\tau,\zeta),\delta(\tau,\zeta)}$$

contains the compact set $Q_{(t^*,x^*),\omega_\epsilon}$, hence, whenever

$$d_{\eta\xi}((t,x),(t^*,x^*)) < \rho,$$

sufficiently small

$$Q_{(t,x),\omega_\epsilon} \subset \mathscr{U},$$

and thus

$$d(y^*, M(\tau,\zeta)(\eta,\xi)) < \epsilon \text{ or } y^* \in \Phi(t,x)(\eta,\xi).$$

Since the map $t \to \Phi(t,x)(\eta,\xi)$ is lower semicontinuous and has closed values, then by Theorem 2.14.2 [1] there exists a measurable selection $v(t)(\eta,\xi)$ of $M(t,x)(\eta,\xi)$, which is the required selection.

Proposition 2. *Assume that the following holds*

(i) For arbitrary $\eta,\xi \in \mathbb{D}\underline{\otimes}\mathbb{E}$, the multivalued map $(t,x) \to G(t,x)(\eta,\xi)$ is lower semicontinuous.

(ii) $g : I \times \widetilde{\mathscr{A}} \to sesq(\mathbb{D}\underline{\otimes}\mathbb{E})$ is continuous single-valued map, and

(iii) $\varepsilon : \widetilde{\mathscr{A}} \to \mathbb{R}_+$ is lower semicontinuous.

Then the map $(t,x) \to \Phi(t,x)(\eta,\xi)$ defined by

$$\Phi(t,x)(\eta,\xi) = B_{\varepsilon(x)}(g(t,x)(\eta,\xi)) \bigcap G(t,x)(\eta,\xi)$$

is lower semicontinuous on its domain.

Proof. Fix (t^*,x^*) in DomΦ, $y^*_{\eta\xi} \in \Phi(t^*,x^*)(\eta,\xi)$ and $\omega > 0$. For some $\sigma > 0$, $\mid y^*_{\eta\xi} - g(t^*,x^*)(\eta,\xi) \mid = \varepsilon(x^*) - \sigma$.
There exists δ_1 such that to any $(t,x) \in I \times \widetilde{\mathscr{A}}$ with $d_{\eta\xi}((t,x),(t^*,x^*)) < \delta_1$, we can associate $y(t,x)(\eta,\xi)$ in $G(t,x)(\eta,\xi)$ so that

$$\mid y_{\eta\xi,(t,x)} - y^*_{\eta\xi} \mid < \min\{\omega, \frac{\sigma}{3}\},$$

and δ_2 such that

$$d_{\eta\xi}((t,x),(t^*,x^*) < \delta_2$$

implies

$$\varepsilon(x) > \varepsilon(x^*) - \frac{\sigma}{3},$$

and δ_3 such that

$$d_{\eta\xi}((t,x),(t^*,x^*) < \delta_3$$

implies $\mid g(t^*,x^*)(\eta,\xi) - g(t,x)(\eta,\xi) \mid < \frac{\sigma}{3}$.
Then when $d_{\eta\xi}((t,x),(t^*,x^*) < \min\{\delta_1,\delta_2,\delta_3\}$,

$$\begin{aligned} \mid y(t,x)(\eta,\xi) - g(t,x)(\eta,\xi) \mid \leq &\mid y(t,x)(\eta,\xi) - y^*_{\eta\xi} \mid + \mid y^*_{\eta\xi} - g(t^*,x^*)(\eta,\xi) \mid \\ &+ \mid g(t^*,x^*)(\eta,\xi) - g(t,x)(\eta,\xi) \mid \\ &< \frac{\sigma}{3} + \varepsilon(x^*) - \sigma + \frac{\sigma}{3} \\ &= \varepsilon(x^*) - \frac{\sigma}{3} < \varepsilon(x) \end{aligned}$$

that is $y(t,x)(\eta,\xi) \in \Phi(t,x)(\eta,\xi)$, and

$$\mid y^*(t,x)(\eta,\xi) - y(t,x)(\eta,\xi) \mid < \omega.$$

We now prove the existence of solution of Lower semicontinuous quantum stochastic differential inclusions.

Theorem 1. *Suppose that the following holds:*

(i) For every $\eta,\xi \in \mathbb{D}\underline{\otimes}\mathbb{E}$, the map $(t,x) \to \mathbb{P}(t,x)(\eta,\xi)$ is a non-empty compact and lower semicontinuous multifunction.

(ii) $(t_0,x_0) \in I \times \widetilde{\mathscr{A}}$, for all $(t,x) \in I \times Q_{x_0,\frac{T}{2}\lambda}$, $\lambda > 0$, such that $\mid \mathbb{P}(t,x)(\eta,\xi) \mid < \lambda$.

Then there exists a set $\mathscr{K}_{\eta\xi}$ and a continuous map $\varphi : \mathscr{K}_{\eta\xi} \to L^1(I)$, a selection of $\mathscr{F}_{\eta\xi}$.

Proof. We shall first show the existence of a finite number $m(0)$ of measurable maps v_i from I into $Q_\lambda(\eta,\xi)$; of a continuous partition of I into $\mathscr{J}^0_i = [\tau^0_{i-1}, \tau^0_i]$ with characteristic functions χ_i such that setting

$$g^0(u)(t)(\eta,\xi) = \sum \chi^0_i(t) v_i(t)(\eta,\xi),$$

we have for every t:

$$d(g^0(u)(t)(\eta,\xi),\mathbb{P}(t,u(t))(\eta,\xi)) < 1. \tag{7}$$

In fact,set in Lemma(1), M to be $\mathbb{P}$, ϵ to be 1 and let ω_0 be the constant provided by (a).
Let $\mathscr{U}^i = Q_{u^i,\omega_0}$, we define $\mathscr{U}^i(\eta,\xi) = \{\langle\eta, x\xi\rangle : x \in \mathscr{U}^i\}$ a finite open covering of the compact $\mathscr{K}_{\eta\xi}$. Let $v_i(t)(\eta,\xi)$, be the corresponding measurable functions as provided by (b). Fix u and t ; where $|\ \chi_i^0(u)(t)(\eta,\xi)\ | > 0$, u is in Q_{u^i,ω_0}, $d(v_i(t)(\eta,\xi),\mathbb{P}(t,u(t))(\eta,\xi)) < 1$, and (6) holds.
We claim that for $n = 0,1,...$ we can define ; $m(n)$ measurable functions $v_i^{(n)}$ from I into $Q_\lambda(\eta,\xi)$, a continuous partition of I , $\mathscr{I}_i^{(n)} = [\tau_{i-1}^{(n)}(u), \tau_i^{(n)}(u)]$ having characteristic functions $\chi_i^{(n)}$ such that setting

$$g^{(n)}(u)(t)(\eta,\xi) = \sum \chi_i^{(n)}(t) v_i^{(n)}(t)(\eta,\xi),$$

we have

(i) for every t,

$$d(g^{(n)}(u)(t)(\eta,\xi),\mathbb{P}(t,u(t))(\eta,\xi)) < \frac{1}{2^n}$$

except on a finite number of intervals , having total length $\frac{1}{2^n}$,

(ii)

$$|\ g^{(n)}(u)(t)(\eta,\xi) - g^{(n-1)}(u)(t)(\eta,\xi)\ | < \frac{1}{2^{n+1}}, n \geq 1.$$

Assume the above to hold up to $n = \nu - 1$, we shall prove that it holds for $n = \nu$.

There exists an open set $\mathscr{S}^\nu$ such that all the maps $t \to v_i^{(\nu-1)}(t)(\eta,\xi)$ are continuous on $I \setminus \mathscr{S}^\nu$, and the measure of $\mathscr{S}^\nu$ is smaller than $\frac{1}{2^{\nu+1}}$.
Let $\delta > 0$ be such that $\|\ w - u\ \|_{\eta\xi} < \delta$ implies that for each i,
$|\ \tau_i^{(\nu-1)}(u) - \tau_i^{(\nu-1)}(w)\ | < (2^{-\nu}(4m(\nu-1)))^{-1}$. A finite number of $Q_{\hat{u}_j,\delta}$ covers $\mathscr{K}$. For each j call E_j the finite union of open intervals $|\ t - \tau_i(\hat{u}_j)\ | < (2^{-\nu}(4(\nu-1)))^{-1}$, $i = 1,...,m(\nu-1)$. Then whenever u is in $Q_{\hat{u}_j,\delta}$, when t is any of the closed intervals whose union is $I \setminus E_j$, $g^{\nu-1}(u)(t)(\eta,\xi) = g^{\nu-1}(\hat{u})(t)(\eta,\xi) = v_i^{\nu-1}(t)(\eta,\xi)$ for some i.
Hence when t belongs to the closed $(I \setminus E_j) \setminus \mathscr{S}^\nu$, the map $t \mapsto g^{\nu-1}u(t)(\eta,\xi)$ is continuous. Set $|\ \rho_j^\nu(t)(\eta,\xi)\ |$ to be 2λ on the open $(E_j \cup \mathscr{S}^\nu)$ and to be $\frac{1}{2^{\nu-1}}$ on the closed $I \setminus (E_j \cup \mathscr{S}^\nu)$. The map $(t,x) \to \mathbb{P}_j(t,x)(\eta,\xi)$ is defined by

$$\mathbb{P}_j(t,x)(\eta,\xi) = Q_{g^{\nu-1}(\hat{u}_j)(t)(\eta,\xi),|\rho_j^\nu(t)(\eta,\xi)|}(\eta,\xi) \bigcap \mathbb{P}(t,x)(\eta,\xi)$$

is strict for (t,x) in $Q_{(t,\hat{u}_j(t)),\delta}$.
In fact, when t is in $(E_j \cup \mathscr{S}^\nu)$. It is enough to remark that both $g^{\nu-1}$ and $\mathbb{P}$ take values in $Q_\lambda(\eta,\xi)$.
Let (t,x) : t in $I \setminus (E_j \cup \mathscr{S}^\nu)$, $\|\ x - \hat{u}_j(t)\ \|_{\eta\xi} < \delta$. Then a translate $u(.)$ of $\hat{u}_j(.)$ is in $Q_{\hat{u}_j(.),\delta}$

and is such that $u(t) = x$. For this u, $g^{\nu-1}(u)(t)(\eta,\xi) = g^{\nu-1}(\hat{u}_j)(t)(\eta,\xi)$ and, by point (i) of the induction

$$d(g^{\nu-1}(u)(t)(\eta,\xi), \mathbb{P}(t,x)(\eta,\xi)) < \frac{1}{2^{\nu-1}}.$$

Since $t \to \rho^\nu(t)$ is lower semicontinuous and $\mathbb{P}_j$ is strict, proposition 2 implies that $(t,x) \to \mathbb{P}_j(t,x)(\eta,\xi)$ is lower semicontinuous. Set in Lemma 1, M to be $\mathbb{P}_j$, ϵ to be $\frac{1}{2^{\nu+1}}$ and call ω_j the constant provided by point (a). A finite number of $Q_{u^i_j,\omega_j}(\eta,\xi)$ covers the compact $\mathscr{K}_{\eta\xi} \cap Q_{\hat{u}_j,\delta}(\eta,\xi)$ By Lemma 1(b), there exists for each i a measurable $v^i_j(t)(\eta,\xi)$ such that $d_{\eta\xi}((t,x),(t,u^i_j(t))) < \omega_i$ implies

$$d(v^i_j(t)(\eta,\xi), \mathbb{P}_j(t,x)(\eta,\xi)) \leq \frac{1}{2^{\nu+1}} < \frac{1}{2^\nu}. \tag{8}$$

The collection of open sets $\mathscr{U}^i_j(\eta,\xi) = Q_{\hat{u}_j,\delta}(\eta,\xi) \cap Q_{u^i_j,\omega_j}(\eta,\xi)$ covers $\mathscr{K}_{\eta\xi}$. Let χ^i_j be the characteristic functions of the corresponding continuous partition $\{\mathscr{F}^j_{\eta\xi,i}\}$ of I. Set

$$g^\nu(u)(t)(\eta,\xi) = \sum_{i,j} \chi^i_j(t) v^i_j(t)(\eta,\xi).$$

We claim that the functions v^i_j and the map g^ν satisfy our induction assumptions.
Fix u and t. Whenever t belongs to $\mathscr{I}^j_i(u)$, $g^\nu(u)(t)(\eta,\xi) = v^i_j(t)(\eta,\xi)$ and u belongs to $Q_{u^i_j,w_j}$, and by (7),

$$\mathbf{d}(v^i_j(t)(\eta,\xi), \mathbb{P}_j(t,u(t))(\eta,\xi)) < \frac{1}{2^\nu}. \tag{9}$$

Since $\mathbb{P}_j(t,u(t))(\eta,\xi) \subset \mathbb{P}(t,u(t))(\eta,\xi)$, (8) check point (i).
To check point (ii) , assume t in $I \setminus (E_j \cup \mathscr{S}^\nu)$. Then $\rho^\nu(t) = \frac{1}{2^{\nu-1}}$,

$$\mathbb{P}_j(t,x)(\eta,\xi)) \subset Q_{g^{\nu-1}(\hat{u}_j)(t)(\eta,\xi),\frac{1}{2^{\nu-1}}} = Q_{g^{\nu-1}(u)(t)(\eta,\xi),\frac{1}{2^{\nu-1}}}$$

hence

$$d(v^i_j(t)(\eta,\xi), Q_{g^{\nu-1}(u)(t)(\eta,\xi),\frac{1}{2^{\nu-1}}}(\eta,\xi)) < \frac{1}{2^\nu} \tag{10}$$

or

$$| g^\nu(u)(t)(\eta,\xi) - g^{\nu-1}(u)(t)(\eta,\xi) | < \frac{1}{2^{\nu+1}} \tag{11}$$

except on an open set $(E_j \cup \mathscr{S}^\nu)$ with measure at most $\frac{1}{2^\nu}$. The sequence of measurable maps $\{g^n(u)(.)(\eta,\xi)\}$ is a Cauchy sequence converging to some measurable function that we denote by $g(u)(.)(\eta,\xi)$ and $g(u)(t)(\eta,\xi) \in \mathbb{P}(t,u(t))(\eta,\xi)$.
Let $K_{\eta\xi}$ be defined by:

$$K_{\eta\xi} = \{u(t)(\eta,\xi) = \langle \eta, u(t)\xi \rangle \in \mathscr{K}_{\eta\xi} : u(t)(\eta,\xi) \text{ is Lipschitzian and } u(t_0)(\eta,\xi) = x_0\}.$$

The continuous map $\varphi : K_{\eta\xi} \to K_{\eta\xi}$ defined by $\varphi(\langle \eta, u(t)\xi\rangle) = \langle \eta, \varphi(u)(t)\xi\rangle$

$$\langle \eta, \varphi(u)(t)\xi\rangle = x_0 + \int_{t_0}^{t} g(u)(s)(\eta,\xi)ds,$$

$\mid \langle \eta, \varphi(u)(t_2)\xi\rangle - \langle \eta, \varphi(u)(t_1)\xi\rangle \mid$ which gives

$$\mid \int_{t_1}^{t_2} g(u)(s)(\eta,\xi)ds \mid <\| g \|| \int_{t_1}^{t_2} u(s)(\eta,\xi)ds \mid$$
$$\in K_{\eta\xi}.$$

Moreover,

$$\frac{d}{dt}\langle \eta, \varphi(u)(t)\xi\rangle = g(u)(t)(\eta,\xi) \in \mathbb{P}(t,u(t))(\eta,\xi).$$

We are left to show the continuity of φ. In fact, we shall show directly that φ is uniformly continuous.
From a(ii) above, for every $\eta, \xi \in \mathbb{D}\underline{\otimes}\mathbb{E}, u(t)(\eta,\xi) \in K_{\eta\xi}$

$$\int_I \mid g^{\nu+1}(u)(s)(\eta,\xi) - g^{\nu}(u)(s)(\eta,\xi) \mid ds \leq \frac{2T}{2^{\nu+1}} + \frac{2M}{2^{\nu+1}}$$
$$= \frac{T+M}{2^{\nu}}$$

so that

$$\int_I \mid g^{n+1}(u)(s)(\eta,\xi) - g^{n}(u)(s)(\eta,\xi) \mid ds + \int_I \mid g^{n+2}(u)(s)(\eta,\xi) - g^{n+1}(u)(s)(\eta,\xi) \mid ds + ...$$
$$\leq (\frac{1}{2^n})(T+M)(1+\frac{1}{2}+\frac{1}{4}+...)$$
$$= \frac{T+M}{2^{n-1}},$$

and since $\int_I \mid g^{\nu}(u)(s)(\eta,\xi) - g^{(}u)(s)(\eta,\xi) \mid ds$ converges to 0,

$$\mid \varphi(\langle \eta, u(t)\xi\rangle) - \varphi(\langle \eta, w(t)\xi\rangle) \mid \leq \int_{t_0}^{t} \mid g(u)(s)(\eta,\xi) - g(w)(s)(\eta,\xi) \mid ds$$
$$\leq \int_I \mid g^{n}(u)(s)(\eta,\xi) - g^{n}(w)(s)(\eta,\xi) \mid ds + \int_I \mid g^{n}(u)(s)(\eta,\xi) - g(u)(s)(\eta,\xi) \mid ds$$
$$+ \int_I \mid g^{n}(w)(s)(\eta,\xi) - g(w)(s)(\eta,\xi) \mid ds$$
$$\leq \int_I \mid g^{n}(u)(s)(\eta,\xi) - g^{n}(w)(s)(\eta,\xi) \mid ds + \frac{4(T+M)}{2^n}.$$

Also, by a(ii)

$$\int_I | g^n(u)(s)(\eta,\xi) - g^n(w)(s)(\eta,\xi) | \, ds \leq \frac{2M}{2^{n+1}} = \frac{M}{2^n}.$$

Therefore $| u(t)(\eta,\xi) - w(t)(\eta,\xi) | \leq \delta$ implies, for every $t \in I$,

$$| \langle \eta, \varphi(u)(t)\xi\rangle - \langle \eta, \varphi(w)(t)\xi\rangle | \leq \frac{T+5M}{2^n} \leq \epsilon.$$

Proving the continuity of φ. Hence φ is the required selection of $\mathscr{F}_{\eta\xi}$.

From above we have the following existence result.

Corollary 1. *For an arbitrary $\eta, \xi \in \mathbb{D}\underline{\otimes}\mathbb{E}$, suppose that multivalued stochastic processes*

$$M : I \times \widetilde{\mathscr{A}} \to 2^{sesq.(\mathbb{D}\underline{\otimes}\mathbb{E})^2}, \quad M \in \{\mu E, \nu F, \sigma G, H\}$$

are compact-valued, lower semicontinuous multifunction.
Let $(t_0, x_0) \in I \times \widetilde{\mathscr{A}}$.
Then the problem

$$\begin{aligned} dX(t) &\in E(t,X(t))d\Lambda_\pi(t) + F(t,X(t))dA_f(t) \\ &\quad + G(t,X(t))dA_g^+(t) + H(t,X(t))dt \text{ almost all } t \in I, \\ X(t_0) &= x_0 \end{aligned} \tag{12}$$

has at least one solution defined on I lying in $Ad(\widetilde{\mathscr{A}})_{wac} \cap L^2_{loc}(\widetilde{\mathscr{A}})$.

References

[1] J.P. Aubin and A. Cellina. *Differential Inclusions*, Springer- Verlag, Berlin, 1984.

[2] E.O. Ayoola. Topological properties of solution sets of Lipschitzian quantum stochastic differential inclusions.*Acta Appl.Math* 100, 15-37, 2008.

[3] E.O. Ayoola. Continuous selections of solution sets of Lipschitzian quantum stochastic differential inclusions.*Int. J. Theor.Phys.*43, 10, 2041-2059, 2004.

[4] K. Deimling. *Multivalued differential equations*, Walter de Gruyter, 1992.

[5] G.O.S. Ekhaguere. Lipschitzian quantum stochastic differential inclusions, *Int. J. Theor. Phys.* 31, 11, 2003-2034, 1992.

[6] G.O.S. Ekhaguere. Quantum stochastic differential inclusions of hypermaximal monotone type *Int. J. Theor. Phys.* 34, 3, 323-353, 1995.

[7] G.O.S. Ekhaguere. Quantum stochastic evolutions *Int. J. Theor. Phys.*35, 9, 1909-1946, 1996.

[8] A. Guichardet, *Symmetric Hilbert spaces and related topics*Lecture Notes in Mathematics, 261, Springer-Verlag, Berlin(1972).

[9] R.L. Hudson, and K.R. Parthasarathy. Quantum Ito's formula and stochastic evolutions, *Comm. Math. Phys.* 93, 3, 301-323, 1984.

[10] M.O. Ogundiran and E.O. Ayoola. Mayer problem for Quantum stochastic control *J. Math. Phys.* 51, 1, 023521-023521-8, 2010.

Mayer problem for quantum stochastic control

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Mayer problem for quantum stochastic control

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This paper is concerned with the value function for optimal quantum stochastic control. We established the Lipschitz continuity of the value function, which is inherent from the Lipschitz property of the set-valued map describing the quantum stochastic control. By using a feedback multivalued map, we also prove a characterization of optimal solutions relating to a noncommutative generalization of Mayer problem. © *2010 American Institute of Physics.* [doi:10.1063/1.3300332]

I. INTRODUCTION

Control theory is one of the major motivations for the development of theory of differential inclusions. Most problems in classical control theory can be reformulated and solved via set-valued approach. An extensive study of differential inclusions and applications can be found in Refs. 1 and 2. Optimal control theory is a systematic approach to controller design, whereby the desired performance objectives are encoded in a cost function, which is subsequently optimized to determine the desired controller. The two fundamental tools of optimal control are dynamic programming and Pontryagin's principle. Dynamic programming is a means by which candidate optimal controls can be verified optimal.[7] The procedure is to find a suitable solution to a dynamic programming equation (DPE), which encodes the optimal performance and to use it to compare the performance of a candidate optimal control. Candidate controls may be determined from Pontryagin's principle or directly from the solution to the DPE. In general, it is difficult to solve DPEs. In continuous time, the DPE is a nonlinear partial differential equation, commonly called the Hamilton–Jacobi–Bellman equation. This equation is generally nonsmooth. For a detailed study of nonsmooth analysis and control theory, see Ref. 6.

In Ref. 5, the control process that minimizes the quadratic performance functional associated with a quantum system whose evolution is described by Hudson–Parthasarathy-type[11] stochastic differential equation on Fock space was explicitly computed. Moreover, it was shown that the noisy-infinite dimensional Ricatti equation associated with this problem has a unique solution.[3]

In an earlier work, Belavkin[4] considered quantum stochastic control, but the Hamilton–Jacobi–Bellman equation for quantum optimal feedback was derived in Ref. 10. The value function is the unique viscosity solution to the Hamilton–Jacobi–Bellman equation. The Mayer problem of control is concerned with the minimization problem arising from the control.

In Ref. 8, Ekhaguere formulated a multivalued analog of quantum stochastic calculus of Hudson and Parthasarathy setting. This formulation is a motivation for the study of quantum stochastic control via set-valued analysis. This work is concerned with the study of regularity property of value function which is inherited from multivalued stochastic processes involved. The Mayer problem associated with the problem is shown to have at least one optimal solution when the value function is directionally differentiable.[9]

In Sec. II, preliminaries on quantum stochastic differential inclusions and quantum stochastic control shall be considered. Section III shall be for the statement and proof of the main result.

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II. PRELIMINARIES

In this section we define the basic notations that shall be employed in sequel. We also state the fundamental result which shall be used in the proof of our main results.

Notations. Let $\mathcal{L}_w^+(D,H)$ be the set of all linear maps x from a pre-Hilbert space D to its completion H. Now let $\mathbb{D}$ be a pre-Hilbert space with completion $\mathcal{H}$. $\mathbb{E}$, $\mathbb{E}_t$, and $\mathbb{E}^t$, $t>0$ be linear spaces generated by the exponential vectors in Fock spaces $\Gamma(L^2_{\mathbb{Y}}(\mathbb{R}_+))$, $\Gamma(L^2_{\mathbb{Y}}([0,t)))$, and $\Gamma(L^2_{\mathbb{Y}}([t,\infty)))$ respectively; where $L^2_{\mathbb{Y}}(\mathbb{R})$ (resp. $L^2_{\mathbb{Y}}([0,t))$ [$L^2_{\mathbb{Y}}([t,\infty))t\in\mathbb{R}_+$)], is the space of square integrable $\mathbb{Y}$-valued maps on $\mathbb{R}_+([0,t),[t,\infty))$ for some fixed Hilbert space $\mathbb{Y}$.

$$\mathcal{A}\equiv\mathcal{L}_w^+(\mathbb{D}\underline{\otimes}\mathbb{E},\mathcal{H}\otimes\Gamma(L^2_{\mathbb{Y}}(\mathbb{R}_+))),$$

$$\mathcal{A}_t\equiv\mathcal{L}_w^+(\mathbb{D}\underline{\otimes}\mathbb{E}_t,\mathcal{H}\otimes\Gamma(L^2_{\mathbb{Y}}([0,t))))\otimes\mathbb{I}^t,$$

$$\mathcal{A}^t\equiv\mathbb{I}_t\otimes\mathcal{L}_w^+(\mathbb{E}^t,\Gamma(L^2_{\mathbb{Y}}([t,\infty)))),\quad t>0,$$

where $\otimes$ denotes algebraic tensor product and $\mathbb{I}_t(\mathbb{I}^t)$ denotes the identity map on $\mathcal{H}\otimes\Gamma(L^2_{\mathbb{Y}}([0,t)))$ $(\Gamma(L^2_{\mathbb{Y}}([t,\infty))))$, $t>0$.

Let the inner product of the Hilbert space $\mathcal{H}\otimes\Gamma(L^2_{\mathbb{Y}}(\mathbb{R}_+))$ be denoted by $\langle\cdot,\cdot\rangle$, and $\|\cdot\|$ the norm induced by $\langle\cdot,\cdot\rangle$.

Let $(\mathbb{D}\underline{\otimes}\mathbb{E})_\infty$ denote the set of all sequences $\eta=\{\eta_n\}_{n=1}^\infty$, $\xi=\{\xi_n\}_{n=1}^\infty$ of members of $\mathbb{D}\underline{\otimes}\mathbb{E}$, such that

$$\sum_{n=1}^\infty|\langle\eta_n,x\xi_n\rangle|<\infty\qquad\forall\, x\in\tilde{\mathcal{A}},$$

then the family of seminorms

$$\{\|\cdot\|_{\eta\xi}:\eta,\xi\in(\mathbb{D}\underline{\otimes}\mathbb{E})_\infty\},$$

$$\text{where }\ \|x\|_{\eta\xi}=\sum_{n=1}^\infty|\langle\eta_n,x\xi_n\rangle|,\quad x\in\tilde{\mathcal{A}},\eta,\xi\in(\mathbb{D}\underline{\otimes}\mathbb{E})_\infty$$

generates a topology $\tau_{\sigma w}$, σ-weak topology.

The completion of the locally convex spaces $(\mathcal{A},\tau_{\sigma w})$, $(\mathcal{A}_t,\tau_{\sigma w})$, and $(\mathcal{A}^t,\tau_{\sigma w})$ are, respectively, $\tilde{\mathcal{A}}$, $\tilde{\mathcal{A}}_t$, and $\tilde{\mathcal{A}}^t$.[12]

A. Quantum stochastic differential inclusions

(a) By a multivalued stochastic process indexed by $I\subseteq\mathbb{R}_+$, we mean a multifunction on I with values in $\mathrm{clos}(\tilde{\mathcal{A}})$.

(b) If Φ is a multivalued stochastic process indexed by $I\subseteq\mathbb{R}_+$, then a selection of Φ is a stochastic process $X:I\to\tilde{\mathcal{A}}$ with the property that $X(t)\in\Phi(t)$ for almost all $t\in I$.

(c) A multivalued stochastic process Φ will be called as follows.

  (i) Adapted if $\Phi(t)\subseteq\tilde{\mathcal{A}}_t$ for each $t\in\mathbb{R}_+$.

  (ii) Measurable if $t\mapsto d_{\eta\xi}(x,\Phi(t))$ is measurable for arbitrary $x\in\tilde{\mathcal{A}}$ $(\eta,\xi\in\mathbb{D}\underline{\otimes}\mathbb{E})_\infty$.

  (iii) Locally absolutely p-integrable if $t\mapsto\|\Phi(t)\|_{\eta\xi}$, $t\in\mathbb{R}_+$, lies in $L^p_{\mathrm{loc}}(I)$ for arbitrary η, $\xi\in(\mathbb{D}\underline{\otimes}\mathbb{E})_\infty$.

  (iv) Let $\mathcal{N}\in\tilde{\mathcal{A}}$, a map

$$\Phi: I \times \mathcal{N} \to 2^{\tilde{\mathcal{A}}}$$

is said to be Lipschitzian if for each $\eta,\ \xi \in (\mathbb{D} \underline{\otimes} \mathbb{E})_\infty$, there exists $K^{\Phi}_{\eta\xi}: I \to (0,\infty)$ in $L^1_{\mathrm{loc}}(I)$, such that

$$\rho_{\eta\xi}(\Phi(t,x),\Phi(t,y)) \leq K^{\Phi}_{\eta\xi}(t)\|x-y\|_{\eta\xi}$$

for all $x,\ y \in \mathcal{N}$ and almost all $t \in I$.

(v) We say that a multifunction Φ is upper semicontinuous at $x_0 \in \tilde{\mathcal{A}}$ if for any open neighborhood N containing $\Phi(x_0)$ there exists a neighborhood M of x_0 such that $\Phi(M) \subset M$.

We state some other notations which shall be employed.

(1) The set of all absolutely p-integrable multivalued stochastic processes will be denoted by $L^p_{\mathrm{loc}}(\tilde{\mathcal{A}})_{\mathrm{mvs}}$.
(2) For $p \in (0,\infty)$ and $I \subseteq \mathbb{R}_+$, $L^p_{\mathrm{loc}}(I \times \tilde{\mathcal{A}})_{\mathrm{mvs}}$ is the set of maps $\Phi: I \times \tilde{\mathcal{A}} \to \mathrm{close}(\tilde{\mathcal{A}})$, such that $t \mapsto \Phi(t,X(t)),\ t \in I$ lies in $L^p_{\mathrm{loc}}(\tilde{\mathcal{A}})_{\mathrm{mvs}}$, for every $X \in L^p_{\mathrm{loc}}(\tilde{\mathcal{A}})$.
(3) If $\Phi \in L^p_{\mathrm{loc}}(I \times \tilde{\mathcal{A}})_{\mathrm{mvs}}$, then

$$L_p(\Phi) \equiv \{\varphi \in L^p_{\mathrm{loc}}(\tilde{\mathcal{A}}) : \varphi \text{ is a selection of } \Phi\}.$$

In the sequel, $f,\ g \in L^\infty_{\Upsilon,\mathrm{loc}}(\mathbb{R}_+)$, $\pi \in L^\infty_{B(\Upsilon),\mathrm{loc}}(\mathbb{R}_+)$, $\mathbb{I}$ is the identity map on $\mathcal{H} \otimes \Gamma(L^2_{\Upsilon}(\mathbb{R}_+))$, and M is any of the stochastic processes A_f, A^+_g, Λ_π, and $s \mapsto s\mathbb{I}$, $s \in \mathbb{R}_+$.

We introduce stochastic integral (differential) expression as follows.

If $\Phi \in L^2_{\mathrm{loc}}(I \times \tilde{\mathcal{A}})_{\mathrm{mvs}}$ and $(t,X) \in I \times L^2_{\mathrm{loc}}(\tilde{\mathcal{A}})$, then we define

$$\int_{t_0}^{t} \Phi(s,X(s))dM(s) \equiv \left\{ \int_{t_0}^{t} \varphi(s)dM(s); \varphi \in L_2(\Phi) \right\}.$$

This leads to the definition of quantum stochastic differential (integral) inclusion which is a multivalued analog of quantum stochastic differential equations of Hudson and Parthasarathy.[11] We adopt the formulation of quantum stochastic differential inclusions of Ref. 8 as follows.

Let $E,\ F,\ G,\ H \in L^2_{\mathrm{loc}}(I \times \tilde{\mathcal{A}})_{\mathrm{mvs}}$, and (t_0,x_0) be a fixed point of $I \times \tilde{\mathcal{A}}$. Then, a relation of the form

$$X(t) \in x_0 + \int_{t_0}^{t} (E(s,X(s))d\Lambda_\pi(s) + F(s,X(s))dA_f(s) + G(s,X(s))dA^+_g(s) + H(s,X(s))ds \quad t \in I)$$

will be called a stochastic integral inclusion with coefficients E, F, G, H, and initial data (t_0,x_0).

The stochastic differential inclusion corresponding to the integral inclusion above is

$$\begin{gathered} dX(t) \in E(t,X(t))d\Lambda_\pi(t) + F(t,X(t))dA_f(t) + G(t,X(t))dA^+_g(t) + H(t,X(t))dt, \\ X(t_0) = x_0 \text{ almost all } t \in I \end{gathered} \tag{1}$$

By a solution of (1), we mean an adapted σ-weakly absolutely continuous stochastic process $\varphi \in L^2_{\mathrm{loc}}(\tilde{\mathcal{A}})$, such that

$$d\varphi(t) \in E(t,\varphi(t))d\Lambda_\pi(t) + F(t,\varphi(t))dA_f(t) + G(t,\varphi(t))dA^+_g(t) + H(t,\varphi(t))dt,$$

$$\varphi(t_0) = x_0 \text{ almost all } t \in I.$$

By using the matrix elements of quantum stochastic calculus of Hudson and Parthasarathy;[11] a sesquilinear equivalent form of (4) was established in Theorem 6.2 of Ekhaguere.[8] These multi-valued sesquilinear maps were defined as follows.

For $\eta, \xi \in (\mathbb{D}\underline{\otimes}\mathbb{E})_\infty$, with $\eta = c \otimes e(\alpha)$, $\xi = d \otimes e(\beta)$, define

$$\mu_{\alpha\beta}, \nu_\beta, \sigma_\alpha : I \to \mathbb{C}, I \subset \mathbb{R}_+$$

by

$$\mu_{\alpha\beta}(t) = \langle \alpha(t), \pi(t)\beta(t)\rangle_{\mathcal{Y}},$$

$$\nu_\beta(t) = \langle f(t), \beta(t)\rangle_{\mathcal{Y}},$$

$$\sigma_\alpha(t) = \langle \alpha(t), g(t)\rangle_{\mathcal{Y}},$$

$t \in I$, f, $g \in L^2_{\mathcal{Y},\mathrm{loc}}(\mathbb{R}_+)$ (locally bounded square integrable $\mathcal{Y}$-valued maps on $\mathbb{R}_+$), $\pi \in L^\infty_{B(\mathcal{Y}),\mathrm{loc}}$ [the space of all measurable, locally bounded maps from $\mathbb{R}_+$ to $B(\mathcal{Y})$, the Banach space of bounded endomorphisms of $\mathcal{Y}$]. To these functions we associate the maps μE, νF, σG, $\mathbb{P}$ from $I \times \tilde{\mathcal{A}}$ with set of sesquilinear forms on $(\mathbb{D}\underline{\otimes}\mathbb{E})_\infty$ defined by

$$(\mu E)(t,x)(\eta,\xi) = \{\langle \eta, \mu_{\alpha\beta}(t)p(t,x)\xi\rangle : p(t,x) \in E(t,x)\},$$

$$(\nu F)(t,x)(\eta,\xi) = \{\langle \eta, \nu_\beta(t)q(t,x)\xi\rangle : q(t,x) \in F(t,x)\},$$

$$(\sigma G)(t,x)(\eta,\xi) = \{\langle \eta, \sigma_\alpha(t)u(t,x)\xi\rangle : u(t,x) \in G(t,x)\},$$

$$\mathbb{P}(t,x)(\eta,\xi) = (\mu E)(t,x)(\eta,\xi) + (\nu F)(t,x)(\eta,\xi) + (\sigma G)(t,x)(\eta,\xi) + H(t,x)(\eta,\xi),$$

$$H(t,x)(\eta,\xi) = \{v(t,x)(\eta,\xi) : v(.,X(.)) \text{ is a selection of } H(.,X(.))\ \forall\, X \in L^2_{\mathrm{loc}}(\tilde{\mathcal{A}})\}.$$

Then problem (1) is equivalent to (2)

$$\frac{d}{dt}\langle \eta, X(t)\xi\rangle \in \mathbb{P}(t,X(t))(\eta,\xi),$$

$$X(t_0) = x_0, \tag{2}$$

for arbitrary $\eta, \xi \in (\mathbb{D}\underline{\otimes}\mathbb{E})_\infty$, almost all $t \in I$.

III. MAYER PROBLEM

Let $T > 0$, $\mathcal{Z}$ a complete separable metric space, called the space of admissible controls; $U:[0,T] \to 2^{\mathcal{Z}}$, and $\mathbb{P}:[0,T] \times \tilde{\mathcal{A}} \times \mathcal{Z} \to 2^{\mathrm{sesq}(\mathbb{D}\underline{\otimes}\mathbb{E})^2}$ are multivalued maps.

We associate with it the control system

$$\frac{d}{dt}\langle \eta, X(t)\xi\rangle = P(t,X(t),u(t))(\eta,\xi), \quad u(t) \in U(t), \tag{3}$$

where P is a selection of $\mathbb{P}$

Let an extended function $g:\tilde{\mathcal{A}} \to \mathbb{R} \cup \{+\infty\}$ and $\xi_0 \in \tilde{\mathcal{A}}$ be given. The Mayer optimal quantum stochastic control problem associated with the control system (3) is the minimization problem,

$$\min \mathbb{E}\{g(X(T)):\text{X is a solution of}(1),X(0)=\xi_0\}.$$

The value function associated with this problem is defined by, for all $(t_0,x_0)\in[0,T]\times\widetilde{\mathcal{A}}$,

$$V(t_0,x_0)=\inf \mathbb{E}\{g(X(T)):\text{X is a solution of}(1),X(t_0)=x_0\}.$$

A. Remark

We remark that quantum stochastic control has vast applications to quantum physics, for example;

(i) to determine the optimal trajectory which minimizes the functional corresponding to quantum stochastic kinetic equations of Vlassov and Boltzmann type, in quantum mechanics;[10]

(ii) control of the evolution of a physical system, such as the position of a quantum particle described by a quantum stochastic differential equation;[5]

(iii) control of a qubit system in quantum information;[10] to mention a few.

B. Lipschitz continuity of the value function

Consider an extended function $g:\widetilde{\mathcal{A}}\to\mathbb{R}\cup\{+\infty\}$, E, F, G, $H\in L^2_{\text{loc}}[0,T]\times\widetilde{\mathcal{A}})$, $\xi_0\in\widetilde{\mathcal{A}})$ and the differential inclusion,

$$dX(t)\in E(t,X(t))d\Lambda_\pi(t)+F(t,X(t))dA_f(t)+G(t,X(t))dA_g^+(t)+H(t,X(t))dt \text{ almost everywhere.} \tag{4}$$

We investigate the minimization problem,

$$\min \mathbb{E}\{g(X(T)):\text{X is a solution to (4), } X(0)=\xi_0\}.$$

The corresponding value function is given by, for all $(t_0,x_0)\in[0,T]\times\widetilde{\mathcal{A}}$,

$$V(t_0,x_0)=\inf \mathbb{E}\{g(X(T)):X \text{ is a solution of (4), } X(t_0)=x_0\}, \tag{5}$$

where minimum is considered over all adapted stochastic control processes $X(\cdot)$.

Let $\mathcal{S}_{[t_0,T]}(x_0)$ denote the set of solutions of (4) starting at x_0 at time t_0 and defined on the interval $[t_0,T]$. The value function is nondecreasing along solutions of (4),

$$\forall x\in\mathcal{S}_{[t_0,T]}(x_0),\quad \forall\, t_0\le t_1\le t_2\le T,$$

$$V(t_1,X(t_1))\le V(t_2,X(t_2))$$

and satisfies the dynamic programming principle,

$$\forall t\in[t_0,T],V(t_0,x_0)=\inf \mathbb{E}\{V(t,X(t)):x\in\mathcal{S}_{[t_0,T]}(x_0)\}, \tag{6}$$

$x\in\mathcal{S}_{[t_0,T]}(x_0)$ is optimal for problem (5) if and only if

$$V(t,X(t))\equiv g(X(T)).$$

We impose the following assumptions on E, F, G, H, and g,

$$\begin{aligned}
&\text{(i) } E,F,G,H \text{ has closed nonempty images},\\
&\text{(ii) } \forall\, X \in \widetilde{\mathcal{A}}, E(\cdot,X),F(\cdot,X),G(\cdot,X),H(\cdot,X) \text{ is measurable},\\
&\text{(iii) } E,F,G,H \text{ is Lipschitz},\\
&\text{(iv) } \exists\, \gamma \in L^1(0,T),\ \forall\, t \in [0,T], \|E(t,0)\|_{\eta\xi} d\Lambda_\pi(t)\\
&\qquad + \|F(t,0)\|_{\eta\xi} dA_f(t) + \|G(t,0)\|_{\eta\xi} dA_g^+(t) + \|H(t,0)\|_{\eta\xi} dt \le \gamma(t),\\
&\text{(v) } g \text{ is locally Lipschitz}.
\end{aligned} \tag{7}$$

Theorem 1: *Assume (7). Then for every $R>0$, there exists $L_R>0$, such that*

(i) for all $(t_0,x_0) \in Q_R$ and every solution

$$X \in \mathcal{S}_{[t_0,T]}(x_0),\ \forall\, t \in [t_0,T], \|X\|_{\eta\xi} \le L_R$$

and the map $[t_0,T] \ni t \mapsto V(t,X(t))$ is absolutely continuous.

Furthermore, for almost every $t \in [t_0,T]$, the directional derivative $[\partial V/\partial(1,X'(t))](t,X(t))$ does exist.

(ii) For all $t \in [t_0,T]$, $V(t,\cdot)$ is L_R-Lipschitz on $B_R(0)$.

Finally, if for all $R>0$, there exists $C_R \ge 0$, such that for

$$a.e.\ t \in [0,T], \quad \forall\, X \in B_R(0), \quad \sup_{Y \in \Phi(t,X)} \|Y\|_{\eta\xi} \le C_R \tag{8}$$

($\Phi \in \{E,F,G,H\}$), then for every $R>0$, there exists $C_R>0$, such that $\forall X \in B_R(0)$, $V(\cdot,X)$ is C_R-Lipschitz.

Proof: Consider any solution $X \in \mathcal{S}_{[t_0,T]}(x_0)$ of differential inclusion (4). Then for almost all $t \in [t_0,T]$,

$$\begin{aligned}
dX(t) &\in E(t,X(t))d\Lambda_\pi(t) + F(t,X(t))dA_f(t) + G(t,X(t))dA_g^+(t) + H(t,X(t))dt \subset \|E(t,0)\|_{\eta\xi} d\Lambda_\pi(t)\\
&\quad + \|F(t,0)\|_{\eta\xi} dA_f(t) + \|G(t,0)\|_{\eta\xi} dA_g^+(t) + \|H(t,0)\|_{\eta\xi} dt + k(t)\|x(t)\|_{\eta\xi} B_{\eta\xi}.
\end{aligned}$$

Then $\forall t \in [t_0,T]$,

$$\|X(t)\|_{\eta\xi} \le \|x_0\|_{\eta\xi} + \int_{t_0}^t \gamma(s)ds + \int_{t_0}^t k(s)\|X(s)\|_{\eta\xi} ds.$$

This and Gronwall's lemma yield the first statement. Since V is locally Lipschitz, we deduce (ii) from Filippov's theorem.

Let $X_1 \in \mathcal{S}_{[t_0,T]}(x_0)$. We claim that the map $t \to V(t,X_1(t))$ is absolutely continuous. Indeed fix $t_0 \le t_1 < t_2 \le T$. By (4), there exists $X_2 \in \mathcal{S}_{[t_1,T]}(X_{t_1})$, such that

$$V(t_2,X_2(t_2)) \le V(t_1,X_1(t_1)) + |t_2 - t_1|.$$

Then from (i) we deduce that for $i=1,2$,

$$\|X_i(t_2) - X_i(t_1)\|_{\eta\xi} \le \int_{t_1}^{t_2} \gamma(s)ds + \int_{t_1}^{t_2} k(s)\|X_i(s)\|_{\eta\xi} ds \le \int_{t_1}^{t_2} \gamma(s)ds + L_{\|x_0\|}\int_{t_1}^{t_2} k(s)ds.$$

Thus, by (ii), for a constant L depending only on $\|x_0\|_{\eta\xi}$,

$$\begin{aligned}
0 &\le V(t_2,X_1(t_2)) - V(t_1,X_1(t_1))\\
&\le V(t_2,X_1(t_2)) - V(t_2,X_2(t_2)) + |t_2 - t_1|\\
&\le L\|X_1(t_2) - X_2(t_2)\|_{\eta\xi} + |t_2 - t_1|
\end{aligned}$$

$$\leq L(\|X_1(t_2) - X_1(t_1) + X_2(t_2) - X_1(t_1)\|_{\eta\xi}) + |t_2 - t_1|$$
$$\leq 2L\int_{t_1}^{t_2} \gamma(s)ds + 2L_{\|x_0\|_{\eta\xi}} L\int_{t_1}^{t_2} k(s)ds + |t_2 - t_1|. \tag{9}$$

Recall the following characterization of absolutely continuous maps.

A function $f:[a,b]\to\mathbb{R}$ is absolutely continuous if and only if

(i) $\exists v(f)>0,\ \forall a=a_1\leq b_1\leq\cdots\leq a_m\leq b_m=b,$

$$\sum_{i=1}^{m} |f(b_i) - f(a_i)| \leq v(f),$$

(ii) $\forall\epsilon>0,\ \exists\delta>0$, such that $\forall a\leq a_i<b_i\leq b,\ i=1,\ldots,m$ satisfying $]a_i,b_i[\cap]a_j,b_j[=\emptyset$ for $i\neq j$, $\Sigma_{i=1}^m(b_i-a_i)\leq\delta$ we have $\Sigma_{i=1}^m|f(b_i)-f(a_i)|\leq\epsilon$.

Thus, by (9) the map $t\to\varphi(t)=V(t,X_1(t))$ is absolutely continuous.

Fix $t\in[t_0,T]$, such that φ and X_1 are differentiable at t. Then from the local Lipschitz continuity of V with respect to the second variable,

$$\lim_{h\to 0+}\frac{V(t+h,X_1(t)+hX_1'(t)-V(t,X_1(t)))}{h} = \lim_{h\to 0+}\frac{\varphi(t+h)-\varphi(t)}{h}.$$

To prove the last statement of the theorem, observe that (6) and (i) imply that for all $R>0$, there exists l_R, such that every $X\in\mathcal{S}_{[t_0,T]}(x_0)$ is l_R Lipschitz whenever $x_0\in B_R(0)$. Fix $0\leq t_0<t_1\leq T$, $x_0\in B_R(0)$. By (6) there exists $X\in\mathcal{S}_{[t_0,T]}(x_0)$, such that $V(t_1,X(t_1))\leq V(t_0,x_0)+|t_1-t_0|$. Then

$$|V(t_1,x_0) - V(t_0,x_0)| \leq |V(t_1,X(t_1)) - V(t_0,x_0)| + |V(t_1,X(t_1)) - V(t_1,x_0)| \leq |t_1 - t_0| + L_R\|X(t_1) - x_0\|_{\eta\xi}$$
$$\leq (L_R l_R + 1)|t_1 - t_0|.$$

Existence of optimal solution. To characterize optimal solutions we introduce the following feedback map, $G:[0,T]\times\tilde{\mathcal{A}}\to 2^{\tilde{\mathcal{A}}}$, defined by

$$\forall (t,x) \in [0,T]\times\tilde{\mathcal{A}}G(t,x) = \left\{v \in \Psi(t,X): \frac{\partial V}{\partial(1,v)}(t,X) = 0\right\},$$

where

$$\Psi(t,X) \equiv E(t,X(t))d\Lambda_\pi(t) + F(t,X(t))dA_f(t) + G(t,X(t))dA_g^+(t) + H(t,X(t))dt.$$

■

Proposition 2: Assume (7). If for some $(t_0,x_0)\in[0,T]\times\tilde{\mathcal{A}}$, E, F, G, H is lower semicontinuous at (t_0,x_0) and for some $v\in\overline{co}(\Psi(t_0,x_0))$, the directional derivative of V at (t_0,x_0) in the direction $(1,v)$ exists, then this directional derivative is non-negative.

Proof: Consider a solution $X(\cdot)$ of quantum stochastic differential inclusion (2) satisfying $X(t_0)=x_0$, $dX(t_0)=v$. Since $V(t,\cdot)$ is Lipschitz on a neighborhood of x_0 with the Lipschitz constant independent of t and since V is nondecreasing along solutions to (2),

$$\lim_{h\to 0+}\frac{V(t_0+h,x_0+hv)-V(t_0,x_0)}{h} = \lim_{h\to 0+}\frac{V(t_0+h,x(t_0+h)-V(t_0,x_0)}{h} \geq 0.$$

■

If E, F, G, H are closed valued, then G has compact nonempty images and is upper semicontinuous, since Graph(G) will be closed.

Theorem 3: *Assume (7) and let $t_0\in[0,T]$. Then the following two statements are equivalent.*

(i) *X is a solution of the quantum stochastic differential inclusion,*

$$dX(t) \in G(t,X(t))dt \quad almost\ \ everywhere\ \ in[t_0,T]. \tag{10}$$

(ii) *X is a solution of quantum stochastic differential inclusion (2) defined on* $[t_0,T]$ *and for every* $t\in[t_0,T]$ $V(t,X(t))=g(x(T))$.

Proof: Fix a solution X of (4) defined on $[t_0,T]$ and set $\varphi(t)=V(t,X(t))$. By Theorem 1, φ is absolutely continuous and for almost all $t\in[t_0,T]$,

$$\varphi'(t)=\frac{\partial V}{\partial(1,X'(t))}(t,X(t)).$$

Assume that (i) holds true. Hence, for almost every $t\in[t_0,T]$, the set $G(t,X(t))$ is nonempty and $\varphi'(t)=0$ almost everywhere in $[t_0,T]$. Consequently $\varphi\equiv V(T,X(T))=g(x(T))$.

Assume next that (ii) is verified. Then, differentiating the map $t\mapsto\varphi(t)$, we obtain that for every $t_0<t<T$, $\varphi'(t)=0$. Therefore, for almost all $t\in[t_0,T]$, $dX(t)\in G(t,X(t))dt$. ■

Corollary 4: Assume (7). Then, a solution $X\in\mathcal{S}_{[t_0,T]}(x_0)$ *is optimal for problem (3) if and only if it is a solution of quantum stochastic differential inclusion (9), satisfying the initial condition* $X(t_0)=x_0$.

[1] Aubin, J. P. and Cellina, A., *Differential Inclusions* (Springer, Berlin, 1984).
[2] Aubin, J. P. and Frankowska, H., *Set-Valued Analysis* (Birkhäuser, Boston, 1990).
[3] Ayoola, E. O., "Topological properties of solution sets of quantum stochastic differential inclusions," Acta Appl. Math. **100**, 15 (2008).
[4] Belavkin, V. P., "On the theory of controlling observable quantum systems," Autom. Remote Control (Engl. Transl.) **442**, 178 (1983).
[5] Boukas, A., "Operator-valued stochastic control in Fock spaces with applications to orbit tracking and noise filtering," Probab. Math. Stat. **16**, 221 (1996).
[6] Clarke, F. H., Ledyaev, Y. S., Stern, R. J., and Wolenski, P. R., *Non Smooth Analysis and Control Theory*, Graduate Texts in Mathematics Vol. 178 (Springer, New York, 1998).
[7] Crandall, M.G. and Lions, P.L., "Viscosity solutions and Hamilton-Jacobi equations," Trans. Am. Math. Soc. **277**, 1 (1983).
[8] Ekhaguere, G. O. S., "Lipschitzian quantum stochastic differential inclusions," Int. J. Theor. Phys. **31**, 2003 (1992).
[9] Frankowska, H. and Plaskacz, S., "Semicontinuous solutions of Hamilton-Jacobi-Bellman equations with degenerate state constraints," J. Math. Anal. Appl. **251**, 818 (2000).
[10] Gough, J., Belavkin, V. P., and Smolyanov, O. G., "Hamilton-Jacobi-Bellman equations for quantum optimal feedback control," J. Opt. **7**, 1 (2005).
[11] Hudson, R. L. and Parthasarathy, K. R., "Quantum Ito's formula and stochastic evolutions," Commun. Math. Phys. **93**, 301 (1984).
[12] Parthasarathy, K. R., *Introduction to Quantum Stochastic Calculus* (Birkhäuser, Berlin, 1992).

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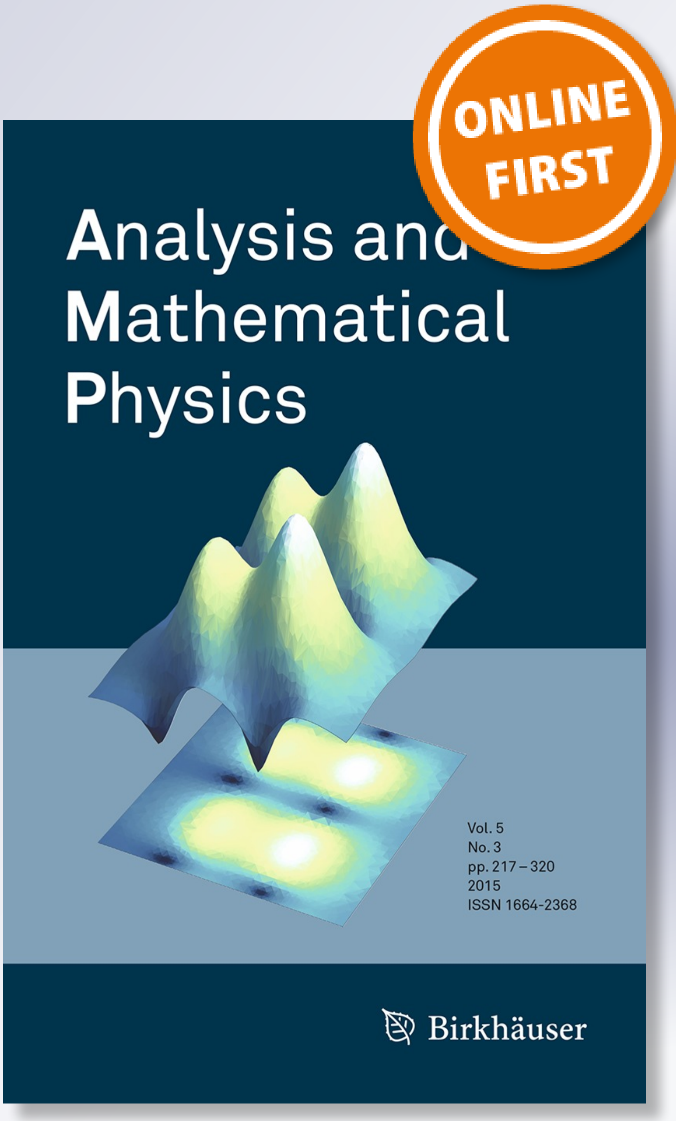

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On topological properties of solution sets of non Lipschitzian quantum stochastic differential inclusions

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Abstract In this paper, we establish results on continuous mappings of the space of the matrix elements of an arbitrary nonempty set of pseudo solutions of non Lipschitz quantum Stochastic differential inclusion (QSDI) into the space of the matrix elements of its solutions. we show that under the non Lipschitz condition, the space of the matrix elements of solutions is still an absolute retract, contractible, locally and integrally connected in an arbitrary dimension. The results here generalize existing results in the literature.

Keywords Non classical ODI · Non-Lipschitz function · Topological properties · Matrix elements

Mathematics Subject Classification 60H10 · 60H20 · 65L05 · 81S25

1 Introduction

We establish some results on topological properties of solution sets of a non Lipschitzian quantum stochastic differential inclusions given by

$$\begin{aligned} dX(t) \in E(X(t), t)d \wedge_\pi (t) + F(X(t), t)dA_g(t) \\ +G(X(t), t)dA_{f^+}(t) + H(X(t), t)dt), \;\; X(t_0) = a, \;\; t \in [t_0, T] \subseteq \mathbb{R}_+ \end{aligned} \tag{1.1}$$

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Inclusion (1.1) is best understood in integral form as

$$X(t) \in a + \int_{t_0}^{t} (E(X(s), s) d \wedge_\pi (s) + F(X(s), s) dA_g(s) \\ + G(X(s), s) dA_{f^+}(s) + H(X(s), s) ds), \; t \in [t_0, T] \subseteq \mathbb{R}_+ \quad (1.2)$$

Inclusion (1.2) is understood in the framework of the Hudson and Parthasarathy formulation of QSDEs, see [10]. Existence and uniqueness of inclusion (1.1) in equation form has been established in [4] under some general Lipschitz condition. We consider the equivalent non-classical QSDI given by

$$\frac{d}{dt}\langle \eta, X(t)\xi \rangle \in P(X, t)(\eta, \xi) \\ X(t_0) \in a, t \in [t_0, T], \quad (1.3)$$

Inclusion (1.3) is a first order non-classical ordinary differential inclusion with a multivalued sesquilinear form valued map P as the right hand side. For more on inclusion (1.3) and the explicit form of the multivalued sesquilinear form map $(X, t) \rightarrow P(X, t)(\eta, \xi)$ on $\mathbb{D} \underline{\otimes} \mathbb{E}$ appearing in inclusion (1.3) see [1,6] and the references there in. In what follows, we employ the locally convex space $\tilde{\mathcal{A}}$ of noncommutative stochastic processes whose topology is generated by the family of seminorms defined in [1].

In this work, we consider QSDI (1.3) where the map $x \rightarrow P(t, x)(\eta, \xi)$ is necessarily not Lipschitzian with values that are closed subsets of the field of complex numbers. In [5], in order to generalize the results in [2], we used the following results to establish some properties of solution sets of non-Lipschitzian QSDI (1.3).

It was established that for the non-Lipschitzian inclusion (1.3) corresponding to each pseudo solution process $Y \in Ad(\tilde{\mathcal{A}})_{wac}$, where

$$d\left(\frac{d}{dt}\langle \eta, Y(t)\xi \rangle, P(t, Y(t))(\eta, \xi)\right) \leq W\rho_{\eta\xi}(t), t \in [t_0, T],$$

$\rho_{\eta\xi} \in L^1_{loc}([t_0, T])$, there corresponds a solution $R(Y) \in S_T(Y(t_0))$ such that

$$\|Y(t) - R(Y)(t)\|_{\eta\xi} \leq \mathbb{E}_{\eta\xi}(t), t \in [t_0, T] \quad (1.4)$$

and

$$\left|\frac{d}{dt}\langle \eta, Y(t)\xi \rangle - \frac{d}{dt}\langle \eta, R(Y)(t)\xi \rangle\right| \\ \leq W(K_{\eta\xi}(t)\mathbb{E}_{\eta\xi}(t) + \rho_{\eta\xi}(t)), t \in [t_0, T] \quad (1.5)$$

where $\mathbb{E}_{\eta\xi}(t)$ and $K_{\eta\xi}(t)$ are as defined in [1] but with $W(t) \neq t$.

This paper is therefore concerned with similar extension of the results in the literature to a class of inclusion that does not depend on the Lipschitz condition $W(t) = t$.

This paper is organized as follows.In Sect. 2, we present some definitions, preliminary results and notations while Sect. 3 is devoted to the main results of our work.

2 Notations and preliminary results

From the references [1,6], we adopt the notations and definitions of the spaces $clos(\tilde{\mathcal{A}})$, $\mathcal{N}$, $clos(\mathcal{N})$, $comp(\mathcal{N})$, wac($\tilde{\mathcal{A}}$), the space of complex functions wac($\tilde{\mathcal{A}}$)(η,ξ). For any element $y \in AC([t_0, T], \mathbb{C})$, we employ the norm defined by

$$|y|_{AC} := |y(t_0)| + \int_{t_0}^{T} \left|\frac{dy}{dt}(t)\right| dt,$$

and for any nonempty set $Y \in clos(AC([t_0, T], \mathbb{C}))$, we employ the point - set distance defined by: $d_{AC}(y, Y) := inf_{z\in Y}|y - z|_{AC}$, the distance $d(x, A)$ of a point x from a set $A \in clos(\mathbb{C})$ is defined by $d(x, A) = inf\{|x - a| : a \in A\}$. Hence, (2.1) in [1] holds in this case.

For $\Phi \in wac(\tilde{\mathcal{A}})$, $M \in clos(wac(\tilde{\mathcal{A}}))$, we define $d_{\eta\xi}(\Phi, M) := inf_{u\in M}|\Phi - u|_{\eta\xi}$ and the set of all solutions of QSDI (1.3) is defined by

$$S_{(T)}(P) := \bigcup_{a\in\tilde{\mathcal{A}}} S_{(T)}(a)$$

and the associated space of absolutely continuous convex valued functions corresponding to each pair of $\eta, \xi \in \mathbb{D}\underline{\otimes}\mathbb{E}$ by

$$S_{(T)}(P) := \left\{\langle \eta, \Phi(.)\xi\rangle : \Phi \in S_{(T)}(P)\right\}.$$

The map $(t, x) \to P(t, x)(\eta, \xi)$ appearing in (1.3) is assumed to satisfy the following conditions .

(1) For each point $(t, x) \in I \times \tilde{\mathcal{A}}$ and $\eta, \xi \in \mathbb{D}\underline{\otimes}\mathbb{E}$, the values of $P(t, x)(\eta, \xi)$ are still nonempty closed, subsets of the field $\mathbb{C}$ of complex numbers.
(2) The map $t \in\to P(t, x)(\eta, \xi)$ is measurable
(3) There exists a measurable map $K^p_{\eta\xi} : [0, T] \to \mathbb{R}_+$ lying in $L^I_{loc}([t_0, T])$ such that

$$\rho(P(t, x)(\eta, \xi) - P(t, y)(\eta, \xi)) \le K^p_{\eta\xi}(t)W\left(\|x - y\|_{\eta\xi}\right)$$

for $t \in [0, T]$, and for each pair $x, y \in \tilde{\mathcal{A}}$.

2.1 Remark

(i) The conditions (1) and (2) are similar to the conditions $S_{(i)}$ and $S_{(ii)}$ in [1], while condition $S_{(iii)}$ in [1] has been been replaced with condition (3) where $W(t) \neq t$.

For arbitrary pair of elements $\eta, \xi \in \mathbb{D}\underline{\otimes}\mathbb{E}$, and for any process $y \in wac(\tilde{\mathcal{A}})$ and a family of processes $\{y_j\}_{j\geq 0} \subseteq wac(\tilde{\mathcal{A}})$, we employ the respective notations $y_{\eta\xi}(.)$, $y_{\eta\xi}, j(.)$, $\quad j = 0, 1, ...$ as in [1]. For any non empty subset $M \subseteq wac(\tilde{\mathcal{A}})$ we define the function space $M(\eta, \xi) := \{\Phi_{\eta\xi}(.) := \langle \eta, \Phi(.)\xi\rangle : \Phi \in M\}$ and for any element $y_{\eta\xi}(.) \in M(\eta, \xi)$, we define the number

$$\Delta(y_{\eta\xi}) := d_{AC}(y_{\eta\xi}(.), S(P)(\eta, \xi)).$$

(ii) We adopt the conditions of definition 2.1 in [1], since the Lipschitz function is not explicitly dependent on time.

Again we adopt the following result established in [1] for the same reason explained in (ii) above. We only state the Lemma and refer the reader to the reference [1]. Hence all results of Lemma 2.2 established in [1] hold in this case.

Remark: Proposition 2.2 has been established in [1], under the Lipschitz condition $W(t) = t$. Hence here, we shall only write out the major changes due to the general Lipschitz condition defined above.

3 Major results

In this section, we present our major results. The method we employ here are simple extension of the methods employed in [1] and the references there in.

Theorem 3.1 *Assume that the map $(t, x) \to P(t, x)(\eta, \xi)$ satisfies conditions (1) to (3). Also assume that a non empty set $M \subset wac(\tilde{\mathcal{A}})$ is given and there exist positive functions $\rho_{\eta\xi}, N_{\eta\xi} : [t_0, T] \to \mathbb{R}_+$ lying in $L^1_{loc}([t_0, T], \mathbb{R}_+)$ such that*

$$d\left(\frac{d}{dt}\langle \eta, Y(t)\xi\rangle, P(t, Y(t))(\eta, \xi)\right) \leq W\rho_{\eta\xi}(t),$$

and

$$\left|\frac{d}{dt}\langle \eta, Y(t)\xi\rangle\right| \leq N_{\eta\xi}(t), t \in [t_0, T], Y \in M.$$

Then for arbitrary $\epsilon > 0$, there exists a continuous map $R : M(\eta, \xi) \to S_{(T)}(P)(\eta, \xi)$ in the norm topology of the space $AC([t_0, T], \mathbb{C})$ such that:

1. $R(y_{\eta\xi}(.))(t_0) = y_{\eta\xi}(t_0)$.
2. $R(y_{\eta\xi}(.)) = y_{\eta\xi}(.)$, for $y_{\eta\xi} \in M(\eta, \xi) \bigcap S_{(T)}(P)(\eta, \xi)$.
3. $|R(y_{\eta\xi}(.))(t) - y_{\eta\xi}(t)| \leq W\mathbb{E}_{\eta\xi}(t) + \epsilon, t \in [t_0, T], \quad y_{\eta\xi} \in M(\eta, \xi) \setminus S_{(T)}(P)(\eta, \xi)$

Proof Let $\epsilon > 0$ be given and for $k = 0, 1, 2, \ldots$ define a sequence of real positive numbers that depend on η, ξ by:

$$a_k = \frac{\epsilon}{2^{k+2} exp\left[2WM_{\eta\xi}(T)\right]} \tag{3.1}$$

Then there exists a number $\sigma_0 = \sigma_0(\eta, \xi)$ depending on η, ξ such tha t

$$a_0 \in \left(0, \frac{\sigma_0}{3 + 4WM_{\eta\xi}(T)}\right)$$

and

$$\int_A N_{\eta\xi}(t)dt < \alpha_0 \tag{3.2}$$

for every measurable set $A \subseteq [t_0, T]$ satisfying $\mu(A) < \sigma_0$. The results (3.3) to (3.18) in [1] hold here. □

Next by (3.18), we have

$$\begin{aligned}
& d\left(G_0(y_{\eta\xi})(t), P(t, y(t_0) + \int_{t_0}^t g_0(y)(s)ds)(\eta, \xi)\right) \\
& \quad = d\left(B(y_{\eta\xi, j})(t), P(t, y(t_0) + \int_{t_0}^t g_0(y)(s)ds)(\eta, \xi)\right) \\
& \quad \le d\left(\frac{d}{dt}\langle \eta, y_j(t)\xi\rangle + B_{\eta\xi, j}(t) - \frac{d}{dt}\langle \eta, y_j(t)\xi\rangle, P(t, y_j(t)(\eta, \xi)\right) \\
& \qquad + \rho\left(P(t, y_j(t)(\eta, \xi), P(t, y(t_0) + \int_{t_0}^t g_0(y)(s)ds)(\eta, \xi)\right) \\
& \quad \le d\left(\frac{d}{dt}\langle \eta, y_j(t)\xi\rangle, P(t, y_j(t))(\eta, \xi)\right) + \left|B_{\eta\xi, j}(t) - \frac{d}{dt}\langle \eta, y_j(t)\xi\rangle\right| \\
& \qquad + K_{\eta\xi}(t)W\left(\left\|y_j(t) - y(t) + y(t) - \int_{t_0}^t g_0(y)(s)ds\right\|_{\eta\xi}\right) \\
& \quad = d\left(\frac{d}{dt}\langle \eta, y_j(t)\xi\rangle, P(t, y_j(t))(\eta, \xi)\right) + \left|B_{\eta\xi, j}(t) - \frac{d}{dt}\langle \eta, y_j(t)\xi\rangle\right| \\
& \qquad + K_{\eta\xi}(t)W|y_{\eta\xi, j}(t) - y_{\eta\xi}(t)| + K_{\eta\xi}(t)W\left|y_{\eta\xi}(t) - y_{\eta\xi}(t_0)\int_{t_0}^t G_0(y_{\eta\xi})(s)ds\right| \\
& \quad \le d\left(\frac{d}{dt}\langle \eta, y_j(t)\xi\rangle, P(t, y_j(t))(\eta, \xi)\right) + \left|B_{\eta\xi, j}(t) - \frac{d}{dt}\langle \eta, y_j(t)\xi\rangle\right| \\
& \qquad + K_{\eta\xi}(t)W\left|y_{\eta\xi, j}(t) - y_{\eta\xi}(t)\right| + 3\sigma_0 W K_{\eta\xi}(t) \ \text{ by (3.18)} \qquad (3.19) \\
& \quad \le d\left(\frac{d}{dt}\langle \eta, y_j(t)\xi\rangle, P(t, y_j(t))(\eta, \xi)\right) + \beta_{\eta\xi}(t) \qquad (3.20)
\end{aligned}$$

where

$$\beta_{\eta\xi}(t) = 4\sigma_0 W K_{\eta\xi}(t) + \sum_{j\ge 1}\left|B_{\eta\xi, j}(t) - \frac{d}{dt}\langle \eta, y_j(t)\xi\rangle\right| \tag{3.21}$$

The estimate given by (3.20) holds by applying (3.17) in [1]. Again by applying (3.2) and (3.3) in [2006] to (3.21) above we get

$$\int_{t_0}^{T} \beta_{\eta\xi}(t)dt := |\beta_{\eta\xi}(.)|_{L^1} < \alpha_0. \tag{3.22}$$

Next, we shall establish a sequence of

1. Continuous decomposition $\{T_j^k(y_{\eta\xi})\}_{j\geq 1}, k \geq 0$ of the interval $[t_0, T]$ corresponding to a stochastic process $y \in M\backslash S_{(T)}(P)$ with the associated matrix element $y_{\eta\xi} \in M(\eta, \xi)\backslash S_{(T)}(P)(\eta, \xi)$.
2. Mappings $g_k : M \to L^1_{loc}(\tilde{\mathcal{A}})$ with the associated continuous sequence of maps $G_k : M(\eta, \xi) \to L^1_{loc}([t_0, T], \mathbb{C})$, and
3. Functions $\zeta_{\eta\xi,k}(.)$, $N_{\eta\xi,k}(.) \in L^1([t_0, T], \mathbb{R}_+)$ corresponding to arbitrary pair of elements $\eta, \xi \in \mathbb{D}\underline{\otimes}\mathbb{E}$.

By induction hypothesis, suppose that (i), (ii) and (iii) have been constructed and let

$$I_k(y)(t) = y(t_0) + \int_{t_0}^{t} g_k(y)(s)ds, \quad y \in M, \quad k \geq 0$$

and the results (3.23), (3.24) and (3.25) in [1] also hold for $k = 0$ in this case.

Similarly by applying Lemma 2.2 to the set $y \in M(\eta, \xi)\backslash S_{(T)}(P)(\eta, \xi)$ and some additional conditions in [1], we see that (3.26) to (2.29) in [1] hold here again.

For $T \in T_j^{k+1}(y_{\eta\xi})$ and by (3.28) in [1], we have:

$$\begin{aligned}
&\left|G_k(y_{\eta\xi})(t) - G_{k+1}(y_{\eta\xi})(t)\right| = \left|G_k(y_{\eta\xi})(t) - U_{\eta\xi,j}(t)\right| \\
&\quad \leq \left|U_{\eta\xi,j}(t) - G_k(y_{\eta\xi,j})(t)\right| + \left|G_k(y_{\eta\xi,j})(t) - G_k(y_{\eta\xi})(t)\right| \\
&\quad = d(G_k(y_{\eta\xi})(t), P(t, I_k(y)(t))(\eta, \xi)) + \left|G_k(y_{\eta\xi,j})(t) - G_k(y_{\eta\xi})(t)\right| \\
&\quad \leq d(G_k(y_{\eta\xi,j})(t), P(t, I_k(y)(t))(\eta, \xi)) + \rho(P(t, I_k(y)(t))(\eta, \xi) \\
&\qquad - P(t, I_k(y_j)(t))(\eta, \xi)) \\
&\qquad + \left|G_k(y_{\eta\xi,j})(t) - G_k(y_{\eta\xi})(t)\right| \\
&\quad \leq d(G_k(y_{\eta\xi})(t), P(t, I_k(y)(t))(\eta, \xi)) + \rho(P(t, I_k(y)(t))(\eta, \xi) \\
&\qquad - P(t, I_k(y_j)(t))(\eta, \xi)) \\
&\qquad + 2\left|G_k(y_{\eta\xi,j})(t) - G_k(y_{\eta\xi})(t)\right| \\
&\quad \leq d(G_k(y_{\eta\xi})(t), P(t, I_k(y)(t))(\eta, \xi)) + K_{\eta\xi}(t)W\left(\left|I_k(y_{\eta\xi})(t) - I_k(y_{\eta\xi,j}(t)\right|\right) \\
&\qquad + 2\sum_{j\geq 1} \chi_{T_j^{k+1}(y_{\eta\xi})\cap(T_i^k(y_{\eta\xi,j})\Delta T_i^k(y_{\eta\xi}))}(t)N_{\eta\xi,k}(t).
\end{aligned} \tag{3.30}$$

Furthermore, we have

$$\begin{aligned}
&d(G_{k+1}(y_{\eta\xi})(t), P(t, I_k(y)(t))(\eta, \xi)) = d(U_{\eta\xi,j})(t), P(t, I_k(y)(t))(\eta, \xi)) \\
&\quad \leq \rho(P(t, I_k(y_j)(t))(\eta, \xi) - P(t, I_k(y)(t))(\eta, \xi)) \\
&\quad \leq K_{\eta\xi}(t)W(|J_k(y_{\eta\xi,j})(t) - J_k(y_{\eta\xi,j}(t)|).
\end{aligned} \tag{3.31}$$

Applying (3.20), (3.30) and (3.29) in [1], we get (3.32) in [1].
Employing (3.29), (3.30) and (3.31), we have

$$\begin{aligned}
&\int_{t_0}^{t} |G_{k+1}(y_{\eta\xi})(s) - G_k(y_{\eta\xi})(s)|ds \leq \int_{t_0}^{t} d(G_k(y_{\eta\xi})(s), P(s, I_k(y_{\eta\xi})(s))ds \\
&\quad +\alpha_{k+1}\left(\frac{1}{6} + \frac{WM_{\eta\xi}(T)+2}{6WM_{\eta\xi}(T)+4}\right) \leq \int_{t_0}^{t} d(G_k(y_{\eta\xi})(s), P(s, I_{k-1}(y_{\eta\xi})(s))ds \\
&\quad + \int_{t_0}^{t} \rho(P(s, I_k(y)(s))(\eta,\xi), P(s, I_{k-1}(y)(s))(\eta,\xi))ds \\
&\quad +\alpha_{k+1}\left(\frac{1}{6} + \frac{WM_{\eta\xi}(T)+2}{6WM_{\eta\xi}(T)+4}\right) \\
&\leq \int_{t_0}^{t} d(G_k(y_{\eta\xi})(s), P(s, I_{k-1}(y_{\eta\xi})(s))ds \\
&\quad + \int_{t_0}^{t} K_{\eta\xi}(s)W\left|J_k(y_{\eta\xi})(s) - J_{k-1}(y_{\eta\xi})(s)\right|ds \\
&\quad +\alpha_{k+1}\left(\frac{1}{6} + \frac{WM_{\eta\xi}(T)+2}{6WM_{\eta\xi}(T)+1}\right) \\
&\leq \int_{t_0}^{t} K_{\eta\xi}(s)W\left|J_k(y_{\eta\xi})(s) - J_{k-1}(y_{\eta\xi})(s)\right|ds + \alpha_{k+1}
\end{aligned} \tag{3.33}$$

Continuing the iteration in (3.33) and using (3.32) and the relation

$$\left|J_k(y_{\eta\xi})(s) - J_{k-1}(y_{\eta\xi})(s)\right| \leq \int_{t_0}^{t} \left|(G_k(y_{\eta\xi})(u), (G_{k-1}(y_{\eta\xi})(u))\right|du$$

by induction, we have

$$\begin{aligned}
&\int_{t_0}^{t} \left|(G_k(y_{\eta\xi})(s), (G_{k-1}(y_{\eta\xi})(s))\right|ds \\
&\leq \int_{t_0}^{t} \frac{W\left(M_{\eta\xi}\right)(t) - \left(M_{\eta\xi}(s)\right)^k}{k!}\rho_{\eta\xi}(s)ds + \alpha_{k+1} \\
&\quad +W\left(M_{\eta\xi}(T)\alpha_k + \cdots + \frac{(M_{\eta\xi})(t))^k}{k!}\alpha_1 + \frac{\left(M_{\eta\xi})(t)\right)^k}{k!}\alpha_0\right) \\
&\leq \int_{t_0}^{t} \frac{W\left(M_{\eta\xi}\right)(t) - \left(M_{\eta\xi}(s)\right)^k}{k!}\rho_{\eta\xi}(s)ds + \frac{\epsilon}{2^{k+3}} + \frac{W(M_{\eta\xi})(t))^k}{k!}\alpha_0
\end{aligned} \tag{3.34}$$

Applying (3.29), (3.31), (3.34) and the Lipschitz condition $W(t) \neq t$, we get:

$$\begin{aligned}
&d(G_{k+1}(y_{\eta\xi})(t), P(t, I_{k+1}(y)(t)(\eta,\xi)) \\
&\quad \leq d(G_{k+1}(y_{\eta\xi})(t), P(t, I_k(y_{\eta\xi})(t)) + K_{\eta\xi}(t)W|J_{k+1}(y_{\eta\xi})(t) - J_k(y_{\eta\xi})(t)|
\end{aligned}$$

$$\leq K_{\eta\xi}(s)W\left(\frac{\alpha_{k+1}}{M_{\eta\xi}(T)}+\int_{t_0}^{t}\frac{(M_{\eta\xi})(t)-(M_{\eta\xi}(s))^k}{k!}\rho_{\eta\xi}(s)ds\right.$$
$$\left.+\frac{\epsilon}{2^{k+3}}+\frac{(M_{\eta\xi})(t))^k}{k!}\alpha_0\right)$$
$$:= W\zeta_{\eta\xi,k+1}(t) \tag{3.35}$$

Showing that the map $\zeta_{\eta\xi,k+1}(.) \in L^1([t_0,T],\mathbb{R}_+)$ with $W(t)\neq t$.

Remark Following the procedure of [1] and the conclusion of Lemma 2.2 we can show in a similar manner that the map $G_{k+1} : M(\eta,\xi) \to L^1([t_0,T],\mathbb{C})$ is continuous, and hence every other results holds concerning the map.

Let $y \in M(\eta,\xi)$. Summing of the inequalities in (3.34) and applying the definition of α_0, we have:

$$\int_{t_0}^{t}\left|G_{k+1}(y_{\eta\xi})(s)-G_0(y_{\eta\xi})(s)\right|ds$$
$$\leq W\int_{t_0}^{t}exp(M_{\eta\xi}(t)-M_{\eta\xi}(s))\rho_{\eta\xi}(s)ds+\sum_{v=0}^{k}\frac{\epsilon}{2^{v+3}}+Wexp(M_{\eta,\xi}(t))\alpha_0$$
$$\leq W\int_{t_0}^{t}exp(M_{\eta\xi}(t)-M_{\eta\xi}(s))\rho_{\eta\xi}(s)ds+\frac{2\epsilon}{2^3}+\frac{\epsilon}{W4exp(2M_{\eta,\xi}(T))}$$
$$\leq W\int_{t_0}^{t}exp(M_{\eta\xi}(t)-M_{\eta\xi}(s))\rho_{\eta\xi}(s)ds+\frac{\epsilon}{2} \tag{3.36}$$

By employing (3.36) and (3.18), we obtain:

$$\left|J_{k+1}(y_{\eta\xi})(t)-(y_{\eta\xi})(t)\right|$$
$$\leq\left|J_{k+1}(y_{\eta\xi})(t)-J_0(y_{\eta\xi})(t)\right|+\left|J_0(y_{\eta\xi})(t)-(y_{\eta\xi})(t)\right|$$
$$\leq W\int_{t_0}^{t}exp(M_{\eta\xi}(t)-M_{\eta\xi}(s))\rho_{\eta\xi}(s)ds+\frac{\epsilon}{2}+\frac{\epsilon}{W4exp(2M_{\eta,\xi}(T))}$$
$$\leq W\mathbb{E}_{\eta\xi}(t)+\epsilon \tag{3.37}$$

Since

$$\left|J_{k+1}(y_{\eta\xi})-J_k(y_{\eta\xi})\right|_{AC}=\left|J_{k+1}(y_{\eta\xi})(t_0)-J_k(y_{\eta\xi})(t_0)\right|$$
$$+\int_{t_0}^{T}\left|g_{k+1}(y_{\eta\xi})(s)-g_k(y_{\eta\xi})(s)\right|ds,\ y_{\eta\xi}\in M(\eta,\xi),$$

then from (3.34), we conclude as in [1] that the sequence of functions $\{J_k(y_{\eta\xi}\}$ converges in $AC([t_0,T],\mathbb{C})$ to a function $R(y_{\eta\xi}) \in AC([t_0,T],\mathbb{C})$. Given that the sequence $\{J_k(y_{\eta\xi})\} \in wac(\tilde{\mathcal{A}})$,

$$\langle\eta, I_k(y)(t)\xi\rangle-\langle\eta, J_k(y)(t)\xi\rangle,\ y\in M$$

and

$$|I_k(y)|_{\eta\xi} = \|I_k(y)(t_0)\|_{\eta\xi} + \int_{t_0}^{T} \left| \frac{d}{dt} \langle \eta, I_k(y)(t)\xi \rangle \right| dt$$
$$|J_k(y_{\eta\xi})(t_0)| + \int_{t_0}^{T} \left| \frac{d}{dt} J_k(y_{\eta\xi})(t) \right| dt$$

Thus by (3.34), the sequence $\{I_k(y_{\eta\xi})\}$ is a Cauchy sequence in $wac(\tilde{\mathcal{A}})$ which converges to a map $F(y) \in wac(\tilde{\mathcal{A}})$ where $\langle \eta, F(y)(t)\xi \rangle = R(y_{\eta\xi})(t)$ and $R(y_{\eta\xi}) \in wac(\tilde{\mathcal{A}})(\eta, \xi)$. By the continuity of the maps G_k, the map R is also continuous and every other results in [1] holds here again.

Next, we show as in [1] that $F(y) \in S_{(T)}(P)$. We have the following:

$$\begin{aligned} & d\left(\frac{d}{dt} \langle \eta, F(y)(t)\xi \rangle , P(t, F(y)(t))(\eta, \xi) \right) \\ & \quad \le d(G_{k+1}(y_{\eta\xi})(t), P(t, I_k(y)(t))(\eta, \xi)) + \left| \frac{d}{dt} \langle \eta, F(y)(t)\xi \rangle - G_{k+1}(y_{\eta\xi})(t) \right| \\ & \quad\quad + K_{\eta\xi}(t)W \big| \langle \eta, F(y)(t)\xi \rangle - \langle \eta, I_k(y)(t)\xi \rangle \big|. \end{aligned} \tag{3.38}$$

Applying (3.31) and (3.29) to $d\left(\frac{d}{dt}\langle \eta, y_j(t)\xi \rangle, P(t, F(y)(t))(\eta, \xi)\right)$ of (3.38) and integrating both sides we get:

$$\begin{aligned} & \int_{t_0}^{T} d\left(\frac{d}{dt} \langle \eta, F(y)(t)\xi \rangle , P(t, F(y)(t))(\eta, \xi) \right) \\ & \quad \le \int_{t_0}^{T} K_{\eta\xi}(t)W \left[\big| y_{\eta\xi, j} - y_{\eta\xi} \big|_{AC} + \int_{t_0}^{t} \sum_{j\ge} \chi_{T_i^k(y_{\eta\xi,j}) \Delta T_i^k(y_{\eta\xi})}(s) N_{\eta\xi,k}(s) ds \right] dt \\ & \quad\quad + \int_{t_0}^{T} \left(\left| \frac{d}{dt} \langle \eta, F(y)(t)\xi \rangle - G_{k+1}(y_{\eta\xi})(t) \right| \right. \\ & \quad\quad + K_{\eta\xi}(t)W(\big| \langle \eta, F(y)(t)\xi \rangle - \langle \eta, I_k(y)(t)\xi \rangle \big|) dt \\ & \quad \le \alpha_{k+1} + \int_{t_0}^{T} \left(\left| \frac{d}{dt} \langle \eta, F(y)(t)\xi \rangle - G_{k+1}(y_{\eta\xi})(t) \right| \right. \\ & \quad\quad \left. + K_{\eta\xi}(t)W \big| \langle \eta, F(y)(t)\xi \rangle - \langle \eta, I_k(y)(t)\xi \rangle \big| \right) dt. \end{aligned}$$

Taking the limit as $k \to \infty$, we have

$$d\left(\frac{d}{dt} \langle \eta, F(y)(t)\xi \rangle , P(t, F(y)(t))(\eta, \xi) \right) = 0$$

Hence $F(y) \in S_{(T)}(P)$ and therefore

$$\langle \eta, F(y)(.)\xi \rangle := R(y_{\eta\xi})(.) \in S_{(T)}(P)(\eta, \xi).$$

We have been able to show that the results in [1] need not be restricted to the Lipschitz condition $W(t) = t$. But can be extended to a general condition as explained above. Hence under this condition it is possible to re - construct a mapping $Y(.) \rightarrow F(Y(.))$ of an arbitrary nonempty set of pseudo solutions in the space $wac(\tilde{\mathcal{A}})$ with the property that the map $Y(.)(\eta, \xi) \rightarrow F(Y(.))(\eta, \xi)$ is continuous in the topology of the space $wac(\tilde{\mathcal{A}})(\eta, \xi)$ for each pair $\eta, \xi \in \mathbb{D}\underline{\otimes}\mathbb{E}$.

Remark The results of corollaries 3.2, 3.3 and 3.4 in [1] concerning the selection from the multifunction $\langle \eta, x\xi \rangle \rightarrow S_{(T)}(x)(\eta, \xi)$ without any restriction on the domain of the selection map also holds in this case without any modification.

References

1. Ayoola, E.O.: Topological properties of solution sets of Lipschitzian quantum stochastic differential inclusions. Acta. Appl. Math. **100**, 15–37 (2006)
2. Ayoola, E.O.: Continuous Selections of Solutions Sets of Lipschitzian quantum stochastic differential inclusions. Int. J. Theor. Phys. **43**(10), 2041–2058 (2004)
3. Ayoola, E.O.: Lipschitzian quantum stochastic differential equations and the associated Kurzweil equations. Stoch. Anal. Appl. **19**(4), 581–603 (2001)
4. Bishop, S.A.: Existence and Variational Stability of Solutions of Kurzweil Equations associated with Quantum Stochastic Differential Equations. PhD Thesis, Covenant University Ota, Ogun State, Nigeria (2012)
5. Bishop, S.A., Anake, T.A.: Extension of continuous selection sets to non Lipschitzian quantum Stochastic differential inclusion. J. Stochastic Anal. Appl. Stochastic Anal. Appl. **31**(6(2013)), 1114–1124 (2013)
6. Ekhaguere, G.O.S.: Lipschitzian quantum stochastic differential inclusions. Int. J. Theo. Phy. **31**(11), 2003–2034 (1992)
7. Smirnov, G.V.: Topological properties of an integral funnel of a differential inclusion with Lischitz right side. Diff. Equ. **27**(2), 157–165 (1991)
8. Kurzweil, J.: Problems which lead to a generalization of the concept of ordinary differential equation. Differential Equations and their Applications. Czech Academy of Science, Prague (1963)
9. Shizan, F., Zhang, T.: A study of a class of stochastic differential equations with non-Lipschitzian coefficients. Probab. Theory Rel. Fields **132**(3), 356–390 (2005)
10. Hudson, R.L., Parthasarathy, K.R.: Quantum Ito's formulae and stochastic evolutions. Comm. Math. Phys. **93**, 301–324 (1984)

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Continuous Selections of Solution Sets of Lipschitzian Quantum Stochastic Differential Inclusions

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A multifunction associated with the set of solutions of Lipschitzian quantum stochastic differential inclusion (QSDI) admits a selection continuous from some subsets of complex numbers to the space of the matrix elements of adapted weakly absolutely continuous quantum stochastic processes. In particular, the solution set map as well as the reachable set of the QSDI admit some continuous representations.

KEY WORDS: QSDI; continuous selection; adapted processes.

1. INTRODUCTION

This paper is concerned with the problems of continuous selections of multivalued solution maps of quantum stochastic differential inclusions in integral form, given by

$$X(t) \in a + \int_0^t E(s, X(s))d \wedge_\pi (s) + F(s, X(s))\, dA_f(s)$$
$$+G(s, X(s))\, dA_g^+(s) + H(s, X(s))\, ds, \quad \text{almost all} \quad t \in [0, T]. \tag{1.1}$$

QSDI (1.1) is understood in the framework of the Hudson and Parthasarathy (1984) formulation of Boson quantum stochastic calculus. In the notations and definitions of various spaces of stochastic processes introduced in the work of Ekhaguere (1992), the coefficients E, F, G, H, lie in $L^2_{\text{loc}}([0, T] \times \tilde{\mathcal{A}})_{\text{mvs}}$, where $\tilde{\mathcal{A}}$ is a locally convex space and $(0, a) \in [0, T] \times \tilde{\mathcal{A}}$ is a fixed point. The maps f, g, π appearing in (1.1) lie in some suitable function spaces. The integrators $\wedge_\pi$, A_g^+ and A_f are the gauge, creation and annihilation processes associated with the

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basic field operators of quantum field theory. As in our previous works (Ayoola, 2001, 2003a,b) concerning some approximation of the reachable sets and solutions of QSDI (1.1), we consider the equivalent form of (1.1) given by

$$\frac{d}{dt}\langle \eta, X(t)\xi\rangle \in P(t, X(t))(\eta, \xi)$$
$$X(0) = a, \quad t \in [0, T]. \tag{1.2}$$

Inclusion (1.2) is a nonclassical ordinary differential inclusion and the map $(\eta, \xi) \to P(t, x)(\eta, \xi)$ is a multivalued sesquilinear form on $(\mathbb{D}\underline{\otimes}\mathbb{E})^2$ for $(t, x) \in [0, T] \times \tilde{\mathcal{A}}$. We refer the reader to the works of Ekhaguere (1992, 1995, 1996) for the explicit forms of the map and the existence results for solutions of QSDI (1.1) of Lipschitz, hypermaximal monotone and of evolution types. We follow the fundamental concepts and structures as in the references by employing the locally convex space $\tilde{\mathcal{A}}$ of noncommutative stochastic processes whose topology is generated by the family of seminorms $\{\|x\|_{\eta\xi} = |\langle \eta, x\xi\rangle|, x \in \mathcal{A}, \eta, \xi \in \mathbb{D}\underline{\otimes}\mathbb{E}\}$. Here, as usual, the underlying elements of $\tilde{\mathcal{A}}$ consists of linear maps from $\mathbb{D}\underline{\otimes}\mathbb{E}$ into $\mathcal{R} \otimes \Gamma(L^2_\gamma(\mathbb{R}_+))$ having domains of their adjoints containing $\mathbb{D}\underline{\otimes}\mathbb{E}$. In particular, the spaces $L^P_{\text{loc}}(\tilde{\mathcal{A}})$, $L^\infty_{\gamma,loc}(\mathbb{R}_+)$, $L^P_{loc}(I \times \tilde{\mathcal{A}})$ for a fixed Hilbert space γ are being adopted as in the above references.

In what follows, we consider QSDI (1.2) where the map $x \to P(t, x)(\eta, \xi)$ is Lipschitzian with values that are closed (not necessarily convex nor bounded) subsets of the field of complex numbers. The point a ranges in a subset A of $\tilde{\mathcal{A}}$ such that the set $A(\eta, \xi) := \{\langle \eta, a\xi\rangle : a \in A\}$ is compact in $\mathbb{C}$.

We denote by $S^{(T)}(a)$ the map that assigns to each point $a \in A$, the set of solutions of QSDI (1.2) and prove a continuous selection theorem from the map $S^{(T)}(a)(\eta, \xi)$ where

$$S^{(T)}(a)(\eta, \xi) = \left\{\langle \eta, \Phi(\cdot)\xi\rangle / \Phi \in S^{(T)}(a)\right\}.$$

An important consequence of our main result is that the set map $\langle \eta, a\xi\rangle \to S^{(T)}(a)(\eta, \xi)$ can be continuously represented in the form

$$g(\langle \eta, a\xi\rangle, \mathcal{U}) = S^{(T)}(a)(\eta, \xi).$$

Similar result holds for the case of the map from $\langle \eta, a\xi\rangle$ to the set $R^{(T)}(a)(\eta, \xi)$, where

$$R^{(T)}(a)(\eta, \xi) = \left\{\langle \eta, \Phi(T)\xi\rangle / \Phi(T) \in R^{(T)}(a)\right\}$$

is the set of complex numbers associated with the reachable set

$$R^{(T)}(a) = \left\{\Phi(T) / \Phi \in S^{(T)}(a)\right\}$$

of QSDI (1.1) at time T (see our previous work, Ayoola (2003b) for details).

Our results in this work are extensions of the results of Cellina and Ornelas (1992) to the present noncommutative quantum setting involving inclusions in certain locally convex spaces. We adapt the arguments employed in the reference to conform with our noncommutative stochastic analysis.

Problems of continuous selections of classical differential inclusions have attracted considerable attention in the literature. Some selection results at the classical setting can be found in the works of Cellina (1988), Aubin and Cellina (1984), Fryszkowski (1983), Antosiewicz and Cellina (1975), Colombo *et al.* (1991) and the book by Repovs and Semenov (1998). As shown in the work of Cellina and Ornelas (1992) selection results have been used to show that the solution set-map and the attainable set admit some continuous parameterizations. Broucke and Arapostathis (2001) have established the existence of a continuous selection from the set of solutions that interpolates a given finite set of trajectories of Lipschitz differential inclusion.

The rest of the paper is organized as follows: In Section 2, we outline some fundamental definitions, notations and results concerning the selection results. Section 3 is devoted to the establishment of the main results of the paper.

2. NOTATIONS AND PRELIMINARY RESULTS

We shall employ the following notations in what follows. If $\mathcal{N}$ is a topological space, then $\text{clos}(\mathcal{N})$ (resp. $\text{comp}(\mathcal{N})$) denotes the collection of all nonempty closed (resp. compact) subsets of $\mathcal{N}$. We shall employ the Hausdorff topology on $\text{clos}(\tilde{\mathcal{A}})$ as explained in Ekhaguere (1992).

We denote by $\rho(A, B)$ the Hausdorff distance between the sets A, B in $\text{clos}(\mathbb{C})$. The distance $d(x, A)$ of a point x from a set $A \in \text{clos}(\mathbb{C})$ is defined by

$$d(x, A) = \inf\{|x - a| : a \in A\}.$$

We denote by I, the interval $[0, T]$ and the characteristic function of a subset E of I by χ_E.

As explained in Ekhaguere (1992), we consider the space $\text{wac}(\tilde{\mathcal{A}})$ the completion of the locally convex space $(\text{Ad}(\tilde{\mathcal{A}})_{\text{wac}}, \tau^{\text{wac}})$ where the topology τ^{wac} is generated by the family of seminorms $\{|\cdot|_{\eta\xi} : \eta, \xi \in \mathbb{D}\underline{\otimes}\mathbb{E}\}$ defined for each $\Phi \in \text{A}d(\tilde{\mathcal{A}})_{\text{wac}}$ by

$$|\Phi|_{\eta\xi} = \|\Phi(0)\|_{\eta\xi} + \int_0^T \left| \frac{d}{ds} \langle \eta, \Phi(s)\xi \rangle \right| ds.$$

Associated with $\text{wac}(\tilde{\mathcal{A}})$, we define for arbitrary $\eta, \xi \in \mathbb{D}\underline{\otimes}\mathbb{E}$, the space of complex valued functions

$$\text{wac}(\tilde{\mathcal{A}})(\eta, \xi) = \{\langle \eta, \Phi(\cdot)\xi \rangle / \Phi \in \text{w}ac(\tilde{\mathcal{A}})\}.$$

We remark that each element $\Phi_{\eta\xi}(\cdot) := \langle \eta, \Phi(\cdot)\xi \rangle$ of $\text{wac}(\tilde{\mathcal{A}})(\eta, \xi)$ is an absolutely continuous complex valued function on the interval $[0, T]$. We assume that A is a subset of $\tilde{\mathcal{A}}$ such that the set of complex numbers

$$A(\eta, \xi) = \{\langle \eta, a\xi \rangle / a \in A\}$$

is compact in $\mathbb{C}$ with diameter $D_{\eta\xi} = \sup_{x,y \in A(\eta,\xi)} |x - y|$.

Furthermore, the map $(t, x) \to P(t, x)(\eta, \xi)$ appearing in (1.2) is assumed to satisfy the following conditions.

$\mathcal{S}_{(i)}$ The values of $P(t, x)(\eta, \xi)$ are nonempty closed, subsets of the field $\mathbb{C}$ of complex numbers.

$\mathcal{S}_{(ii)}$ The map $t \to P(t, x)(\eta, \xi)$ is measurable.

$\mathcal{S}_{(iii)}$ There exists a map $K^P_{\eta\xi} : [0, T] \to \mathbb{R}_+$ lying in $L^1_{loc}([0, T])$ such that

$$\rho(P(t, x)(\eta, \xi), P(t, y)(\eta, \xi)) \le K^P_{\eta\xi}(t)\|x - y\|_{\eta\xi}$$

for $t \in [0, T]$, and for each pair $x, y \in \tilde{\mathcal{A}}$.

$\mathcal{S}_{(iv)}$ There exists a stochastic process $Y: [0, T] \to \tilde{\mathcal{A}}$ lying in $\mathrm{Ad}(\tilde{\mathcal{A}})_{\mathrm{wac}}$ such that the map

$$t \to d\left(\frac{d}{dt}\langle \eta, Y(t)\xi \rangle, P(t, Y(t))(\eta, \xi)\right)$$

lies in $L^1_{\mathrm{loc}}([0, T])$.

From the result of Ekhaguere (1992), it is known that under the conditions $\mathcal{S}_{(i)}$ to $\mathcal{S}_{(iv)}$, QSDI (1.1) admits at least one adapted weakly absolutely continuous solution for each $a \in A$. We denote the set of all such solutions, with the topology of $\mathrm{wac}(\tilde{\mathcal{A}})$ by $S^{(T)}(a)$.

To prove our main result in Section 3, we need an important notion of partition of unity.

Definition 2.1. Let A be a subset of $\tilde{\mathcal{A}}$ such that for arbitrary elements $\eta, \xi \in \mathbb{D} \underline{\otimes} \mathbb{E}$, the set $A(\eta, \xi)$ is compact in the field of complex numbers.

Let $\{\Omega_i\}_{i \in J}$ be an open covering for $A(\eta, \xi)$ with a finite open subcovering $\{\Omega_i\}$, $i = 1, 2, \cdots m$. A family of functions $\{P_i(\cdot)\}$, $i = 1, 2 \cdots m$ defined on $A(\eta, \xi)$ is called a Lipschitzian partition of unity subordinated to the finite subcovering if:

(i) $P_i(\cdot)$ is Lipschitzian for all $i = 1, 2 \cdots m$. That is, there exists constant $L_{\eta\xi}$ such that for any pair $a_{\eta\xi},\ a'_{\eta\xi} \in A(\eta, \xi)$, we have

$$|P_i(a_{\eta\xi}) - P_i(a'_{\eta\xi})| \le L_{\eta\xi}|a_{\eta\xi} - a'_{\eta\xi}|.$$

(ii) $P_i(a_{\eta\xi}) > 0$ for $a_{\eta\xi} \in \Omega_i \cap A(\eta, \xi)$ and $P_i(a_{\eta\xi}) = 0$ for $a_{\eta\xi} \in A(\eta, \xi) \setminus \Omega_i$.

(iii) For each $a_{\eta\xi} \in A(\eta, \xi)$, $\sum_{i=1}^{m} P_i(a_{\eta\xi}) = 1$.

Lemma 2.1. *Let $A \subseteq \tilde{A}$ such that for arbitrary $\eta, \xi \in \mathbb{D} \underline{\otimes} \mathbb{E}$, the set $A(\eta, \xi)$ is compact in $\mathbb{C}$. Then there exists a Lipschitzian partition of unity subordinated to any finite subcovering of an open covering for the set $A(\eta, \xi)$.*

Proof: Let $\{\Omega_i\}$, $i = 1, 2 \cdots m$ be a finite open subcovering of an open covering $\{\Omega_i\}_{i \in J}$ of $A(\eta, \xi)$ in the field of complex numbers. First, by Lemma 2.1 in Smirnov (2002) for $Q \subseteq \mathbb{C}$, the map $q : \mathbb{C} \to \mathbb{R}_+$ defined by $q(x) = d(x, Q)$ satisfies for $x_1, x_2 \in \mathbb{C}$

$$|q(x_1) - q(x_2)| \leq |x_1 - x_2|.$$

For $i = 1, 2 \cdots m$, define the functions $q_i : A(\eta, \xi) \to \mathbb{R}_+$ by

$$q_i(a_{\eta\xi}) = d(a_{\eta\xi}, A(\eta, \xi) \backslash \Omega_i)$$

and functions $P_i : A(\eta, \xi) \to \mathbb{R}_+$ by

$$P_i(a_{\eta\xi}) = \frac{q_i(a_{\eta\xi})}{\sum_{j=1}^{m} q_j(a_{\eta\xi})} \tag{2.1}$$

For at least one $j \in \{1, 2 \cdots m\}$, $a_{\eta\xi} \in \Omega_j$. Therefore $\displaystyle\sum_{j=1}^{m} q_j(a_{\eta\xi}) > 0$.

Consequently, (2.1) is well defined.

Moreover, for each $a_{\eta\xi} \in A(\eta, \xi)$, $\displaystyle\sum_{i=1}^{m} P_i(a_{\eta\xi}) = 1$ and $P_i(a_{\eta\xi}) > 0$ for $a_{\eta\xi} \in \Omega_i \cap (A(\eta, \xi))$, and $P_i(a_{\eta\xi}) = 0$ for $a_{\eta\xi} \in A(\eta, \xi) \backslash \Omega_i$.

Next we show that each function P_i is Lipschitzian on $A(\eta, \xi)$.

Since the set $A(\eta, \xi)$ is compact, there exist numbers $M_{\eta\xi}, m_{\eta\xi} > 0$ such that

$$m_{\eta\xi} < \sum_{j=1}^{m} q_j(a_{\eta\xi}) < M_{\eta\xi}$$

for any element $a_{\eta\xi} \in A(\eta, \xi)$.

For any pair $a_{\eta\xi}, a'_{\eta\xi} \in A(\eta, \xi)$, we have

$$\begin{aligned}
|P_i(a_{\eta\xi}) - P_i(a'_{\eta\xi})| &= \frac{\left|q_i(a'_{\eta\xi}) \sum_{j=1}^{m} q_j(a_{\eta\xi}) - q_i(a_{\eta\xi}) \sum_{j=1}^{m} q_j(a'_{\eta\xi})\right|}{\sum_{j=1}^{m} q_j(a_{\eta\xi}) \sum_{j=1}^{m} q_j(a'_{\eta\xi})} \\
&\leq \frac{\left|q_i(a'_{\eta\xi}) \sum_{j=1}^{m} q_j(a_{\eta\xi}) - q_i(a_{\eta\xi}) \sum_{j=1}^{m} q_j(a'_{\eta\xi})\right|}{m_{\eta\xi}^2} \\
&\leq \frac{1}{m_{\eta\xi}^2} \sum_{j=1}^{m} (|q_i(a'_{\eta\xi}) q_j(a_{\eta\xi}) - q_i(a_{\eta\xi}) q_j(a_{\eta\xi})|
\end{aligned}$$

$$+ |q_i(a_{\eta\xi})q_j(a_{\eta\xi}) - q_i(a_{\eta\xi})q_j(a'_{\eta\xi})|)$$
$$\leq \frac{1}{m^2_{\eta\xi}}\left(\sum_{j=1}^{m} q_j(a_{\eta\xi})|a_{\eta\xi} - a'_{\eta\xi}| + q_i(a_{\eta\xi})\sum_{j=1}^{m}|a_{\eta\xi} - a'_{\eta\xi}|\right)$$
$$\leq \frac{(1+m)M_{\eta\xi}}{m^2_{\eta\xi}}|a_{\eta\xi} - a'_{\eta\xi}|,$$

where we have used the inequality

$$|q_i(a'_{\eta\xi}) - q_i(a_{\eta\xi})| \leq |a'_{\eta\xi} - a_{\eta\xi}|$$

satisfied by each function $q_i(\cdot)$, $i = 1, 2 \cdots m$ on $A(\eta, \xi)$.

Thus $P_i(\cdot)$ is Lipschitzian with Lipschitz constant $L_{\eta\xi} = \frac{(1+m)M_{\eta\xi}}{m^2_{\eta\xi}}$. □

Next, we present a proposition which we shall frequently use in the proof of our selection theorem. We obtained the result by adapting its classical analogue presented in Cellina and Ornelas (1992) to the present noncommutative quantum setting.

Proposition 2.2. *Let $V_0, V_1, \cdots V_m$ be stochastic processes in $L^1_{\text{loc}}(\tilde{\mathcal{A}})$ and for any pair of points $\eta, \xi \in \mathbb{D}\underline{\otimes}\mathbb{E}$, let $\{I_j(a_{\eta\xi})\}$ be a partition of the interval $I = [0, T]$ into a finite number of subintervals with endpoints depending continuously on the point $a_{\eta\xi} := \langle\eta, a\xi\rangle, a \in A$.*

Consider the map

$$W : a_{\eta\xi} \to a_{\eta\xi} + \int_0^t \sum_{j=0}^{m} \chi_{I_j(a_{\eta\xi})}(s)\langle\eta, V_j(s)\xi\rangle ds.$$

Then there exists a map $R_{\eta\xi}(t)$ lying in $L^1_{loc}([0, T])$ such that for every $\epsilon > 0$, there exists $\partial > 0$ such that $|a_{\eta\xi} - a'_{\eta\xi}| < \partial$ implies that

$$\left|\frac{d}{dt}W(a_{\eta\xi})(t) - \frac{d}{dt}W(a'_{\eta\xi})(t)\right| \leq R_{\eta\xi}(t)\chi_E(t),$$

for some set $E \subseteq I$ with measure $\mu(E) < \epsilon$.

Proof: First, we assume the hypothesis of Lemma 2.1. By the conclusion of the lemma, there exists a Lipschitzian partition of unity $P_j(\cdot)$ subordinated to a finite open subcovering of $A(\eta, \xi)$.

Let $\epsilon > 0$ be given. Define for each $a_{\eta\xi}$ in $A(\eta, \xi)$,

$$t_0(a_{\eta\xi}) = 0, \quad t_j(a_{\eta\xi}) = t_{j-1}(a_{\eta\xi}) + TP_j(a_{\eta\xi}), \quad 1 \leq j \leq m.$$

Then for each j, t_j is continuous on $A(\eta, \xi)$. This can be shown as follows:

For any pair $a_{\eta\xi}, a'_{\eta\xi} \in A(\eta, \xi)$ we have

$$|t_j(a_{\eta\xi}) - t_j(a'_{\eta\xi})| \leq |t_{j-1}(a_{\eta\xi}) - t_{j-1}(a'_{\eta\xi})| + T\,|P_j(a_{\eta\xi}) - P_j(a'_{\eta\xi})| \leq |t_{j-1}(a_{\eta\xi}) - t_{j-1}(a'_{\eta\xi})| + TL_{\eta\xi}|a_{\eta\xi} - a'_{\eta\xi}|,$$

since each $P_j(\cdot)$ is Lipschitzian.

At the jth iteration, we have

$$|t_j(a_{\eta\xi}) - t_j(a'_{\eta\xi})| \leq jTL_{\eta\xi}|a_{\eta\xi} - a'_{\eta\xi}|.$$

Putting $\delta_j = \frac{\epsilon}{jTL_{\eta\xi}}$, then whenever $|a_{\eta\xi} - a'_{\eta\xi}| \leq \delta_j$, we have

$$|t_j(a_{\eta\xi}) - t_j(a'_{\eta\xi})| \leq \epsilon.$$

Now we define the intervals $I_j(a_{\eta\xi}) = [t_{j-1}(a_{\eta\xi}), t_j(a_{\eta\xi}))$ for $j = 0, 1, \cdots m$. Thus

$$I = [0, T] = \bigcup_{j=0}^{m} I_j(a_{\eta\xi}).$$

We observe that for any two points $a_{\eta\xi}, a'_{\eta\xi} \in A(\eta, \xi)$, we have the following estimate

$$\begin{aligned} \left| \frac{d}{dt} W(a_{\eta\xi})(t) - \frac{d}{dt} W(a'_{\eta\xi})(t) \right| &\leq \sum_{j=0}^{m} \chi_{I_j(a_{\eta\xi}) \triangle I_j(a'_{\eta\xi})(t)}(t) \|V_j(t)\|_{\eta\xi} \\ &= \sum_{j=0}^{m} \chi_{I_j(a_{\eta\xi}) \triangle I_j(a'_{\eta\xi})}(t) R_{\eta\xi}(t) \end{aligned} \tag{2.2}$$

where $R_{\eta\xi}(t) = \max_j \|V_j(t)\|_{\eta\xi}$, for each $t \in [0, T]$ and

$$I_j(a_{\eta\xi}) \triangle I_j(a'_{\eta\xi}) = [I_j(a_{\eta\xi}) \cap (I_j(a_{\eta\xi}) \setminus I_j(a'_{\eta\xi}))] \bigcup [(I_j(a'_{\eta\xi}) \setminus I_j(a_{\eta\xi})) \cap I_j(a'_{\eta\xi})].$$

We remark that the family of sets $\{I_j(a_{\eta\xi}) \triangle I_j(a'_{\eta\xi})\}$ are pairwise disjoint. Putting $E = \bigcup_{j=0}^{m} I_j(a_{\eta\xi}) \triangle I_j(a'_{\eta\xi})$, then from (2.2),

$$\left| \frac{d}{dt} W(a_{\eta\xi})(t) - \frac{d}{dt} W(a'_{\eta\xi})(t) \right| \leq \chi_E(t) R_{\eta\xi}(t)$$

where by the properties of characteristic functions (see for example, Halmos, 1988),

$$\chi_E(t) = \sum_{j=1}^{m} \chi_{I_j(a_{\eta\xi}) \triangle I_j(a'_{\eta\xi})}(t).$$

Since the set $A(\eta, \xi)$ is compact in $\mathbb{C}$, then the family of functions $\{t_j\}$ is a uniformly equicontinuous family of real valued functions. Thus for every $\epsilon > 0$, there exists $\delta > 0$ such that for every $a_{\eta\xi}, a'_{\eta\xi} \in A(\eta, \xi)$ satisfying

$$|a_{\eta\xi} - a'_{\eta\xi}| < \delta,$$

then

$$|t_j(a_{\eta\xi}) - t_j(a'_{\eta\xi})| < \frac{\epsilon}{2(m+1)}.$$

Hence,

$$\mu(I_j(a_{\eta\xi}) \triangle I_j(a'_{\eta\xi})) \leq 2|t_j(a_{\eta\xi}) - t_j(a'_{\eta\xi})| \leq \frac{\epsilon}{m+1}.$$

Consequently,

$$\mu(E) = \sum_{j=0}^{m} \mu\left(I_j(a_{\eta\xi}) \triangle I_j(a'_{\eta\xi})\right) < \epsilon.$$

□

3. MAIN RESULTS

We present in this section, our main results. In what follows, for points $a_k \in A,\ k = 0, 1, 2 \cdots$, and $\Phi_k \in S^{(T)}(a_k)$, we set $a_{\eta\xi,k} = \langle \eta, a_k \xi \rangle$ and $\Phi_{\eta\xi,k}(\cdot) = \langle \eta, \Phi_k(\cdot)\xi \rangle$.

Our method of proof for the main results below, is an adaptation of the arguments employed in Cellina and Ornelas (1992) concerning similar result for classical differential inclusions. In addition, we employ successive approximations similar to what we have in Ekhaguere (1992) for proving the existence of solutions of quantum stochastic differential inclusion (1.1).

Theorem 3.1. *Assume that the map $(t, x) \to P(t, x)(\eta, \xi)$ satisfies conditions $\mathcal{S}_{(i)} - \mathcal{S}_{(iv)}$.*

Let $\Phi_0 \in S^{(T)}(a_0)$ for a fixed point $a_0 \in A$. Then there exists a continuous map $W : A(\eta, \xi) \to \mathrm{wac}(\tilde{\mathcal{A}})(\eta, \xi)$, a selection from $S^{(T)}(a)(\eta, \xi)$ such that

$$W(a_{\eta\xi,0}) = \Phi_{\eta\xi,0}.$$

Proof: Since $\Phi_0 \in S^{(T)}(a_0)$, then $\Phi_0 \in \mathrm{Ad}(\tilde{\mathcal{A}})_{\mathrm{wac}}$. By the properties of the solution established in Ekhaguere (1992), there exists a stochastic process $V_0 : [0, T] \to \tilde{\mathcal{A}}$ lying in $L^1_{\mathrm{loc}}(\tilde{\mathcal{A}})$ such that for almost all $t \in [0, T]$,

$$\Phi_0(t) = a_0 + \int_0^t V_0(s)ds \tag{3.1}$$

and for arbitrary $\eta, \xi \in \mathbb{D}\underline{\otimes}\mathbb{E}$, we have

$$\langle \eta, V_0(s)\xi \rangle = \frac{d}{ds}\langle \eta, \Phi_0(s)\xi \rangle \in P(s, \Phi_0(s))(\eta, \xi), \quad s \in [0, T].$$

Now for arbitrary element $a \in A$, set

$$Y : A \to \text{wac}(\tilde{\mathcal{A}})$$

to be

$$Y(a)(t) = a + \int_0^t V_0(s)ds.$$

Associated with the map Y, we set the map

$$W : A(\eta, \xi) \to \text{wac}(\tilde{\mathcal{A}})(\eta, \xi)$$

to be

$$W(a_{\eta\xi})(t) = a_{\eta\xi} + \int_0^t \langle \eta, V_0(s)\xi \rangle ds.$$

We remark that the map W is well defined and continuous on $A(\eta, \xi)$ and that

$$\frac{d}{dt}W(a_{\eta\xi})(t) = \langle \eta, V_0(t)\xi \rangle.$$

Furthermore,

$$\begin{aligned}
&d\left(\frac{d}{dt}W(a_{\eta\xi})(t), P(t, Y(a)(t))(\eta, \xi)\right) \\
&\quad = d\left(\frac{d}{dt}\langle \eta, \Phi_0(t)\xi \rangle, P(t, Y(a)(t))(\eta, \xi)\right) \\
&\quad \leq \rho\left(P(t, Y(a_0)(t))(\eta, \xi), P(t, Y(a)(t))(\eta, \xi)\right) \\
&\quad \leq K^P_{\eta\xi}(t)\|Y(a_0)(t) - Y(a)(t)\|_{\eta\xi} \\
&\quad = K^P_{\eta\xi}(t)\|a_0 - a\|_{\eta\xi}.
\end{aligned}$$

Since the map $t \to P(t, Y(a)(t))(\eta, \xi)$ is measurable with closed values in the complex field, then by Theorem 2, Chapter 1, Section 14 in the book of Aubin and Cellina (1984) (see also Ekhaguere, 1992), we can choose $U_0(a)(t)(\eta, \xi)$ to be a measurable selection from $P(t, Y(a)(t))(\eta, \xi)$ such that

$$\begin{aligned}
\left|\frac{d}{dt}W(a_{\eta\xi})(t) - U_0(a)(t)(\eta, \xi)\right| &= d\left(\frac{d}{dt}W(a_{\eta\xi})(t),\ P(t, Y(a)(t))(\eta, \xi)\right) \\
&\leq K^P_{\eta\xi}(t)\|a_0 - a\|_{\eta\xi}. \qquad (3.2)
\end{aligned}$$

As the map $(\eta, \xi) \to U_0(a)(t)(\eta, \xi)$ is a sesquilinear form on $\mathbb{D}\underline{\otimes}\mathbb{E}$ for almost all $t \in [0, T]$ and by the adaptedness of $Y(a)$, there exists an adapted stochastic process $U_0(a) : [0, T] \to \tilde{\mathcal{A}}$ such that

$$\langle \eta, U_0(a)(t)\xi \rangle = U_0(a)(t)(\eta, \xi).$$

By Equation (3.2), the process $U_0(a)$ lies in $L^1_{\text{loc}}(\tilde{\mathcal{A}})$, for any $a \in A$. This assertion follows from the fact that from (3.2), we have

$$\|V_0(t) - U_0(a)(t)\|_{\eta\xi} \leq K^P_{\eta\xi}(t)\|a_0 - a\|_{\eta\xi}$$

where $V_0 \in L^1_{loc}(\tilde{\mathcal{A}})$.

Next we fix some positive real number θ and define for any $a_{\eta\xi} \in A(\eta, \xi)$,

$$\delta(a_{\eta\xi}) = \min\left\{2^{-3}\theta, \ \frac{|a_{\eta\xi} - a_{\eta\xi,0}|}{2}\right\} a_{\eta\xi} \neq a_{\eta\xi,0}$$

and

$$\delta(a_{\eta\xi,0}) = 2^{-3}\theta.$$

Next, we define the open balls

$$B(a_{\eta\xi}, \ \delta(a_{\eta\xi})) = \{x \in \mathbb{C}/|x - a_{\eta\xi}| < \delta\}.$$

Then the family of open sets $\{B(a_{\eta\xi}, \ \delta(a_{\eta\xi})), \ a_{\eta\xi} \in A(\eta, \xi)\}$ covers the set $A(\eta, \xi)$.

By the compactness of $A(\eta, \xi)$ let $B(a_{\eta\xi,j}, \ \delta(a_{\eta\xi,j}))$, $j = 0, 1, 2 \cdots m$ be a finite open subcovering . We notice that the point $a_{\eta\xi,0}$ belongs only to the set $B(a_{\eta\xi,0}, \ \delta(a_{\eta\xi,0}))$.

Let $P_j(\cdot)$, $j = 0, 1, 2 \cdots m$ be a Lipschitzian partition of unity subordinate to the covering. We define the following intervals:

$$I_0(a_{\eta\xi}) = [0, \ TP_0(a_{\eta\xi})]$$

and for $j > 0$,

$$I_j(a_{\eta\xi}) = [T(P_0(a_{\eta\xi}) + \cdots + P_{j-1}(a_{\eta\xi})), \ T(P_0(a_{\eta\xi}) + \cdots P_j(a_{\eta\xi}))].$$

Next we set

$$Y_1(a)(t) = a + \int_0^t \sum_{j=0}^m \chi_{I_j(a_{\eta\xi})}(s) U_0(a_j)(s) ds \tag{3.3}$$

and

$$W_1(a_{\eta\xi})(t) = a_{\eta\xi} + \int_0^t \sum_{j=0}^m \chi_{I_j(a_{\eta\xi})}(s) \langle \eta, U_0(a_j)(s)\xi \rangle ds. \tag{3.4}$$

By Proposition (2.1), the map $W_1 : A(\eta, \xi) \to \mathrm{wac}(\tilde{\mathcal{A}})(\eta, \xi)$ is continuous. Moreover, since $a_{\eta\xi,0}$ belongs only to the set $B(a_{\eta\xi,0}, \ \delta(a_{\eta\xi,0}))$, $P_0(a_{\eta\xi,0}) = 1$ and therefore we have $I_0(a_{\eta\xi,0}) = [0, T]$. Since $\chi_{I_j(a_{\eta\xi,0})} U_0(a_j)(s) = 0, \ \ j \neq 0$, we have from (3.3) and (3.4),

$$Y_1(a_0)(t) = a_0 + \int_0^t U_0(a_0)(s) ds$$

and

$$W_1(a_{\eta\xi,0})(t) = a_{\eta\xi,0} + \int_0^t \langle \eta, U_0(a_0)(s)\xi \rangle ds.$$

But by (3.2),

$$\left| \frac{d}{dt} W(a_{\eta\xi,0})(t) - \langle \eta, U_0(a_0)(t)\xi \rangle \right| = 0$$

That is

$$|\langle \eta, (V_0(t) - U_0(a_0)(t))\xi \rangle| = 0$$

holds for arbitrary $\eta, \xi \in \mathbb{D} \underline{\otimes} \mathbb{E}$.

Hence,

$$V_0(t) = U_0(a_0)(t), \quad t \in [0, T].$$

Therefore by (3.1)

$$Y_1(a_0)(t) = \Phi_0(t)$$

and

$$W_1(a_{\eta\xi,0})(t) = \Phi_{\eta\xi,0}(t) \tag{3.5}$$

Next we have by (3.2)

$$\begin{aligned} &\int_0^t \left| \frac{d}{ds} W_1(a_{\eta\xi})(s) - \frac{d}{ds} W(a_{\eta\xi})(s) \right| ds \\ &\quad \le \int_0^t \sum_{j=0}^m \chi_{I_j(a_{\eta\xi})}(s) \left| \langle \eta, U_0(a_j)(s)\xi \rangle - \frac{d}{ds} W(a_{\eta\xi})(s) \right| ds \\ &\quad \le \int_0^t \sum_{j=0}^m \chi_{I_j(a_{\eta\xi})} K_{\eta\xi}^P(s) \|a_0 - a_j\|_{\eta\xi} ds \le D_{\eta\xi} M_{\eta\xi}(t) \end{aligned} \tag{3.6}$$

where

$$M_{\eta\xi}(t) = \int_0^t K_{\eta\xi}^P(s) ds$$

and

$$\sum_{j=0}^{m} \chi_{I_j(a_{\eta\xi})}(s) = \chi_I(s) = 1, \quad I = \bigcup_{j=0}^{m} I_j(a_{\eta\xi}) = [0, T].$$

Let $t \in [0, T]$ be fixed and let $j \in \{0, 1, 2 \cdots m\}$ be such that $t \in I_j(a_{\eta\xi})$. Then by the definition of $W_1(a_{\eta\xi})(t)$,

$$\begin{aligned} & d\left(\frac{d}{dt} W_1(a_{\eta\xi})(t),\ P(t, Y(a)(t))(\eta\xi)\right) \\ & = d(\langle \eta, U_0(a_j)(t)\xi \rangle, P(t, Y(a)(t))(\eta, \xi)) \\ & \leq \rho(P(t, Y(a_j)(t))(\eta, \xi), P(t, Y(a)(t))(\eta, \xi)) \\ & \leq K^P_{\eta\xi}(t) \|Y(a_j)(t) - Y(a)(t)\|_{\eta\xi} \\ & = K^P_{\eta\xi}(t)\|a_j - a\|_{\eta\xi} = K^P_{\eta\xi}(t)|a_{\eta\xi, j} - a_{\eta\xi}| \\ & \leq 2^{-3}\theta K^P_{\eta\xi} \end{aligned} \tag{3.7}$$

We remark here that inequality (3.7) holds since $a_{\eta\xi} \in B(a_{\eta\xi, j}, \delta(a_{\eta\xi, j}))$ for some $0 \leq j \leq m$ and $|a_{\eta\xi} - a_{\eta\xi, j}| < \delta(a_{\eta\xi, j}) \leq 2^{-3}\theta$.

The estimate holds on the whole interval $[0, T]$ since it is independent of j. Similarly, we have for $t \in I_j(a_{\eta\xi})$,

$$\begin{aligned} & d\left(\frac{d}{dt} W_1(a_{\eta\xi})(t),\ P(t, Y_1(a)(t))(\eta, \xi)\right) \\ & \leq d\left(\frac{d}{dt} W_1(a_{\eta\xi})(t), P(t, Y(a)(t))(\eta, \xi)\right) \\ & + \rho\,(P(t, Y(a)(t))(\eta, \xi),\ P(t, Y_1(a)(t))(\eta, \xi)) \\ & \leq K^P_{\eta\xi}(t)2^{-3}\theta + K^P_{\eta\xi}(t)\|Y(a)(t) - Y_1(a)(t)\|_{\eta\xi} \\ & \leq K^P_{\eta\xi}(t)[2^{-3}\theta + D_{\eta\xi} M_{\eta\xi}(t)] \end{aligned} \tag{3.8}$$

Inequality (3.8) follows from the following estimates:

$$\begin{aligned} & \|Y(a)(t) - Y_1(a)(t)\|_{\eta\xi} = |W(a_{\eta\xi})(t) - W_1(a_{\eta\xi})(t)| \\ & \leq \int_0^t \left| \frac{d}{ds} W(a_{\eta\xi})(s) - \langle \eta, U_0(a_j)(s)\xi \rangle \right| ds \\ & \leq \int_0^t K^P_{\eta\xi}(s)\|a_0 - a_j\|_{\eta\xi} \leq D_{\eta\xi} M_{\eta\xi}(t) \end{aligned}$$

by (3.2)

In general, we claim that for $n = 1, 2 \cdots$, we can define sequences of maps: $Y_n : A \to \text{wac}(\tilde{\mathcal{A}})$ and $W_n : A(\eta, \xi) \to \text{wac}(\tilde{\mathcal{A}})(\eta, \xi)$ such that W_n is continuous on $A(\eta,\xi)$ and for each $a \in A$, $a_{\eta\xi} \in A(\eta,\xi)$, $Y_n(a)$ lies in $L^1_{\mathrm{loc}}(\tilde{\mathcal{A}})$, $Y_n(a_0) = \Phi_0$ and $W_n(a_{\eta\xi,0}) = \Phi_{\eta\xi,0}$.

Moreover,

(i)
$$\int_0^t \left| \frac{d}{ds} W_n(a_{\eta\xi})(s) - \frac{d}{ds} W_{n-1}(a_{\eta\xi})(s) \right| ds \le D_{\eta\xi} \frac{M^n_{\eta\xi}(t)}{n!} + \theta 2^{-n-1} \left[2^{-2} + \sum_{i=1}^{n} \frac{(2M_{\eta\xi}(t))^i}{i!} \right]$$

(ii)
$$d\left(\frac{d}{dt} W_n(a_{\eta\xi})(t),\, P(t, Y_{n-1}(a)(t))(\eta,\xi) \right) \le \theta 2^{-n-2} K^P_{\eta\xi}(t)$$

(iii)
$$d\left(\frac{d}{dt} W_n(a_{\eta\xi})(t),\;\; P(t, Y_n(a)(t))(\eta,\xi) \right) \le D_{\eta\xi} K^P_{\eta\xi}(t) \frac{M^n_{\eta\xi}(t)}{n!} + \theta 2^{-n-1} K^P_{\eta\xi}(t) \sum_{i=0}^{n} \frac{(2M_{\eta\xi}(t))^i}{i!}.$$

(iv) There exists maps $R_{\eta\xi,n} : [0,T] \to \mathbb{R}_+$ lying in $L^1_{\mathrm{loc}}([0,T])$ such that for every $\epsilon > 0$, there exists $\delta > 0$, such that $|a'_{\eta\xi} - a_{\eta\xi}| < \delta$ implies that
$$\left| \frac{d}{dt} W_n(a_{\eta\xi})(t) - \frac{d}{dt} W_n(a'_{\eta\xi})(t) \right| \le R_{\eta\xi,n}(t) \chi_E(t),$$
for some subset E of $[0,T]$ with measure $\mu(E) \le \epsilon$.

Our claim (i)–(iv) above hold for the case $n = 1$ from the definition of the maps Y_1 and W_1 and by applying Proposition (2.1). Assume that the claim holds for $n-1$, we show that it holds for n as follows:

Choose $U_{n-1}(a)(t)(\eta,\xi) \in P(t, Y_{n-1}(a)(t))(\eta,\xi)$ such that
$$\begin{aligned} &\left| \frac{d}{dt} W_{n-1}(a_{\eta\xi})(t) - U_{n-1}(a)(t)(\eta,\xi) \right| \\ &\quad = d\left[\frac{d}{dt} W_{n-1}(a_{\eta\xi})(t),\, P(t, Y_{n-1}(a)(t))(\eta,\xi) \right] \\ &\quad \le D_{\eta\xi} K^P_{\eta\xi}(t) \frac{M^{n-1}_{\eta\xi}(t)}{(n-1)!} + \theta 2^{-n} K^P_{\eta\xi}(t) \sum_{i=0}^{n-1} \frac{(2M_{\eta\xi}(t))^i}{i!}. \end{aligned} \tag{3.9}$$

As $(\eta,\xi) \to U_{n-1}(a)(t)(\eta,\xi)$ is a sesquilinear form and $Y_{n-1}(a)$ is adapted, then there exists an adapted process $U_{n-1}(a) : [0,T] \to \tilde{\mathcal{A}}$ such that
$$U_{n-1}(a)(t)(\eta,\xi) = \langle \eta, (U_{n-1}(a)(t))\xi \rangle.$$

By inequality (3.9) and the assumption that $Y_{n-1}(a)$ lies in $L^1_{\text{loc}}(\tilde{\mathcal{A}})$, then the process $U_{n-1}(a) \in L^1_{\text{loc}}(\tilde{\mathcal{A}})$ for each $a \in A$.

By (iv) of the recursive hypothesis, there exists $\delta_n > 0$ such that $|a'_{\eta\xi} - a_{\eta\xi}| < \delta_n$ implies

$$\left|\frac{d}{dt}W_{n-1}(a'_{\eta\xi})(t) - \frac{d}{dt}W_{n-1}(a_{\eta\xi})(t)\right| \leq R_{\eta\xi,n-1}(t)\chi_E(t) \tag{3.10}$$

for some $E \subseteq [0, T]$ satisfying

$$\int_E R_{\eta\xi,n-1}(t)dt \leq \theta 2^{-n-3}. \tag{3.11}$$

Next we define for any $a \in A$, $a_{\eta\xi} \in A(\eta, \xi)$

$$\delta_n(a_{\eta\xi}) = \min\left\{\delta_n, \theta 2^{-n-3}, \frac{|a_{\eta\xi} - a_{\eta\xi,0}|}{2}\right\}, a_{\eta\xi} \neq a_{\eta\xi,0}$$

and

$$\delta_n(a_{\eta\xi,0}) = \min\{\delta_n, \theta 2^{-n-3}\}.$$

We cover the set $A(\eta, \xi)$ with the balls $B(a_{\eta\xi}, \delta_n(a_{\eta\xi}))$ and let $B(a^n_{\eta\xi,j}, \delta_n(a^n_{\eta\xi,j}))$, $j = 0, 1, 2 \cdots m_n$, be a finite subcover, where we have put $a^n_{\eta\xi,0} = a_{\eta\xi,0}$. We remark that by the inequality

$$\left|a_{\eta\xi,0} - a^n_{\eta\xi,j}\right| > \frac{1}{2}\left|a_{\eta\xi,0} - a^n_{\eta\xi,j}\right|$$

the point $a_{\eta\xi,0}$ belongs only to the ball $B(a_{\eta\xi,0}, \delta_n(a_{\eta\xi,0}))$. Let $\{P^n_j(\cdot)\}^{j=m_n}_{j=0}$ be a continuous partition of unity subordinate to this covering and define the following intervals:

$$I^n_0(a_{\eta\xi}) = \left[0, T P^n_0(a_{\eta\xi})\right]$$

and for $j > 0$,

$$I^n_j(a_{\eta\xi}) = \left[T\left(P^n_0(a_{\eta\xi}) + \cdots + P^n_{j-1}(a_{\eta\xi})\right), \qquad T(P_0(a_{\eta\xi}) + \cdots + P^n_j\left(a_{\eta\xi}\right))\right].$$

Define the maps Y_n, W_n as follows:

$$Y_n(a)(t) = a + \int_0^t \sum_{j=0}^{m_n} \chi_{I^n_j(a_{\eta\xi})}(s)U_{n-1}\left(a^n_j\right)(s)ds \tag{3.12}$$

$$W_n(a_{\eta\xi})(t) = a_{\eta\xi} + \int_0^t \sum_{j=0}^{m_n} \chi_{I^n_j(a_{\eta\xi})}(s)\langle\eta, U_{n-1}\left(a^n_j\right)(s)\xi\rangle ds \tag{3.13}$$

Since $Y_{n-1}(a_0) = \Phi_0$ and by the properties of the process $U_{n-1}(a)$ and the fact that $I^n_0(a_{\eta\xi,o}) = [0, T]$, we have

$$Y_n(a_0) = \Phi_0$$

and by Proposition (2.1), the map $W_n : A(\eta, \xi) \to \mathrm{wac}(\tilde{\mathcal{A}})(\eta, \xi)$ given by (3.12) is continuous and $W_n(a_{\eta\xi,0}) = \Phi_{\eta\xi,0}$. Furthermore, we have the following estimates:

$$
\begin{aligned}
&\int_0^t \left| \frac{d}{ds} W_n(a_{\eta\xi})(s) - \frac{d}{ds} W_{n-1}(a_{\eta\xi})(s) \right| ds \\
&\quad \le \int_0^t \sum_j \chi_{I_j^n(a_{\eta\xi})}(s) \left| \langle \eta, U_{n-1}\left(a_j^n\right)(s)\xi \rangle - \frac{d}{ds} W_{n-1}(a_{\eta\xi})(s) \right| ds \\
&\quad \le \int_0^t \sum_j \chi_{I_j^n(a_{\eta\xi})}(s) \left| \langle \eta, U_{n-1}\left(a_j^n\right)(s)\xi \rangle - \frac{d}{ds} W_{n-1}\left(a_{\eta\xi,j}^n\right)(s) \right| ds \\
&\quad + \int_0^t \sum_j \chi_{I_j^n(a_{\eta\xi})}(s) \left| \frac{d}{ds} W_{n-1}\left(a_{\eta\xi,j}^n\right)(s) - \frac{d}{ds} W_{n-1}(a_{\eta\xi})(s) \right| ds \\
&\quad \le \int_0^t \left(\sum_j \chi_{I_j^n(a_{\eta\xi})}(s) \right) \left[D_{\eta\xi} K_{\eta\xi}^P(t) \frac{M_{\eta\xi}^{n-1}(t)}{n!} + \theta 2^{-n} K_{\eta\xi}^P(t) \sum_{i=0}^{n-1} \frac{(2M_{\eta\xi}(t))^i}{i!} \right] \\
&\quad + \int_0^t \left(\sum_j \chi_{I_j^n(a_{\eta\xi}}(s) \right) R_{\eta\xi,n-1}(s)\chi_E(s) ds \\
&\quad \le D_{\eta\xi} \frac{M_{\eta\xi}^n(t)}{n!} + \theta 2^{-n-1} \sum_{i=1}^{n} \frac{(2M_{\eta\xi}(t)^i}{i!} + \theta 2^{-n-3}.
\end{aligned}
$$

Hence, item (i) of the recursive hypothesis holds for all n. Next we fix $t \in [0, T]$ and let j be such that $t \in I_j(a_{\eta\xi})$. Then by (3.11) and (3.12),

$$
\begin{aligned}
&d\left(\frac{d}{dt} W_n(a_{\eta\xi})(t),\ P(t, Y_{n-1}(a)(t))(\eta, \xi) \right) \\
&\quad = d\left(\langle \eta, U_{n-1}(a_j^n)(t)\xi \rangle,\ P(t, Y_{n-1}(a)(t))(\eta, \xi) \right) \\
&\quad \le \rho\left(P(t, Y_{n-1}(a_j^n)(t))(\eta, \xi), P(t, Y_{n-1}(a)(t))(\eta, \xi) \right) \\
&\quad \le K_{\eta\xi}^P(t) \| Y_{n-1}(a_j^n)(t) - Y_{n-1}(a)(t) \|_{\eta\xi}.
\end{aligned}
\tag{3.14}
$$

By applying (3.10), we have

$$
\begin{aligned}
&\| Y_{n-1}(a_j^n)(t) - Y_{n-1}(a)(t) \|_{\eta\xi} \\
&\quad \le |a_{\eta\xi,j}^n - a_{\eta\xi} + \int_0^t | \frac{d}{ds} W_{n-1}\left(a_{\eta\xi,j}^n\right)(s) - \frac{d}{ds} W_{n-1}(a_{\eta\xi})(s)| ds \\
&\quad \le \theta 2^{-n-3} + \theta 2^{-n-3} = \theta 2^{-n-2}
\end{aligned}
\tag{3.15}
$$

Combining inequalities (3.13) and (3.14), we have

$$d\left(\frac{d}{dt}W_n(a_{\eta\xi})(t),\;\; P(t, Y_{n-1}(a)(t))(\eta,\xi)\right) \leq \theta 2^{-n-2} K^P_{\eta\xi}(t).$$

The estimate is independent of j and so it holds on the whole interval $[0, T]$. This proves item (ii) of our claim. To establish item (iii) we proceed as follows:

$$\begin{aligned}
& d\left(\frac{d}{dt}W_n(a_{\eta\xi})(t), P(t, Y_n(a)(t))(\eta,\xi)\right) \\
& \quad \leq d\left(\frac{d}{dt}W_n(a_{\eta\xi})(t), P(t, Y_{n-1}(a)(t))(\eta,\xi)\right) \\
& \quad +\rho\left(P(t, Y_{n-1}(a)(t))(\eta,\xi), P(t, Y_n(a)(t))(\eta,\xi)\right) \\
& \quad \leq \theta 2^{-n-2} K^P_{\eta\xi}(t) + K^P_{\eta\xi}(t)\|Y_{n-1}(a)(t) - Y_n(a)(t)\|_{\eta\xi} \\
& \quad \leq \theta 2^{-n-2} K^P_{\eta\xi}(t) + K^P_{\eta\xi}(t)\int_0^t \left|\frac{d}{ds}W_{n-1}(a_{\eta\xi})(s) - \frac{d}{ds}W_n(a_{\eta\xi})(s)\right| ds \\
& \quad \leq K^P_{\eta\xi}(t)\left[\theta 2^{-n-2} + D_{\eta\xi}\frac{M^n_{\eta\xi}(t)}{n!} + \theta 2^{-n-1}\sum_{i=1}^{n}\frac{(2M_{\eta\xi}(t))^i}{i!} + \theta 2^{-n-3}\right], \\
& \qquad \text{by (i) and (ii)} \\
& \quad \leq D_{\eta\xi}K^P_{\eta\xi}(t)\frac{M^n_{\eta\xi}(t)}{n!} + \theta 2^{-n-1}K^P_{\eta\xi}(t)\sum_{i=0}^{n}\frac{(2M_{\eta\xi}(t))^i}{i!},
\end{aligned}$$

(obtained by applying the inequality; $\theta 2^{-n-2} + \theta 2^{-n-3} \leq \theta 2^{-n-1}$).

The last estimate proves item (iii) of our claim. Item (iv) of the claim is established by the application of Proposition (2.1) to the map $W_n : A(\eta,\xi) \to \mathrm{wac}(\tilde{\mathcal{A}})(\eta,\xi)$.

From item (i), we have

$$\begin{aligned}
& |Y_n(a) - Y_{n-1}(a)|_{\eta\xi} \\
& \quad = \|Y_n(a)(0) - Y_{n-1}(a)(0)\|_{\eta\xi} + \int_0^T \left|\frac{d}{dt}\langle\eta, Y_n(a)(t)\xi\rangle\right. \\
& \qquad \left. - \frac{d}{dt}\langle\eta, Y_{n-1}(a)(t)\xi\rangle\right| dt \\
& \quad = |W_n(a_{\eta\xi})(0) - W_{n-1}(a_{\eta\xi})(0)| + \int_0^T \left|\frac{d}{ds}W_n(a_{\eta\xi})(s) - \frac{d}{ds}W_{n-1}(a_{\eta\xi})(s)\right| ds \\
& \quad \leq D_{\eta\xi}\frac{M^n_{\eta\xi}(T)}{n!} + \theta 2^{-n-1}e^{2M_{\eta\xi}(T)} \qquad (3.16)
\end{aligned}$$

It follows by (3.15) that the sequence $\{Y_n(a)\}$ is uniformly Cauchy in wac$(\tilde{\mathcal{A}})$ and thus converges uniformly to a map $\Phi : A \to$ wac$(\tilde{\mathcal{A}})$. Again,

$$\lim_{n\to\infty} W_n(a_{\eta\xi})(t) = \lim_{n\to\infty} \langle \eta, Y_n(a)(t)\xi\rangle = \langle \eta, \Phi(a)(t)\xi\rangle$$

The map $a_{\eta\xi} \to \langle \eta, \Phi(a)(t)\xi\rangle$ is continuous since the map $a_{\eta\xi} \to W_n(a_{\eta\xi})$ is continuous for each n.

As the stochastic process $\Phi(a)$ is the limit of a sequence of adapted weakly absolutely continuous processes in $L^1_{\text{loc}}(\tilde{\mathcal{A}})$, $\Phi(a)$ lies in Ad$(\tilde{\mathcal{A}})_{\text{wac}} \bigcap L^1_{loc}(\tilde{\mathcal{A}})$ and $\Phi(a_0) = \Phi_0$.

By (iii) of the recursive formula,

$$d\left(\frac{d}{dt}\langle \eta, \Phi(a)(t)\xi\rangle, P(t, \Phi(a)(t))(\eta, \xi)\right) = 0.$$

Therefore

$$\Phi(a) \in S^{(T)}(a), \langle \eta, \Phi(a)(\cdot)\xi\rangle \in S^{(T)}(a)(\eta, \xi), \langle \eta, \Phi(a_0)(\cdot)\xi\rangle = \Phi_{\eta\xi,o}. \qquad \square$$

The following corollaries show that the set-map $S^{(T)}(a)(\eta, \xi)$ and the set of complex numbers $R^{(T)}(a)(\eta, \xi)$ associated with the reachable set of QSDI (1.1) at the final time T admit some continuous parameterizations.

Corollary 3.2. *There exists a subspace $\mathcal{U}$ of the space of all continuous maps from $A(\eta, \xi)$ into* wac$(\tilde{\mathcal{A}})(\eta, \xi)$ *and a continuous function*

$$g : A(\eta, \xi) \times \mathcal{U} \to wac(\tilde{\mathcal{A}})(\eta, \xi)$$

such that for any $a \in A,\ a_{\eta\xi} \in A(\eta, \xi)$,

$$g(a_{\eta\xi}, \mathcal{U}) = S^{(T)}(a)(\eta, \xi).$$

Proof: We remark first that the space wac$(\tilde{\mathcal{A}})(\eta, \xi)$ is a subspace of the space $AC[0, T]$ of all absolutely continuous complex valued functions on $[0, T]$, a separable Banach space with the usual sup norm.

We put X to be the set of continuous maps from the compact set $A(\eta, \xi)$ into wac$(\tilde{\mathcal{A}})(\eta, \xi)$ and define the subspace $\mathcal{U}$ of X by

$$\mathcal{U} = \{W : A(\eta, \xi) \to wac(\tilde{\mathcal{A}})(\eta, \xi)/W \quad \text{is continuous and} \quad W(a_{\eta\xi}) \in S^{(T)}(a)(\eta, \xi)\}$$

the set of all continuous selections from the map $a_{\eta\xi} \to S^{(T)}(a)(\eta, \xi)$.

Define the map g by

$$g(a_{\eta\xi}, W) = W(a_{\eta\xi}).$$

Then by the continuity of each $W \in \mathcal{U}$, g is continuous and by Theorem 3.1 above,

$$g(a_{\eta\xi}, \mathcal{U}) = S^{(T)}(a)(\eta, \xi).$$

□

Corollary 3.3. *There exists a subset $\mathcal{U}$ of the space of all continuous maps from $A(\eta, \xi)$ into $\mathrm{wac}(\tilde{\mathcal{A}})(\eta, \xi)$ and a continuous function*

$$h : A(\eta, \xi) \times \mathcal{U} \to \mathbb{C}$$

such that for any $a \in A$, $a_{\eta\xi} \in A(\eta, \xi)$

$$h(a_{\eta\xi}, \mathcal{U}) = R^{(T)}(a)(\eta, \xi).$$

Proof: Adopting the notation employed in the proof of Corollary 3.2, we define the map h by

$$h(a_{\eta\xi}, W) = W(a_{\eta\xi})(T).$$

Since

$$W(a_{\eta\xi}) \in S^{(T)}(a)(\eta, \xi),$$

then $W(a_{\eta\xi})(\cdot)$ is of the form

$$W(a_{\eta\xi})(\cdot) = \langle \eta, \tilde{W}(\cdot)\xi \rangle$$

for some $\tilde{W} \in S^{(T)}(a)$. Hence, we have

$$W(a_{\eta\xi})(T) = \langle \eta, \tilde{W}(T)\xi \rangle \in R^{(T)}(a)(\eta, \xi).$$

By Theorem 3.1, it follows that

$$h(a_{\eta\xi}, \mathcal{U}) = R^{(T)}(a)(\eta, \xi).$$

□

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REFERENCES

Antosiewicz, H. A. and Cellina, A. (1975). Continuous selections and differential relations. *Journal of Differential Equations* **19**, 386–398.

Aubin, J. P. and Cellina, A. (1984). *Differential Inclusions*, Springer-Verlag, Berlin.

Ayoola, E. O. (2001). Construction of approximate attainability sets for Lipschitzian quantum stochastic differential inclusions. *Stochastic Analysis and Applications* **19**(3), 461–471.

Ayoola, E. O. (2003a). Error estimates for discretized quantum stochastic differential inclusions. *Stochastic Analysis and Applications* **21**(6) 1215–1230.

Ayoola, E. O. (2003b). Exponential formula for the reachable sets of quantum stochastic differential inclusions. *Stochastic Analysis and Applications* **21**(3), 515–543.

Broucke, M. and Arapostathis, A. (2001). Continuous interpolation of solutions of Lipschitz inclusions. *Journal of Mathematical Analysis and Applications* **258**, 565–572.

Cellina, A. (1988). On the set of solutions to Lipschitzian differential inclusions. *Differential and Integral Equations* **1**(4), 495–500.

Cellina, A. and Ornelas, A. (1992). Representation of the attainable set for Lipschitzian differential inclusions. *Rocky Mountain Journal of Mathematics* **22**(1), 117–124.

Colombo, R. M., Fryszkowski, A., Rzezuchowski, T., and Staicu, V. (1991). Continuous selections of solution sets of Lipschitzian differential inclusions. *Funkcialaj Ekvacioj* **34**, 321–330.

Ekhaguere, G. O. S. (1992). Lipschitzian quantum stochastic differential inclusions. *International Journal of Theoretical Physics* **31**(11), 2003–2034.

Ekhaguere, G. O. S. (1995). Quantum stochastic differential inclusions of hypermaximal monotone type. *International Journal of Theoretical Physics* **34**(3), 323–353.

Ekhaguere, G. O. S. (1996). Quantum stochastic evolutions. *International Journal of Theoretical Physics* **35**(9) 1909–1946.

Fryszkowski, A. (1983). Continuous selections for a class of non-convex multivalued maps. *Studia Mathematica, T. LXXVI*, 163–174.

Halmos, P. R. (1988). *Measure Theory, Graduate Texts in Mathematics*, Springer-Verlag, Berlin.

Hudson, R. L. and Parthasarathy, K. R. (1984). Quantum Ito's formulae and stochastic evolutions. *Communications in Mathematical Physics* **93**, 301–324.

Repovs, D. and Semenov, P. V. (1998). *Continuous Selections of Multivalued Mappings, Mathematics and its Applications*, Kluwer, Dordrecht.

Smirnov, G. V. (2002). *Introduction to the Theory of Differential Inclusions, Graduate Studies in Mathematics*, American Mathematical Society, Providence, RI, Vol. 41.

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QUANTUM STOCHASTIC DIFFERENTIAL INCLUSIONS SATISFYING A GENERAL LIPSCHITZ CONDITION

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ABSTRACT. We establish further results concerning the existence and non-uniqueness of solutions of quantum stochastic differential inclusions in the framework of Hudson and Parthasarathy formulation of quantum stochastic calculus. Our results are established by considering a general Lipschitz condition on the coefficients of the inclusion. We present examples of continuous multivalued maps satisfying the general Lipschitz condition in the sense of this paper.

Key Words: Fock space, Exponential vectors, Lipschitzian quantum stochastic differential inclusions

AMS (MOS) Subject Classification. 81S25, 60H10

1. INTRODUCTION

Some very important preoccupations of classical analysis are the numerical and analytical characterizations of solutions of classical differential inclusions defined in finite dimensional Euclidean spaces. Indeed the existence and non uniqueness of solutions of such inclusions have been thoroughly investigated (see, for example, [1, 11, 13, 16]). Indeed, many features of reachable sets, the solution sets and their selection theorems have been studied to a great extent [6, 7, 11, 13, 15, 16].

However, in the non commutative quantum setting, the situation is different. The analysis of quantum stochastic differential inclusions (QSDI) concerns quantum stochastic processes as solutions that live in certain infinite dimensional locally convex spaces. In addition, there are several locally convex operator topologies that may be defined on the space of such processes arising from several theories of noncommutative stochastic integration. There are several variants of topological conditions depending on the underlying properties of the locally convex spaces of the integrands that may be required of the coefficients of the quantum stochastic differential inclusions. The objective of this paper is to further investigate the existence and non-uniqueness of

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solutions of quantum stochastic differential inclusions of the form:

$$X(t) \in X_0 + \int_0^t (E(s,X(s))d\wedge_\pi(s) + F(s,X(s))dA_f(s) + G(s,X(s))dA_g^+(s) + H(s,X(s))ds), \quad t \in [0,T], \tag{1.1}$$

under a Lipschitz condition that generalizes similar condition employed in [8]. In the framework of the Hudson and Parthasarathy [12, 14] formulation of quantum stochastic calculus, we consider a more general class of Lipschitzian coefficients E, F, G, H appearing in (1.1). The Lipschiz condition in [8] is a special case of the present formulation. The integral in (1.1) is understood in the sense of Hudson and Parthasarathy [12] and the maps f, g, π belong to appropriate function spaces as described in [8]. The integrator processes $\wedge_\pi$, A_f^+, A_g are the gauge, creation and annihilation processes associated with the basic field operators of quantum field theory. In [8], under the Lipschitz condition of that paper, the existence of solutions and the equivalent form of QSDI (1.1) have been established. We establish a wider class of Lipschitzian QSDI (1.1) to cover some important multivalued maps that are Lipschitzian in the general sense of this paper. This class of maps was not covered by the notion of Lipschitz maps due to [8]. In particular, we present a class of Lipschitzian multivalued maps associated with the space of continuous endomorphisms of the locally convex space of our quantum stochastic processes as an important example of multivalued maps satisfying the Lipschitz condition in our sense. This work therefore extends the class of QSDI investigated in [2, 3, 4, 5, 8]. We remark that a very strong motivation for studying QSDI (1.1) among others, concerns the need for sufficient information and knowledge about the dynamics and fluctuations of the systems described by discontinuous quantum stochastic differential equations which may be reformulated as regularized QSDI. QSDI of the form (1.1) plays a central role in quantum stochastic control theory and quantum dynamical systems (see [3, 4, 8]).

The rest of the paper is organized as follows: We present in Section 2, the description of some very important relevant spaces, some fundamental assumptions and some results. Our main results concerning the existence, and non-uniqueness of solutions of QSDI (1.1) are established in Section 3.

2. PRELIMINARY RESULTS AND ASSUMPTIONS

Our framework in this paper relies largely on the formulation in [8, 9, 10]. Detailed definitions of various spaces that appear below can be found in [8]. In what follows, γ is a fixed Hilbert space, $\mathbb{D}$ is an inner product space with $\mathcal{R}$ as its completion, and $\Gamma(L^2_\gamma(\mathbb{R}_+))$ is the Boson Fock Space determined by the function space $L^2_\gamma(\mathbb{R}_+)$. The set $\mathbb{E}$ is the subset of the Fock space generated by the exponential vectors. If $\mathcal{N}$ is a topological space, then we denote by $clos(\mathcal{N})$ (resp. $comp(\mathcal{N})$), the family of all nonempty closed subsets of $\mathcal{N}$ (resp. compact members of $clos(\mathcal{N})$).

In our formulations, quantum stochastic processes are $\tilde{\mathcal{A}}$-valued maps on $[0, T]$. The space $\tilde{\mathcal{A}}$ is the completion of the linear space

$$\mathcal{A} = L^+_W(\mathbb{D}\underline{\otimes}\mathbb{E}, \mathcal{R} \otimes \Gamma(L^2_\gamma(\mathbb{R}_+)))$$

endowed with the locally convex operator topology generated by the family of seminorms $\{x \to \|x\|_{\eta\xi} = |\langle \eta, x\xi\rangle|,\ \eta, \xi \in \mathbb{D} \underline{\otimes} \mathbb{E}\}$. Here, $\mathcal{A}$ consists of linear operators from $\mathbb{D} \underline{\otimes} \mathbb{E}$ into $\mathcal{R} \otimes \Gamma(L^2_\gamma(\mathbb{R}_+))$ with the property that the domain of the operator adjoint contains $\mathbb{D} \underline{\otimes} \mathbb{E}$. We adopt the notation and the definitions of Hausdorff topology on $clos(\tilde{\mathcal{A}})$ as explained in [8, 9, 10].

For any pair of $\eta, \xi \in \mathbb{D} \underline{\otimes} \mathbb{E}$ such that $\eta = c \otimes e(\alpha)$,$\xi = d \otimes e(\beta)$, $\alpha, \beta \in L^2_\gamma(\mathbb{R}_+)$, we shall in what follows, employ the equivalent form of (1.1) established in [8] and given by the nonclassical ordinary differential inclusion:

$$\begin{aligned} &\frac{d}{dt}\langle \eta, X(t)\xi\rangle \in P(t, X(t))(\eta, \xi), \\ &X(0) = X_0, \ \ t \in [0, T]. \end{aligned} \tag{2.1}$$

The multivalued map P appearing in (2.1) is of the form

$$P(t, x)(\eta, \xi) = \langle \eta, P_{\alpha\beta}(t, x)\xi\rangle$$

where the map $P_{\alpha\beta} : [0, T] \times \tilde{\mathcal{A}} \to 2^{\tilde{\mathcal{A}}}$ is given by

$$P_{\alpha\beta}(t, x) = \mu_{\alpha\beta}(t)E(t, x) + \nu_\beta(t)F(t, x) + \sigma_\alpha(t)G(t, x) + H(t, x).$$

The complex valued functions $\mu_{\alpha\beta}, \nu_\beta, \sigma_\alpha : [0, T] \to \mathbb{C}$ are defined by

$$\mu_{\alpha\beta}(t) = \langle \alpha(t), \pi(t)\beta(t)\rangle_\gamma,\ \nu_\beta(t) = \langle f(t), \beta(t)\rangle_\gamma,$$
$$\sigma_\alpha(t) = \langle \alpha(t), g(t)\rangle_\gamma,\ t \in [0, T]$$

for all $(t, x) \in [0, T]\times\tilde{\mathcal{A}}$ and the coefficients E, F, G, H belong to the space $L^2_{loc}([0, T]\times \tilde{\mathcal{A}})_{mvs}$ of multivalued stochastic processes with closed values.

As explained in [8], the map P cannot in general be written in the form:

$$P(t, x)(\eta, \xi) = \tilde{P}(t, < \eta, x\xi >)$$

for some complex valued multifunction defined on $[0, T] \times \mathbb{C}$, for $t \in [0, T]$, $x \in \tilde{\mathcal{A}}$, $\eta, \xi \in \mathbb{D}\underline{\otimes}\mathbb{E}$.

Definition 2.1. (a) Let Fin$[A]$ denote the family of all finite subsets of a nonempty set A. For $x \in \mathcal{A}$, and $\Theta \in \text{Fin}[(\mathbb{D}\underline{\otimes}\mathbb{E})^2]$, define $\|x\|_\Theta$ by

$$\|x\|_\Theta = \max_{(\eta,\xi)\in\Theta} \|x\|_{\eta\xi}. \tag{2.2}$$

Then, the set $\{\|\cdot\|_\Theta : \Theta \in \text{Fin}[(\mathbb{D}\underline{\otimes}\mathbb{E})^2]\}$ is a family of seminorms on $\mathcal{A}$. We denote by τ the topology generated by this family of seminorms and we let $\tilde{\mathcal{A}}'$ represents the completion of the topological space $(\mathcal{A}, \tau)$.

(b) Let $I = [0, T] \subseteq \mathbb{R}_+$. A multivalued map $\Phi : I \times \tilde{\mathcal{A}} \to clos(\tilde{\mathcal{A}})$ will be called Lipschitzian if for any pair $(\eta, \xi) \in (\mathbb{D}\underline{\otimes}\mathbb{E})^2$, the map satisfies an estimate of the type

$$\rho_{\eta\xi}\left(\Phi(t,x), \Phi(t,y)\right) \leq K^{\Phi}_{\eta\xi}(t)\|x-y\|_{\Theta_\Phi(\eta,\xi)} \tag{2.3}$$

for all $x, y \in \tilde{\mathcal{A}}$ and almost all $t \in I$ and where $K^{\Phi}_{\eta\xi} : I \to (0, \infty)$ lies in $L^1_{loc}(I)$ and Θ_Φ is a map from $(\mathbb{D}\underline{\otimes}\mathbb{E})^2$ into $\mathrm{Fin}[(\mathbb{D}\underline{\otimes}\mathbb{E})^2]$. Similar definition holds for a map of the form $\Phi : I \times \tilde{\mathcal{A}} \to clos(\mathbb{C})$ where the Hausdorff metric $\rho(\cdot,\cdot)$ on $clos(\mathbb{C})$ replaces the pseudo metric $\rho_{\eta\xi}(\cdot,\cdot)$ on $clos(\tilde{\mathcal{A}})$ (see [8]).

Remark. In [8], the map $(\eta,\xi) \to \Theta_\Phi(\eta,\xi)$ that appears in (2.3) is just the identity map. Let $L(\tilde{\mathcal{A}})$ denote the linear space of all continuous endomorphisms of $\tilde{\mathcal{A}}$. Then the above definition enables us to exhibit a class of Lipschitzian multivalued maps which are continuous from the space $\mathbb{R}_+ \times \tilde{\mathcal{A}}$ to the Hausdorff topological space $\left(clos(\tilde{\mathcal{A}}), \tau_H\right)$. The multivalued maps in this class are not Lipschitzian in the sense of [8].

Theorem 2.2. Let $A : \mathbb{R}_+ \to L(\tilde{\mathcal{A}})$ be a single valued map on $\mathbb{R}_+$. For arbitrary $\eta, \xi \in \mathbb{D}\underline{\otimes}\mathbb{E}$ and a fixed closed ball $S \in comp(\tilde{\mathcal{A}})$ with centre at the origin, define for any $x \in \tilde{\mathcal{A}}$,

$$F(t,x) = \|A(t)x\|_{\eta\xi} S.$$

Then the map $(t,x) \to F(t,x)$ is Lipschitzian.

Proof: For $x,\ y \in \tilde{\mathcal{A}},\ t \in \mathbb{R}_+$, we employ some basic results similar to Lemma (II.1.5) and Corollary (II.1.2) in [13] as follows:

$$\begin{aligned}
\rho_{\eta\xi}\left(F(t,x), F(t,y)\right) &= \rho_{\eta\xi}\left(\|A(t)x\|_{\eta\xi}S, \|A(t)y\|_{\eta\xi}S\right) \\
&\leq |\|A(t)x\|_{\eta\xi} - \|A(t)y\|_{\eta\xi}|\rho_{\eta\xi}\left(S, \{0\}\right) \\
&\leq \|A(t)x - A(t)y\|_{\eta\xi}\rho_{\eta\xi}\left(S,\ \{0\}\right) \\
&= \|A(t)(x-y)\|_{\eta\xi}\rho_{\eta\xi}\left(S, \{0\}\right) \\
&\leq \|S\|_{\eta\xi} C^A_{\eta\xi}(t)\|x-y\|_{\Theta_A(\eta,\xi)} \\
&= K^F_{\eta\xi}(t)\|x-y\|_{\Theta_A(\eta,\xi)},
\end{aligned}$$

where $\|S\|_{\eta\xi} = \rho_{\eta\xi}\left(S, \{0\}\right)$, $K^F_{\eta\xi}(t) = \|S\|_{\eta\xi}C^A_{\eta\xi}(t)$, Θ_A is a map from $(\mathbb{D}\underline{\otimes}\mathbb{E})^2$ into $\mathrm{Fin}(\mathbb{D}\underline{\otimes}\mathbb{E})^2$ and $C^A_{\eta\xi}(t)$ is a positive function depending on the map $A(t)$ and elements $\eta, \xi \in \mathbb{D}\underline{\otimes}\mathbb{E}$.

The continuity of the multivalued map $(t,x) \to F(t,x)$ follows from the last inequality.

Remark. (a) Since Θ is a finite set, we see that $\|x\|_\Theta = \|x\|_{\eta'\xi'}$, for some $(\eta', \xi') \in \Theta$. Thus, in what follows, we employ in the proof of our main results the fact that a map $\Phi\ :\ I \times \tilde{\mathcal{A}} \to clos(\tilde{\mathcal{A}})$ is Lipschitzian if given any $(\eta, \xi) \in (\mathbb{D}\underline{\otimes}\mathbb{E})^2$, there corresponds

$(\eta', \xi') \in (\mathbb{D}\underline{\otimes}\mathbb{E})^2$ such that

$$\rho_{\eta\xi}(\Phi(t,x), \Phi(t,y)) \leq K^{\Phi}_{\eta\xi}(t)\|x-y\|_{\eta'\xi'} \tag{2.4}$$

for all $x, y \in \tilde{\mathcal{A}}$ and $t \in I$.

(b) By the definition of the map $(t,x) \to P(t,x)(\eta,\xi)$ that appears in (2.1), and by the remark above, it is straightforward to show that if the coefficients of (1.1) are Lipschitzian in the sense of (2.4), then the complex valued multifunction $(t,x) \to P(t,x)(\eta,\xi)$ is also Lipschitzian. That is, there exists $(\eta', \xi') \in (\mathbb{D}\underline{\otimes}\mathbb{E})^2$ such that for all $x, y \in \tilde{\mathcal{A}}$,

$$\rho(P(t,x)(\eta,\xi), P(t,y)(\eta,\xi)) \leq K^{P}_{\eta\xi}(t)\|x-y\|_{\eta'\xi'} \tag{2.5}$$

where the map $K^P_{\eta\xi} : [0,T] \to \mathbb{R}_+$ lies in $L^1_{loc}([0,T])$ and $\rho(\cdot,\cdot)$ is the Hausdorf distance function on $clos(\mathbb{C})$.

(c) Using the definition in (a), we see that if $P : \mathbb{R}_+ \to comp(\tilde{\mathcal{A}})$ such that $P(t)$ is a closed ball with centre at the origin and $(\eta', \xi') \in (\mathbb{D}\underline{\otimes}\mathbb{E})^2$ is a fixed point, then the multivalued map F defined by

$$F(t,x) = |\langle \eta', x\xi' \rangle| P(t)$$

is Lipschitzian. This follows, since for any $t \in \mathbb{R}_+$, $x, y, \in \tilde{\mathcal{A}}$,

$$\begin{aligned}\rho_{\eta\xi}(F(t,x), F(t,y)) &= \rho_{\eta\xi}\left(\langle \eta', x\xi' \rangle P(t), \langle \eta', y\xi' \rangle P(t)\right) \\ &\leq |\ \|x\|_{\eta'\xi'} - \|y\|_{\eta'\xi'}\ | \rho_{\eta\xi}\left(P(t), \{0\}\right) \leq \|P(t)\|_{\eta\xi}\|x-y\|_{\eta'\xi'},\end{aligned}$$

where

$$\|P(t)\|_{\eta\xi} = \rho_{\eta\xi}(P(t), \{0\}).$$

3. EXISTENCE AND NON UNIQUENESS OF SOLUTIONS

Subject to the conditions below, we shall establish the existence and non-uniqueness of solutions of QSDI (1.1) in this section. By a solution of (1.1) we mean a quantum stochastic process $\Phi : [0,T] \to \tilde{\mathcal{A}}$ lying in $Ad(\tilde{\mathcal{A}})_{wac} \bigcap L^2_{loc}(\tilde{\mathcal{A}})$ satisfying QSDI (1.1).

In what follows, we consider, without loss of generality, quantum stochastic processes and the related inclusions defined on the interval $[0,1]$. We employ the notion of adaptedness of quantum stochastic processes as explained in [8]. In connection with the subsequent results, we list the following statements and assumptions.

$(\mathcal{S}_{(1)})$ $Z : [0,1] \to \tilde{\mathcal{A}}$ is a stochastic process in $Ad(\tilde{\mathcal{A}})_{wac}$ with the property that for each pair $\eta,\ \xi \in \mathbb{D}\underline{\otimes}\mathbb{E}$, and almost all $t \in [0,1]$, there exists a positive function $W_{\eta\xi}(t)$ lying in $L^1_{loc}([0,1])$ such that

$$d\left(\frac{d}{dt}\langle \eta, Z(t)\xi \rangle, P(t, Z(t))(\eta,\xi)\right) \leq W_{\eta\xi}(t)$$

$(\mathcal{S}_{(2)})$ $\gamma > 0$ is an arbitrary but fixed number and $Q_{Z,\gamma}$ is the set

$$Q_{Z,\gamma} = \{(t,x) \in [0,1] \times \tilde{\mathcal{A}} : \|x - Z(t)\|_{\eta\xi} \le \gamma,\ \forall \eta, \xi \in \mathbb{D}\underline{\otimes}\mathbb{E}\}.$$

$(\mathcal{S}_{(3)})$ Each of the coefficients E, F, G, H appearing in (1.1) is Lipschitzian from $Q_{Z,\gamma}$ to the Hausdorf topological space $(clos(\tilde{\mathcal{A}}), \tau_H)$, i.e, for each $M \in \{E, F, G, H\}$ there exists a positive map $K^M_{\eta\xi} : [0,1] \to \mathbb{R}_+$ lying in $L^1_{loc}([0,1])$ corresponding to each pair η, ξ such that

$$\rho_{\eta\xi}\left(M(t,x), M(t,y)\right) \le K^M_{\eta\xi}(t)\|x - y\|_{\Theta_M(\eta,\xi)}$$

for some map

$$\Theta_M : (\mathbb{D}\underline{\otimes}\mathbb{E})^2 \to \ \mathrm{Fin}[(\mathbb{D}\underline{\otimes}\mathbb{E})^2].$$

$(\mathcal{S}_{(4)})$ For each pair $\eta, \xi \in \mathbb{D}\underline{\otimes}\mathbb{E}$,

$$\delta_{\eta\xi} \equiv \|x_0 - Z(0)\|_{\eta\xi} \quad \text{and} \quad \delta_{\eta\xi} \le \gamma.$$

$(\mathcal{S}_{(5)})$

$$R_{\eta\xi} := \max(\delta_{\eta\xi},\ W_{\eta\xi})$$

for all $\eta, \xi \in \mathbb{D}\underline{\otimes}\mathbb{E}$ where

$$W_{\eta\xi} = ess \sup_{[0,1]} W_{\eta\xi}(t).$$

$(\mathcal{S}_{(6)})$ For any countably infinite sequence of points $\{(\eta_n, \xi_n) \subseteq (\mathbb{D}\underline{\otimes}\mathbb{E})^2, n = 1, 2, \dots\}$,

$$\sup_{n\in\mathbb{N}} \left\{ ess \sup_{t\in[0,1]} K^P_{\eta_n\xi_n}(t) \right\} < \infty.$$

$(\mathcal{S}_{(7)})$ $\{L_{\eta_j\xi_j}\}_{j=1}^{j=\infty}$ is a sequence of positive real numbers indexed by a countably infinite sequence of elements $\{(\eta_j, \xi_j)\}_{j=1}^{\infty} \subseteq (\mathbb{D}\underline{\otimes}\mathbb{E})^2$ that depends on an arbitrary pair $(\eta, \xi) \in \mathbb{D}\underline{\otimes}\mathbb{E}$ and defined as follows:

$$L_{\eta_1\xi_1} = R_{\eta_1\xi_1}$$

and

$$L_{\eta_j\xi_j} := ess \sup_{[0,1]} K^P_{\eta_j\xi_j}(t), \quad j \ge 2.$$

$(\mathcal{S}_{(8)})$ From $(\mathcal{S}_{(7)})$ above, we set

$$L_{\eta\xi,n} = \max_{j=1,2,\dots,n} \{L_{\eta_j\xi_j}\} \ \text{ and } \ L_{\eta\xi} = \sup_{n\in\mathbb{N}} \{L_{\eta\xi,n}\}.$$

$(\mathcal{S}_{(9)})$ For arbitrary $\eta, \xi \in \mathbb{D}\underline{\otimes}\mathbb{E}$ and $t \in [0,1]$, we define

$$\mathcal{E}_{\eta\xi}(t) = 2L_{\eta\xi} + 2L_{\eta\xi} \int_0^t \left(K^P_{\eta\xi}(s) e^{L_{\eta\xi}s} \right) ds,$$

where the constant $L_{\eta\xi}$ is given by $\mathcal{S}_{(8)}$ above.

$(\mathcal{S}_{(10)})$ J is the subset of the interval $[0,1]$ defined by

$$J = \{t \in [0,1] : \mathcal{E}_{\eta\xi}(t) \le \gamma,\ \forall \eta, \xi \in \mathbb{D}\underline{\otimes}\mathbb{E}\}.$$

Next we present a proposition which is useful for the proof of the existence result that follows.

Proposition 3.1. Let $\{\Phi_i\}_{i=1}^{\infty}$ be a sequence of weakly absolutely continuous maps from $[0,1]$ to $\tilde{\mathcal{A}}$ which satisfy the following conditions:
(i) $(t, \Phi_i(t)) \in Q_{Z,\gamma}$, $i \geq 1$, for almost all $t \in J$.
(ii) There exists a sequence $\{V_i\}_{i=1}^{\infty}$ such that for arbitrary $\eta, \xi \in \mathbb{D}\underline{\otimes}\mathbb{E}$ and a constant $L_{\eta\xi} > 0$,
(a) $\Phi_i(t) = X_0 + \int_0^t V_{i-1}(s)ds$, $i \geq 1$
(b) $|\frac{d}{dt}\langle\eta, \Phi_i(t)\xi\rangle - \frac{d}{dt}\langle\eta, \Phi_{i-1}(t)\xi\rangle| \leq 2L_{\eta\xi}^{i-1}K_{\eta\xi}^{P}(t)\frac{t^{i-2}}{(i-2)!}$, for almost all $t \in J$. Then,
(c) $\|\Phi_i(t) - \Phi_{i-1}(t)\|_{\eta\xi} \leq 2L_{\eta\xi}\int_0^t K_{\eta\xi}^{P}(s)\frac{(L_{\eta\xi}s)^{i-2}}{(i-2)!}ds$, $t \in J$, $i \geq 2$.

Proof. Let (i) and (ii) hold. Then

$$\begin{aligned}
\|\Phi_i(t) - \Phi_{i-1}(t)\|_{\eta\xi} &= |\int_0^t \langle\eta, (V_{i-1}(s) - V_{i-2}(s))\xi\rangle ds|, \text{ by (ii)(a)} \\
&= |\int_0^t \{\frac{d}{ds}\langle\eta, \Phi_i(s)\xi\rangle - \frac{d}{ds}\langle\eta, \Phi_{i-1}(s)\xi\rangle\}ds|, \text{ by (ii)(a)} \\
&\leq \int_0^t |\frac{d}{ds}\langle\eta, \Phi_i(s)\xi\rangle - \frac{d}{ds}\langle\eta, \Phi_{i-1}(s)\xi\rangle| ds \\
&\leq 2L_{\eta\xi}^{i-1}\int_0^t K_{\eta\xi}^{P}(s)\frac{s^{i-2}}{(i-2)!}ds \\
&= 2L_{\eta\xi}\int_0^t K_{\eta\xi}^{P}(s)\frac{(L_{\eta\xi}s)^{i-2}}{(i-2)!}ds, \quad t \in J, \quad i \geq 2, \quad \text{by (ii)(b).}
\end{aligned}$$

This concludes the proof.

Next, we present our result on the existence of solution of QSDI (1.1) subject to the conditions $(\mathcal{S}_{(1)}) - (\mathcal{S}_{(10)})$ above. The result shall be established by employing a similar line of argument as in the proof of Theorem (8.2) in [8].

Theorem 3.2. Suppose that the conditions $\mathcal{S}_{(1)} - \mathcal{S}_{(10)}$ hold and the coefficients E, F, G, H are continuous from $[0,1] \times \tilde{\mathcal{A}}$ to $\big(clos(\tilde{\mathcal{A}}), \tau_H\big)$.
Then there exists a solution Φ of (1.1) such that

$$\|\Phi(t) - Z(t)\|_{\eta\xi} \leq \mathcal{E}_{\eta\xi}(t), \;\; t \in J, \tag{3.1}$$

and

$$|\frac{d}{dt}\langle\eta, \Phi(t)\xi\rangle - \frac{d}{dt}\langle\eta, Z(t)\xi\rangle| \leq L_{\eta\xi}\left(1 + 2K_{\eta\xi}^{P}(t)e^{L_{\eta\xi}t}\right). \tag{3.2}$$

Proof. In what follows, $\eta, \xi \in \mathbb{D}\underline{\otimes}\mathbb{E}$ are arbitrary elements. Our proof will be established by constructing a Cauchy sequence $\{\Phi_n\}_{n\geq 0}$ in $\tilde{\mathcal{A}}$ of successive approximations of Φ in such a way that the sequence $\{\frac{d}{dt}\langle\eta, \Phi_n(t)\xi\rangle\}$ is also Cauchy in the field of complex numbers.

Define $\Phi_0(t) = Z$, then Φ_0 is adapted. By Theorem (1.14.2) in [1] (See also [8]), there exists a measurable selection $V_0(\cdot)(\eta,\xi) \in P(\cdot, \Phi_0(\cdot))(\eta,\xi)$ such that

$$|V_0(t)(\eta,\xi) - \frac{d}{dt}\langle \eta, \Phi_0(t)\xi\rangle| = d\left(\frac{d}{dt}\langle \eta, \Phi_0(t)\rangle, P(t, \Phi_0(t))(\eta,\xi)\right) \leq W_{\eta\xi}(t). \tag{3.3}$$

As the map $(\eta,\xi) \rightarrow V_0(t)(\eta,\xi)$ is a sesquilinear form on $\mathbb{D}\underline{\otimes}\mathbb{E}$, for almost all $t \in J$, then there exists $V_0(t) \in \tilde{\mathcal{A}}$ such that $V_0(t)(\eta,\xi) = \langle \eta, V_0(t)\xi\rangle$. Since $V_0(\cdot)(\eta,\xi)$ is locally absolutely integrable, then $V_0 \in L^1_{loc}(\tilde{\mathcal{A}})$.

Next we define

$$\Phi_1(t) = X_0 + \int_0^t V_0(s)ds, \quad t \in J.$$

As $V_0(t) \in \tilde{\mathcal{A}}$ for almost all $t \in J$, it follows that $\Phi_1(t) \in \tilde{\mathcal{A}}_t$, i.e Φ_1 is adapted.

Furthermore, for $t \in J$,

$$\|\Phi_1(t) - \Phi_0(t)\|_{\eta\xi} \leq \|X_0 - \Phi_0(t_0)\|_{\eta\xi} + \int_0^t |V_0(s)(\eta,\xi) - \frac{d}{ds}\langle \eta, \Phi_0(s)\xi\rangle| ds$$
$$\leq \delta_{\eta\xi} + \int_0^t W_{\eta\xi}(s)ds \tag{3.4}$$

Notice that by (3.3),

$$|\frac{d}{dt}\langle \eta, \Phi_1(t)\xi\rangle - \frac{d}{dt}\langle \eta, \Phi_0(t)\xi\rangle| \leq W_{\eta\xi}(t). \tag{3.5}$$

Again there exists a measurable selection $V_1(\cdot)(\eta,\xi) \in P(\cdot, \Phi_1(\cdot))(\eta,\xi)$ such that

$$\begin{aligned} |V_1(t)(\eta,\xi) - \frac{d}{dt}\langle \eta, \Phi_1(t)\xi\rangle| &= d\left(\frac{d}{dt}\langle \eta, \Phi_1(t)\xi\rangle,\ P(t, \Phi_1(t))(\eta,\xi)\right) \\ &\leq \rho\left(P(t, \Phi_0(t))(\eta,\xi),\ P(t, \Phi_1(t))(\eta,\xi)\right) \\ &\leq K^P_{\eta\xi}(t)\|\Phi_0(t) - \Phi_1(t)\|_{\eta_1\xi_1} \\ &\leq K^P_{\eta\xi}(t)\left(\delta_{\eta_1\xi_1} + \int_0^t W_{\eta_1\xi_1}(s)ds\right), \end{aligned} \tag{3.6}$$

for some $\eta_1, \xi_1 \in \mathbb{D}\underline{\otimes}\mathbb{E}$ that depend on η, ξ.

By a similar argument as for the existence of $V_0(\cdot)$, there exists $V_1 \in L^1_{loc}(\tilde{\mathcal{A}})$ such that for almost all $t \in J$,

$$V_1(t)(\eta,\xi) = \langle \eta, V_1(t)\xi\rangle.$$

Next we define,

$$\Phi_2(t) = X_0 + \int_0^t V_1(s)ds, \quad t \in J.$$

Again, $\Phi_2(t) \in \tilde{\mathcal{A}}_t$ since $V_1(t) \in \tilde{\mathcal{A}}$ for almost all $t \in J$, i.e. Φ_2 is adapted.

Furthermore, for $t \in J$,

$$\|\Phi_2(t) - \Phi_1(t)\|_{\eta\xi} = \|\int_0^t (V_1(s) - V_0(s))\, ds\|_{\eta\xi}$$

$$
\begin{aligned}
&= |\int_0^t \langle \eta, (V_1(s) - V_0(s))\, \xi \rangle ds| \\
&\leq \int_0^t |\langle \eta, V_1(s)\xi \rangle - \langle \eta, V_0(s)\xi \rangle| ds \\
&\leq \int_0^t \rho\left(P(s, \Phi_1(s))(\eta, \xi),\ P(s, \Phi_0(s))(\eta, \xi)\right) ds \\
&\leq \int_0^t K^P_{\eta\xi}(s) \|\Phi_1(s) - \Phi_0(s)\|_{\eta_1\xi_1} ds
\end{aligned}
$$

By applying (3.4), we have the estimate

$$
\|\Phi_2(t) - \Phi_1(t)\|_{\eta\xi} \leq \int_0^t \left(K^P_{\eta\xi}(s) \left[\delta_{\eta_1\xi_1} + \int_0^s W_{\eta_1\xi_1}(r) dr \right] \right) ds. \tag{3.7}
$$

We may write (3.6) as

$$
|\frac{d}{dt}\langle \eta, \Phi_2(t)\xi \rangle - \frac{d}{dt}\langle \eta, \Phi_1(t)\xi \rangle| \leq K^P_{\eta\xi}(t) \left(\delta_{\eta_1\xi_1} + \int_0^t W_{\eta_1\xi_1}(r) dr \right). \tag{3.8}
$$

Continuing the procedure, there exists a measurable selection $V_2(\cdot)(\eta, \xi) \in P(\cdot, \Phi_2(\cdot))(\eta, \xi)$ and a pair of elements $\eta_2,\ \xi_2 \in \mathbb{D}\underline{\otimes}\mathbb{E}$ depending on η, ξ such that

$$
\begin{aligned}
|V_2(t)(\eta, \xi) - \frac{d}{dt}\langle \eta, \Phi_2(t)\xi \rangle| &= d\left(\frac{d}{dt}\langle \eta, \Phi_2(t)\xi \rangle,\ P(t, \Phi_2(t))(\eta, \xi) \right) \\
&\leq \rho\left(P(t, \Phi_1(t))(\eta, \xi),\ P(t, \Phi_2(t))(\eta, \xi)\right) \\
&\leq K^P_{\eta\xi}(t) \|\Phi_1(t) - \Phi_2(t)\|_{\eta_2\xi_2} \\
&\leq K^P_{\eta\xi}(t) \int_0^t \left(K^P_{\eta_2\xi_2}(s) \left[\delta_{\eta_1\xi_1} + \int_0^s W_{\eta_1\xi_1}(r) dr \right] \right) ds, \qquad (3.9)
\end{aligned}
$$

on account of (3.7).

Again, (3.9) may be written as

$$
\begin{aligned}
&|\frac{d}{dt}\langle \eta, \Phi_3(t)\xi \rangle - \frac{d}{dt}\langle \eta, \Phi_2(t)\xi \rangle| \\
&\qquad \leq K^P_{\eta\xi}(t)\delta_{\eta_1\xi_1} \int_0^t K^P_{\eta_2\xi_2}(s) ds + K^P_{\eta\xi}(t) \int_0^t K^P_{\eta_2\xi_2}(s) \int_0^s W_{\eta_1\xi_1}(r) dr\, ds \qquad (3.10)
\end{aligned}
$$

As before, it is straightforward to show that there exist $V_3, V_2 \in L^1_{loc}(\tilde{\mathcal{A}})$ defining adapted processes $\Phi_3,\ \Phi_4$ for $t \in J$ by

$$
\begin{aligned}
\Phi_3(t) &= X_0 + \int_0^t V_2(s) ds,\ t \in J \\
\Phi_4(t) &= X_0 + \int_0^t V_3(s) ds,\ t \in J \qquad (3.11)
\end{aligned}
$$

and satisfy the following inequalities

$$
\begin{aligned}
&\|\Phi_3(t) - \Phi_2(t)\|_{\eta\xi} \\
&\quad \leq \int_0^t \left(K^P_{\eta\xi}(s) \left[\int_0^s K^P_{\eta_2\xi_2}(s^{'}) \left[\delta_{\eta_1\xi_1} + \int_0^{s^{'}} W_{\eta_1\xi_1}(r)dr \right] ds^{'} \right] \right) ds \\
&\quad = \int_0^t K^P_{\eta\xi}(s) \int_0^s \delta_{\eta_1\xi_1} K^P_{\eta_2\xi_2}(s^{'}) ds^{'} ds \\
&\qquad + \int_0^t K^P_{\eta\xi}(s) \int_0^s K^P_{\eta_2\xi_2}(s') \int_0^{s'} W_{\eta_1\xi_1}(r)dr\, ds' ds. \qquad (3.12)
\end{aligned}
$$

and

$$
\begin{aligned}
&\|\Phi_4(t) - \Phi_3(t)\|_{\eta\xi} \\
&\quad \leq \int_0^t K^P_{\eta\xi}(s) \int_0^s K^P_{\eta_3\xi_3}(s') \int_0^{s'} \delta_{\eta_1\xi_1} K^P_{\eta_2\xi_2}(s'') ds'' ds' ds \\
&\qquad + \int_0^t K^P_{\eta\xi}(s) \int_0^s K^P_{\eta_3\xi_3}(s') \int_0^{s'} K^P_{\eta_2\xi_2}(s'') \int_0^{s''} W_{\eta_1\xi_1}(r)drds^{''}ds^{'}ds. \qquad (3.13)
\end{aligned}
$$

Furthermore

$$
\begin{aligned}
\left|\frac{d}{dt}\langle \eta, \Phi_4(t)\xi\rangle - \frac{d}{dt}\langle \eta, \Phi_3(t)\xi\rangle\right| &\leq K^P_{\eta\xi}(t) \int_0^t K^P_{\eta_3\xi_3}(s) \int_0^s \delta_{\eta_1\xi_1} K^P_{\eta_2\xi_2}(s')ds'ds \\
&\quad + K^P_{\eta\xi}(t) \int_0^t K^P_{\eta_3\xi_3}(s) \int_0^s K^P_{\eta_2\xi_2}(s') \int_0^{s'} W_{\eta_1\xi_1}(r)drds'ds \qquad (3.14)
\end{aligned}
$$

so that from (3.13) and (3.14)

$$
\begin{aligned}
\|\Phi_4(t) - \Phi_3(t)\|_{\eta\xi} &\leq \int_0^t K^P_{\eta\xi}(s) \int_0^s L_{\eta_3\xi_3} \int_0^{s'} \delta_{\eta_1\xi_1} L_{\eta_2\xi_2} ds'' ds' ds \\
&\quad + \int_0^t K^P_{\eta\xi}(s) \int_0^s L_{\eta_3\xi_3} \int_0^{s'} L_{\eta_2\xi_2} \int_{t_0}^{s''} W_{\eta_1\xi_1} dr\, ds'' ds' ds \\
&\leq 2L^3_{\eta\xi} \int_0^t K^P_{\eta\xi}(s) \frac{s^2}{2} ds,
\end{aligned}
$$

and

$$
\begin{aligned}
&\left|\frac{d}{dt}\langle \eta, \Phi_4(t)\xi\rangle - \frac{d}{dt}\langle \eta, \Phi_3(t)\xi\rangle\right| \\
&\leq K^P_{\eta\xi}(t) \left[\delta_{\eta_1\xi_1} L_{\eta_2\xi_2} L_{\eta_3\xi_3} \int_0^t \int_0^s ds' ds \right. \\
&\qquad \left. + W_{\eta_1\xi_1} L_{\eta_2\xi_2} L_{\eta_3\xi_3} \int_0^t \int_0^s \int_0^{s'} dr\, ds' ds \right] \\
&= K^P_{\eta\xi}(t) \left[\delta_{\eta_1\xi_1} L_{\eta_2\xi_2} L_{\eta_3\xi_3} \frac{t^2}{2} + W_{\eta_1\xi_1} L_{\eta_2\xi_2} L_{\eta_3\xi_3} \frac{t^3}{6} \right] \\
&\leq K^P_{\eta\xi}(t) \left[L^3_{\eta\xi,3} \frac{t^2}{2} + L^3_{\eta\xi,3} \frac{t^3}{6} \right] \leq 2K^P_{\eta\xi}(t) L^3_{\eta\xi} \frac{t^2}{2}, \qquad t \in [0,1]. \qquad (3.15)
\end{aligned}
$$

Indeed, there exists a sequence $\{\Phi_i\}_{i\geq 0}$ of weakly absolutely continuous processes from $[0,1]$ to $\tilde{\mathcal{A}}$ satisfying the hypothesis (i) and (ii) of Proposition (3.1) and hence its conclusion.

To prove this claim, we assume that the sequence $\{\Phi_i\}$ has already been defined and satisfies the hypothesis (i) and (ii) of the proposition for $i = 0, 1, 2 \ldots, n$. We shall show that there exists a map $\Phi_{n+1} : J \to \tilde{\mathcal{A}}$ for which (i) and (ii) of the proposition also hold.

Again by Theorem (1.14.2) in [1], there exists

$$V_n(\cdot)(\eta,\xi) \in P(\cdot, \Phi_n(\cdot))(\eta,\xi)$$

such that

$$|\frac{d}{dt}\langle \eta, \Phi_n(t)\xi\rangle - V_n(t)(\eta,\xi)| = d\left(\frac{d}{dt}\langle \eta, \Phi_n(t)\xi\rangle,\ P(t,\Phi_n(t))(\eta,\xi)\right), \text{ a.e. on } J.$$

As $(\eta,\xi) \to V_n(t)(\eta,\xi)$ is a sesquilinear form on $\mathbb{D}\underline{\otimes}\mathbb{E}$, for almost all $t \in J$, there exist $V_n \in L^1_{loc}(\tilde{\mathcal{A}})$ such that

$$V_n(t)(\eta,\xi) = \langle \eta, V_n(t)\xi\rangle, \text{ a.e on } J$$

Define

$$\Phi_{n+1}(t) = X_0 + \int_0^t V_n(s)ds,\ t \in J.$$

Then, for some pair of elements $\eta_n,\ \xi_n \in \mathbb{D}\underline{\otimes}\mathbb{E}$ depending on $\eta,\ \xi$, we have the following estimates:

$$\begin{aligned}
|\frac{d}{dt}\langle \eta, \Phi_{n+1}(t)\xi\rangle - \frac{d}{dt}\langle \eta, \Phi_n(t)\xi\rangle| &= |\langle \eta, V_n(t)\xi\rangle - \langle \eta, V_{n-1}(t)\xi\rangle| \\
&\leq \rho\left(P(t,\Phi_n(t))(\eta,\xi),\ P(t,\Phi_{n-1}(t))(\eta,\xi)\right) \\
&\leq K^P_{\eta\xi}(t)\|\Phi_n(t) - \Phi_{n-1}(t)\|_{\eta_n\xi_n} \\
&\leq K^P_{\eta\xi}(t)\left[2L_{\eta\xi}\int_0^t K^P_{\eta_n\xi_n}(s)\frac{(L_{\eta\xi}s)^{n-2}}{(n-2)!}ds\right] \\
&\leq 2L^n_{\eta\xi}K^P_{\eta\xi}(t)\frac{t^{n-1}}{(n-1)!},
\end{aligned}$$

which proves (ii)(b) of Proposition (3.1).

Furthermore, for $t \in J$,

$$\begin{aligned}
\|\Phi_{n+1}(t) - \Phi_0(t)\|_{\eta\xi} &\leq \|\Phi_1(t) - \Phi_0(t)\|_{\eta\xi} + \|\Phi_2(t) - \Phi_1(t)\|_{\eta\xi} \\
&\quad + \cdots + \|\Phi_{n+1}(t) - \Phi_n(t)\|_{\eta\xi} \\
&\leq 2L_{\eta\xi} + 2L_{\eta\xi}\sum_{k=0}^{n-1}\int_0^t K^P_{\eta\xi}(s)\frac{(L_{\eta\xi}s)^k}{k!}ds \\
&\leq 2L_{\eta\xi}\left(1 + \int_0^t K^P_{\eta\xi}(s)e^{L_{\eta\xi}s}ds\right) \leq \gamma. \qquad (3.16)
\end{aligned}$$

This shows that $(t,\Phi_{n+1}(t)) \in Q_{Z,\gamma}$ and therefore proves (ii)(c) of Proposition 3.1. It follows that the sequence $\{\Phi_n(t)\}$ is a τ_ω-Cauchy sequence and therefore converges to some $\Phi(t) \in \tilde{\mathcal{A}}$. We conclude that $\Phi(t)$ is a solution of (1.1) for almost all $t \in J$ in the same way as in the proof of Theorem 8.2 in [8].

Finally, by using (ii)(b) of Proposition 3.1, we have the following:

$$|\frac{d}{dt}\langle\eta,\Phi_{n+1}(t)\xi\rangle - \frac{d}{dt}\langle\eta,\Phi_0(t)\xi\rangle| \leq |\frac{d}{dt}\langle\eta,\Phi_1(t)\xi\rangle - \frac{d}{dt}\langle\eta,\Phi_0(t)\xi\rangle| + \sum_{k=0}^{n-1} 2L_{\eta\xi}K^P_{\eta\xi}(t)\frac{[L_{\eta\xi}t]^k}{k!} \leq L_{\eta\xi} + 2L_{\eta\xi}K^P_{\eta\xi}(t)e^{L_{\eta\xi}t}.$$

Taking the limit as $n \to \infty$, we obtain inequality (3.2). Similarly, inequality (3.1) follows from (3.16) above.

Corollary 3.3. Suppose that the conditions $\mathcal{S}_{(1)} - \mathcal{S}_{(10)}$ hold in the region

$$Q_{X_0,\gamma} = \{(t,x) \in [0,1] \times \tilde{\mathcal{A}} : \|x - X_0\|_{\eta\xi} \leq \gamma\},$$

then the solution $X(t)$ of (1.1) exists on the segment.

Proof. The conditions of Theorem (3.2) will be satisfied if we set $Z(t) \equiv X_0$, a trivially adapted quasi solution, and the function

$$W_{\eta\xi}(t) = d\,(0,\ P(t,X_0)(\eta,\xi))$$

is continuous, by the continuity of the map $(t,x) \to P(t,x)(\eta,\xi)$.

Our next result shows that new solutions of QSDI (1.1) exist in some neighbourhoods of a solution. This establishes the nonuniqueness of solutions as in the case of Lipschitz differential inclusions in finite dimensional Euclidean spaces (see [1]).

Theorem 3.4. Let $\Phi_0(t)$ be a solution of problem (1.1). Suppose that in the region Q_{Φ_0,ϵ_0}, the conditions of Theorem (3.2) are satisfied with Lipschitz constant $K_{\eta\xi}$ that depends only on arbitrary elements $\eta,\xi \in \mathbb{D}\underline{\otimes}\mathbb{E}$, for some constant $\epsilon_0 > 0$.

Then for any

$$\epsilon > 2L_{\eta\xi} + 2K_{\eta\xi}(e^{L_{\eta\xi}} - 1) \tag{3.17}$$

valid for all $\eta,\ \xi \in \mathbb{D}\underline{\otimes}\mathbb{E}$, a solution $\Phi(t)$ of QSDI (1.1) exists such that

$$\|\Phi(t) - \Phi_0(t)\|_{\eta\xi} < \epsilon, \quad \text{on } [0,1].$$

Suppose in addition that the map $t \to \frac{d}{dt}\langle\eta,\Phi_0(t)\xi\rangle$ is continuous on the interval $[0,1]$, then there exists a constant $M_{\eta\xi} > 0$ depending on η,ξ such that

$$|\frac{d}{dt}\langle\eta,\Phi(t)\xi\rangle| < M_{\eta\xi}, \ \text{ almost all } \ t \in [0,1]. \tag{3.18}$$

Proof. We employ an adaptation of the argument in the proof of Theorem 2 in [11] as follows:

We consider the region $Q_{\Phi_0,\epsilon}$ for ϵ_0 big enough such that $0 < \epsilon < \epsilon_0$ in view of the constraint (3.17). By the continuity of the map $(t, x) \to d\,(0, P(t, x)(\eta, \xi))$ on the region Q_{Φ_0,ϵ_0}, we have

$$\sup_{Q_{\Phi_0,\epsilon_0}} d\,(0,\ P(t, x)(\eta, \xi)) = S_{\eta\xi} < \infty.$$

Define the number

$$A_{\eta\xi} = \frac{2\epsilon}{\epsilon - 2L_{\eta\xi} - 2K_{\eta\xi}(e^{L_{\eta\xi}} - 1)}.$$

Then in view of (3.17), $A_{\eta\xi} > 0$. Thus, by a similar reason as in [11], we can find numbers $b \geq K_{\eta\xi}\epsilon$, $b \geq S_{\eta\xi}$ such that

$$\int_B |\frac{d}{dt}\langle\eta, \Phi_0(t)\xi\rangle| dt < \frac{\epsilon}{A_{\eta\xi}} \tag{3.19}$$

where

$$B = \{t \in [0, 1] : |\frac{d}{dt}\langle\eta, \Phi_0(t)\xi\rangle| > b\}.$$

By the argument in the proof of Theorem 3.2, since $\Phi_0(t)$ is a solution of (1.1), there exists an element $V_0 \in L^1_{loc}(\tilde{\mathcal{A}})$ such that

$$\Phi_0(t) = X_0 + \int_0^t V_0(s) ds$$

and

$$\frac{d}{dt}\langle\eta, \Phi_0(t)\xi\rangle = \langle\eta, V_0(t)\xi\rangle,\ \text{almost all}\ \ t \in [0, 1].$$

Next we define

$$\begin{aligned} V(t) &= V_0(t),\ \ t \in ([0, 1]\backslash B) \\ &= 0, \qquad t \in B. \end{aligned}$$

and

$$Y(t) = X_0 + \int_0^t V(s) ds.$$

We note here that the process Y lies in $Ad(\tilde{\mathcal{A}})_{wac}$.

For $t \in ([0, 1]\backslash B)$,

$$\langle\eta, Y(t)\xi\rangle = \langle\eta, X_0\xi\rangle + \int_0^t \langle\eta, V(s)\xi\rangle ds = \langle\eta, \Phi_0(t)\xi\rangle.$$

For $t \in B$,

$$\langle\eta, Y(t)\xi\rangle = \langle\eta, X_0\xi\rangle.$$

Therefore we have for both cases using (3.19)

$$|\langle\eta, Y(t)\xi\rangle - \langle\eta, \Phi_0(t)\xi\rangle| = \|Y(t) - \Phi_0(t)\|_{\eta\xi} \leq \frac{\epsilon}{A_{\eta\xi}},$$

and

$$d\left(\frac{d}{dt}\langle\eta, Y(t)\xi\rangle,\ P(t, Y(t))(\eta, \xi)\right) = W_{\eta\xi}(t),$$

almost everywhere on $[0,1]$.

Furthermore,

$$W_{\eta\xi}(t) \leq S_{\eta\xi} < b, \quad \text{for} \;\; t \in B, \tag{3.20}$$

For $t \in ([0,1]\backslash B)$, we have

$$\frac{d}{dt}\langle \eta, Y(t)\xi\rangle = \frac{d}{dt}\langle \eta, \Phi_0(t)\xi\rangle \in P(t, \Phi_0(t))(\eta,\xi).$$

Thus

$$W_{\eta\xi}(t) = d\left(\frac{d}{dt}\langle \eta, \Phi_0(t)\xi\rangle,\ P(t, Y(t))(\eta,\xi)\right) = 0 \leq K_{\eta\xi}\epsilon. \tag{3.21}$$

Hence by Theorem 3.2, there exists a solution Φ of (1.1) satisfying

$$\|\Phi(t) - Y(t)\|_{\eta\xi} \leq \mathcal{E}_{\eta\xi}(t), \quad t \in J,$$

and where $\mathcal{E}_{\eta\xi}(t)$ is given by $\mathcal{S}_{(9)}$.

By the definition of the set $B \subseteq [0,1]$, and the estimate (3.19) above, we have

$$\int_B b ds < \int_B |\frac{d}{ds}\langle \eta, \Phi_0(s)\xi\rangle| ds \leq \frac{\epsilon}{A_{\eta\xi}}. \tag{3.22}$$

By (3.19), (3.22) and $\mathcal{S}_{(9)}$, we have,

$$\begin{aligned}\mathcal{E}_{\eta\xi}(t) &< \int_B S_{\eta\xi} ds + 2L_{\eta\xi} + 2L_{\eta\xi}\int_0^t \left(K_{\eta\xi}e^{L_{\eta\xi}s}\right) ds \\ &< \frac{\epsilon}{A_{\eta\xi}} + 2L_{\eta\xi} + 2K_{\eta\xi}(e^{L_{\eta\xi}} - 1).\end{aligned}$$

Hence, we have

$$\begin{aligned}\|\Phi(t) - \Phi_0(t)\|_{\eta\xi} &\leq \|\Phi(t) - Y(t)\|_{\eta\xi} + \|Y(t) - \Phi_0(t)\|_{\eta\xi} \\ &\leq \frac{2\epsilon}{A_{\eta\xi}} + 2L_{\eta\xi} + 2K_{\eta\xi}(e^{L_{\eta\xi}} - 1) = \epsilon.\end{aligned}$$

Again by Equation (3.2), $\Phi(t)$ satisfies

$$\begin{aligned}|\frac{d}{dt}\langle \eta, \Phi(t)\xi\rangle - \frac{d}{dt}\langle \eta, Y(t)\xi\rangle| &\leq L_{\eta\xi}(1 + 2K_{\eta\xi}e^{L_{\eta\xi}t}) \\ &\leq L_{\eta\xi}(1 + 2K_{\eta\xi}U_{\eta\xi}) := N_{\eta\xi},\end{aligned} \tag{3.23}$$

where

$$U_{\eta\xi} = \sup_{t\in[0,1]} (e^{L_{\eta\xi}t}).$$

Thus by definition, $\frac{d}{dt}\langle \eta, Y(t)\xi\rangle = 0$ for $t \in B$ and for $t \in ([0,1]\backslash B)$,

$$\frac{d}{dt}\langle \eta, Y(t)\xi\rangle = \frac{d}{dt}\langle \eta, \Phi_0(t)\xi\rangle.$$

Putting

$$\sup_{[0,1]} |\frac{d}{dt}\langle \eta, \Phi_0(t)\xi\rangle| = T_{\eta\xi},$$

then from (3.23)

$$|\frac{d}{dt}\langle\eta, \Phi(t)\xi\rangle| \leq N_{\eta\xi}, \quad t \in B,$$

and

$$|\frac{d}{dt}\langle\eta, \Phi(t)\xi\rangle| \leq T_{\eta\xi} + N_{\eta\xi}, \quad t \in ([0,1]\backslash B).$$

Inequality (3.18) follows by defining

$$M_{\eta\xi} = T_{\eta\xi} + N_{\eta\xi}.$$

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REFERENCES

[1] Aubin, J. P; Cellina, A. *Differential Inclusions*, Springer-Verlag, Berlin, 1984.

[2] Ayoola, E. O. Continuous selections of solution sets of Lipschitzian quantum stochastic differential inclusions, *International Journal of Theoretical Physics* **43**(10), (2004) 2041–2059.

[3] Ayoola, E. O. Error estimates for discretized quantum stochastic differential inclusions, *Stochastic Analysis and Applications* **21**(6) (2003) 1215–1230.

[4] Ayoola, E. O. Exponential formula for the reachable sets of quantum stochastic differential inclusions, *Stochastic Analysis and Applications* **21**(3) (2003) 515–543.

[5] Ayoola, E. O. Construction of approximate attainability sets for Lipschitzian quantum stochastic differential inclusions, *Stochastic Analysis and Applications* **19**(3) (2001) 461–471.

[6] Dontchev, A. L; Farkhi, E. M: Error estimates for discretized differential inclusions. **Computing 41** (1989) 349–358.

[7] Dontchev, A; Lempio, F.: Difference methods for differential inclusions: A survey. **SIAM Reviews 34 (2)** (1994) 263–294.

[8] Ekhaguere, G. O. S. Lipschitzian quantum stochastic differential inclusions, *International Journal of Theoretical Physics* Vol. **31**(11) (1992) 2003–2034.

[9] Ekhaguere, G. O. S. Quantum stochastic differential inclusions of hypermaximal monotone type, *International Journal of Theoretical Physics* **34**(3) (1995) 323–353.

[10] Ekhaguere, G.O.S. Quantum stochastic evolutions, *International Journal of Theoretical Physics* **35**(9) (1996) 1909–1946.

[11] Filippov, A. F. Classical solutions of differential equations with multivalued right hand side, SIAM Journal on Control **5** (1967) 609–621.

[12] Hudson, R. L; Parthasarathy, K. R. Quantum Ito's formulae and stochastic evolutions, *Communications in Mathematical Physics* **93** (1984) 301–324

[13] Michal Kisielewicz: *Differential Inclusions and Optimal Control*, Mathematics and its Applications (East European Series), Volume 44, Kluwer Academic Publishers, Dordrecht, Boston, London, 1991.

[14] Parthasarathy, K. R. An introduction to quantum stochastic calculus, Monographs in Mathematics, Vol. **85**, Birkhause Verlag, 1992.

[15] Repovs, D; Semenov, P.V. : Continuous selections of multivalued mappings. Mathematics and its Applications. Kluwer, Dordrecht, 1998.

[16] Smirnov, G. V.: Introduction to the Theory of Differential Inclusions. Graduate Studies in Mathematics, AMS, Providence, RI Volume 41, 2002.

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In honour of Prof. Ekhaguere at 70

Review of academic research works of Professor G.O.S. Ekhaguere: 1976–2017

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Abstract. The contributions of Professor G.O.S. Ekhaguere between 1976 and 2017 can broadly be classified into 4 groups as follows: (a) contributions to mathematical physics, (b) contributions to non-commutative stochastic analysis, (c) contributions to ∗-algebras and (d) contributions to mathematical finance. All the contributions are significant, breaking new grounds at the frontiers, lead to new enquiries, questions and applications to physical problems. The groups are not mutually exclusive. Results in some of the groups are often applied or employed in other groups.

Keywords: mathematical physics, non-commutative stochastic analysis, ∗-algebras, mathematical finance.

1. Contribution to Mathematical Physics

(a) In 1977, Professor G.O.S. Ekhaguere (GOSE, hereinafter) worked on the Markov properties of stochastic processes due to Nelson and Wong. The study furnished important applications to Mathematics and Physics problems. Further details can be found in **Journal of Mathematical Physics, 18(1977)2104-2107** and reviewed by T. Neabrunn for the AMS published MR.

(b) In 1978, GOSE did not only develop the theory of superselection rules, but also established a wide class of inequivalent irreducible ∗-representations of the canonical commutations relations of the electromagnetic field. He employed the method of C^*-algebra for the representation. These contributions were published in **J. Mathematical Physics, Volume 19,1751-1757 2**, reviewed for the MR by Y. Kato.

He Continued his study of superselection rules in **J. Mathematical Physics, 25 (1984), 678-683** and characterized a subclass of the ∗-representation consisting of positive ∗-representation. He exhibited new superselection sectors.

1.1 *Gaussian fields of Markov types*

In 1979, GOSE established necessary and sufficient conditions for a class of Guassian generalized field to have a Markov property. He showed that Wong's notion of Markov property is weaker than that of Nelson in certain cases. More so, he showed that his results have applications to quantum field theory by employing the theory of Markovian generalized stochastic fields. These eesults were published in **Physics A 99(3) 545-568 1979**, reviewed by Koichiro Matsuno for the MR.

Furthermore he established a theorem in the same year 1979, that new Markov fields may be obtained from old ones by the use of multiplicative measurable operators. He employed the Gudder-Marchand formulation of noncommutative integration. Results were published in **J. Maths Physic. 20(8) 1679-1683 (1979)**, reviewed by Paul Benioff for the MR.

In 1980, GOSE established a characterization of Markovian homogenous multicomponent Gaussian fields. He gave a necessary and sufficient condition for Markov property. Results were published in **Communications in Maths Physics, 63-77 (1980)**.

In 1982, GOSE formulated a noncommutative stochastic process over complete locally convex ∗-algebra and discussed quantum fields as examples. The results were published in **J. Physics A 15(11) 3453-3463** and reviewed by H. Araki for the MR.

1.2 *Central limit theorems*

In 1985, GOSE established some central limit theorems in probability space. He extended the results of Urbanik to the case where the random variables are densely defined self adjoint linear operators on a separable Hilbert space. A 50 page-long contribution was published in **Publications Research Institute Maths Science (1985) V21(3) 541-591**.

2. Contributions to non-commutative stochastic analysis

(a) In 1982, GOSE made a very significant contribution to the theory of non-commutative stochastic integration. He developed stochastic integration with respect to certain Martingales of non-commuting measurable operators and showed by some calculations that his formulation extends the classical Ito integration. The results were published in **J. Nigerian Maths Society Vol 1 (1982), 11-23**.

(b) In 1985, GOSE reformulated some results of Hudson and Patasarathy in the language of Op*-algebra. He established a notion of differentiability within the context of the algebra. Then he established a chain rule for stochastic differentials of suitable integrands. Results were published in **Lecture Notes in Physics 262, Springer, 1986** and reviewed by David Applebaum.

GOSE also established the existence and some properties of solutions of quantum integral equations in **COMO, 1985, 453-455**.

(c) In 1990, GOSE established a major and complicated theorem on the functional Ito formula in quantum stochastic calculus. A noncommutative analogue of the Ito formula for Boson quantum stochastic integrals was developed. His algebraic approach allowed the validity of results for unbounded operators. He introduced an operator algebra of unbounded linear operators on certain Hilbert spaces and considered the locally convex completions of the algebra. Results were published in **J. Maths Physics 31(1990) 2921-2929** and reviewed by Koichiro Matsuno for the MR. It should be noted that the paper was submitted in 1986 but published after 4 years under editorial review.

(d) In 1994, GOSE established stochastic integration in $*$-algebras without Doob-Meyer decomposition theorems. He defined the integration with respect to square integrable Martingales in unital $*$-algebras. He generalized some of his previous results in this direction. Earlier, he had worked on decomposition theorems and established a Doob-Mayer decomposition theorems. The theory was shown to be applicable to algebras generated by annihilation and creation operators on symmetric Fock spaces. Results were published in **J. Nigerian Maths Soc 13 (1994), 9-22 and 81-101** and reviewed by Stanislaw Goldstein.

(e) In 1992, he began to publish series of papers on quantum stochastic differential inclusions of the form

$$\begin{aligned} dx(t) &\in E(t,x(t))d\wedge_\pi(t) + F(t,x(t))dA_f(t) \\ &\quad + G(t,x(t))dA_g^+(t) + H(t,x(t))dt \\ x(t_0) &= x_0, \quad \text{almost all} \quad t \in [t_0,T] \end{aligned}$$

which is understood in integral form as:

$$\begin{aligned} x(t) \in x_0 + \int_{t_0}^{t} (E(s,x(s))d\wedge_\pi(s) + F(s,x(s))dA_f(s) \\ + G(s,x(s))dA_g^+(s) + H(s,x(s))ds) \end{aligned}$$

where the integral is in the sense of Hudson and Pathasarathy.

GOSE first generalised the standard Fock space quantum stochastic calculus to multivalued stochastic integrands. His approach have applications to stochastic controls and stochastic differential equations with discontinuous coefficients. In addition, GOSE established the existence of solutions to Lipschitzian QSDI and those of their convexifications.

In 1995, GOSE proved that a QSDI of hypermaximal monotone type has a unique adapted solution. As examples, he exhibited a large class of QSDI of hypermaximal monotone type arising as perturbations of certain quantum stochastic equations by some multivaled processes.

In 1996, GOSE developed the theory of Quantum stochastic evolutions in parallel with the classical theory. His 3 papers in this area were published in **International Journal on Theoretical Physics, Vol 31 (1992) 2003-2027, Vol 34 (1995) 323-353, Vol 35 (1996) 1909-1946** and reviewed by 2 world class mathematicians K.R Pathasarathy and Camillo Trapani for the AMS MR.

In the year 2007, by endowing the space $\mathcal{A}$ of quantum stochastic processes consisting of a class of linear maps from a preHilbert space to its completion, with seven different types of topologies generated by diverse families of seminorms, GOSE established properties of topological solutions of noncommutative stochastic differential equations in integral form given by:

$$\begin{aligned} dx(t) &= E(t,x(t))d\wedge_\pi(t) + F(t,x(t))dA_f(t) \\ &\quad + G(t,x(t))dA_g^+(t) + H(t,x(t))dt \\ x(t_0) &= x_0, \quad \text{almost all} \quad t \in [t_0, T] \end{aligned}$$

Results were published in **Stochastic Analysis and Applications, 25, 961-991 (2007)**, and reviewed for the MR by Raja Bhat.

3. Contributions to $*$-algebras

In 1988, GOSE worked on the Dirichlet forms of partial $*$-algebras. Some results on C*-algebras were extended to partial $*$-algebras.

He defined on $L^2(\mathcal{A}, \mathcal{B}, \tau)$ over the triple consisting of a partial $*$algebra and Ideal of $\mathcal{A}$ and τ a sesquilinear form satisfying some assumptions. GOSE established Dirichlet forms on $L^2(\mathcal{A}, \mathcal{B}, \tau)$ which are sesquilinear forms whose domain is closed under the actions of Lipschitzian maps. He examined relationship between between Markovian operators and Dirichlet forms. These results were published in **Maths Proceedings, Cambridge Philosophical Society, 104, (1988) 129-140** and reviewed by Camillo Trapani.

In 1989, GOSE worked on unbounded partial Hilbert algebras. He introduced this notion and studied some properties and examples. Results were published in **J. Maths Physics 30, (1989), 1957-1964** and reviewed by Konrad Schmudgen.

In 1991, GOSE worked on Partial W*-dynamical systems and on completely positive conjugate - bilinear maps on partial $*$-algebras. He introduced and studied the partial W*-dynamical systems $(M, \{\Phi_t\}, t \in \mathbb{R}_+)$, where Φ_t is a semigroup of a completely positive conjugate bilinear map on M. He solved the dilation problems and described the associated Markov process. He established major results on the generalization to a particular partial $*$-algebra of Stinesprings concerning completely positive maps on C^*-algebras and on generalization of Radon-Nikodym for $*$ algebras. Results were published in **J. Maths Physics, 32 (1991), 2951-2958**.

In 1993, GOSE worked on non-commutative mean ergodic theorem for partial W^* -dynamical semigroups. He furnished applications to statistical mechanics and quantum field theory and proved the Mean Ergodic Theorem for semigroups of maps on the algebra. The result which generalized Watanabe's Mean Ergodic Theorem was published in **Internal. Journal of Theoretical Physics 32 (1993), 1187-1196** and was reviewed by A.I. Danilenko.

In the year 2001, GOSE established an algebraic representation theory of partial algebras. He employed the notion of operator set. Results were published in the **Annals of Henri Poincare 2, 377-385** and reviewed by Camillo Trapani.

In the year 2007, GOSE examined Bitraces on Partial O*-algebras. He studied some properties of $*$-representations determined by bitraces. He furnished the notion of partial W*-algebras as generalization of W*-algebras. Results were published in **International Journal of Mathematics and Mathematical Sciences, 2007**, and reviewed by Francesco Tschinke for the MR.

In the year 2008, GOSE embarked on characterizations of partial algebras. He established that every locally convex algebra is an inductive limit of locally convex partial algebras. He identified partial algebras that can be represented as partial algebras of unbounded operators. Results were published in **J. Mathematical Analysis and Applications, 337, 1295-1301**.

In the year 2015, GOSE carried out a study on partial W*-dynamical systems and their dilations. He introduced the concepts of a partial O* -algebras and a partial W^*-algebras whose elements are linear operators on a Hilbert space. He described the infinitesimal generators of a $*$-biautomorphism groups and $*$-biderivations of a partial W^*-algebra. He established a relationship between the generators of $*$-biderivatives. Results were published in **Contemporary Maths 645, AMS, Provedence, RI** and reviewed by Chul Ki ko for the MR.

4. Contributions to mathematical finance

In the year 2004, GOSE examined some aspects of the mathematical foundations of the theory of contingent claims in financial markets. He described the classical theory of the pricing of contingent claims in an ideal financial markets and subsequently highlighted some ways of relaxing assumptions of an ideal market in the case of an imperfect or real world financial markets. The article appeared in **Publications of the ICMCS 1, 197-214**.

Viable solutions of lower semicontinuous quantum stochastic differential inclusions

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Viable solutions of lower semicontinuous quantum stochastic differential inclusions

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Abstract
We establish the existence and some properties of viable solutions of lower semicontinuous quantum stochastic differential inclusions within the framework of the Hudson–Parthasarathy formulations of quantum stochastic calculus. The main results here are accomplished by establishing a major auxiliary selection result. The results here extend the classical Nagumo viability theorems,valid on finite dimensional Euclidean spaces, to the present infinite dimensional locally convex space of non-commutative stochastic processes.

Keywords Lower semicontinuous · Nagumo viability · Tangent cone · Fock spaces

Mathematics Subject Classification 60H10 · 60H20 · 81S25

1 Introduction

This paper continues our previous works in [4–9,12–16,19,20]. On this occasion, the existence and some properties of viable solutions of lower semicontinouos quantum stochastic differential inclusions (QSDI) are established. In our previous considerations, existence of solutions were sought and established globally in the locally convex space of solutions. In this work, the global requirement are removed by restricting the solution space to a subset of the entire space satisfying some topological conditions. By employing the multivalued analogue of quantum stochastic calculus developed by Hudson and Parthasarathy [17], in the framework of [13,19] the main results of this paper are established.

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It is well known that classical differential inclusions could be solved by reducing them to differential equations through selection theorems. By employing a similar idea, this paper employs a non commutative generalization of Michael selection result established in [20] to transform the lower semicontinuous quantum stochastic differential inclusions under consideration to a quantum stochastic differential equation.

The existence of viable solutions of differential equations and inclusions defined on finite dimensional Euclidean spaces have been well studied, see, for example [1–3,10,11,18,21,22]. However, similar classes of problems have not been well studied for QSDI. This is a major motivation for this work. In the classical finite dimensional setting, a necessary and sufficient condition for the existence of viable solutions was established by Nagumo [22] in which the closed subset K, which is the viability subset is bounded and satisfy the tangential condition. Other researchers have similarly worked on further developments, and the extensions of Nagumo theorems and applications see [10,11,18,22].

However, for the Nagumo-type fixed point results to work in the present non commutative settings, this paper first established an auxiliary result by circumventing certain difficulties using the unique properties of the family of seminorms that defines the topology for the underling locally convex space of non commutative stochastic processes.

The rest of the paper is organized as follows: Sect. 2 is devoted to the preliminaries and some notations. The main results on viability of solutions including the convergence of approximate solutions are established in Sect. 3.

2 Preliminaries

Let D be an inner product space and H, the completion of D. We denote by $L^+(D, H)$, the set $\{X : D \to H : X$ is a linear map satisfying $Dom\ X^* \supseteq D$, where X^* is the operator adjoint of $X\}$.
We remark that $L^+(D, H)$ is a linear space under the usual notions of addition and scalar multiplication of operators.
In what follows, $\mathbb{D}$ is some inner product space with R as its completion, and γ is some fixed Hilbert space.

For each $t \in \mathbb{R}_+$, we write $L^2_\gamma(\mathbb{R}_+)$, (resp. $L^2_\gamma([0, t))$ resp. $L^2_\gamma([t, \infty))$), for the Hilbert spaces of square integrable, γ-valued maps on $\mathbb{R}_+ \equiv [0, \infty)$, (resp. $[0, t)$; resp. $[t, \infty)$). Then we introduce the following spaces:

(i) $\mathcal{A} \equiv L^+(\mathbb{D}\underline{\otimes}\,\mathbb{E},\ \mathcal{R} \otimes \Gamma(L^2_\gamma(\mathbb{R}_+))$.
(ii) $\mathcal{A}_t \equiv L^+(\mathbb{D}\underline{\otimes}\,\mathbb{E}_t, \mathcal{R} \otimes \Gamma(L^2_\gamma([0, t))) \otimes 1^t$.
(iii) $\mathcal{A}^t \equiv 1_t \otimes L^+(\mathbb{D}\underline{\otimes}\,\mathbb{E}^t, \mathcal{R} \otimes \Gamma(L^2_\gamma([t, \infty))), t > 0$

where $\underline{\otimes}$ denotes algebraic tensor product and 1_t (resp. 1^t) denotes the identity map on $\mathcal{R} \otimes \Gamma(L^2_\gamma([0, t)))(\text{resp.}\,\Gamma(L^2_\gamma([t, \infty)))$, $t > 0$. We note that $\mathcal{A}^t$ and $\mathcal{A}_t$, $t > 0$, may be naturally identified with subspaces of $\mathcal{A}$. For $\eta, \xi \in \mathbb{D}\underline{\otimes}\,\mathbb{E}$, define $\|\cdot\|_{\eta\xi}$ on $\mathcal{A}$ by $\|x\|_{\eta\xi} = |\langle\eta, x\xi\rangle|$, $x \in \mathcal{A}$. Then $\{\|\cdot\|_{\eta\xi}, \eta, \xi \in \mathbb{D}\underline{\otimes}\,\mathbb{E}\}$ is a family of seminorms on $\mathcal{A}$; we write τ_w for the locally convex Hausdorff topology on $\mathcal{A}$ determined by this family. We denote by $\tilde{\mathcal{A}}$, $\tilde{\mathcal{A}}_t$ and $\tilde{\mathcal{A}}^t$ the completions of the locally convex topological spaces $(\mathcal{A}, \tau_w)$, $(\mathcal{A}_t, \tau_w)$ and $(\mathcal{A}^t, \tau_w)$, $t > 0$, respectively.

We define the Hausdorff topology on clos($\widetilde{\mathcal{A}}$) as follows: For $x \in \widetilde{\mathcal{A}}$, $\mathcal{M}, \mathcal{N} \in$ clos($\widetilde{\mathcal{A}}$) and $\eta, \xi \in \mathbb{D}\underline{\otimes}\mathbb{E}$, define

$$\rho_{\eta\xi}(\mathcal{M}, \mathcal{N}) \equiv \max(\delta_{\eta\xi}(\mathcal{M}, \mathcal{N}), \delta_{\eta\xi}(\mathcal{N}, \mathcal{M}))$$

where

$$\delta_{\eta\xi}(\mathcal{M}, \mathcal{N}) \equiv \sup_{x\in\mathcal{M}} \mathbf{d}_{\eta\xi}(x, \mathcal{N}) \text{ and}$$
$$\mathbf{d}_{\eta\xi}(x, \mathcal{N}) \equiv \inf_{y\in\mathcal{N}} \parallel x - y \parallel_{\eta\xi} .$$

The Hausdorff topology which shall be employed in what follows, denoted by, τ_H , is generated by the family of pseudometrics $\{\rho_{\eta\xi}(.) : \eta\xi \in \mathbb{D}\underline{\otimes}\mathbb{E}\}$.
Moreover, if $\mathcal{M} \in clos(\widetilde{\mathcal{A}})$, then $\parallel \mathcal{M} \parallel_{\eta\xi}$ is defined by

$$\parallel \mathcal{M} \parallel_{\eta\xi} \equiv \rho_{\eta\xi}(\mathcal{M}, \{0\});$$

for arbitrary $\eta, \xi \in \mathbb{D}\underline{\otimes}\mathbb{E}$.
For $A, B \in clos(C)$ *and* $x \in C$, a complex number, define

$$\mathbf{d}(x, \mathcal{B}) \equiv \inf_{y\in\mathcal{B}} | x - y |$$
$$\delta_{\xi}(\mathcal{A}, \mathcal{B}) \equiv \sup_{x\in\mathcal{A}} \mathbf{d}_{\xi}(x, \mathcal{B}) \text{ and}$$
$$\rho(\mathcal{A}, \mathcal{B}) \equiv max(\delta(\mathcal{A}, \mathcal{B}), \delta(\mathcal{A}, \mathcal{B}))$$

Then ρ is a metric on $clos(C)$ and induces a metric topology on the space.
We define the Hausdorff topology on clos($\widetilde{\mathcal{A}}$) as follows: For $x \in \widetilde{\mathcal{A}}$, $\mathcal{M}, \mathcal{N} \in$ clos($\widetilde{\mathcal{A}}$) and $\eta, \xi \in \mathbb{D}\underline{\otimes}\mathbb{E}$, define

$$\rho_{\eta\xi}(\mathcal{M}, \mathcal{N}) \equiv \max(\delta_{\eta\xi}(\mathcal{M}, \mathcal{N}), \delta_{\eta\xi}(\mathcal{N}, \mathcal{M}))$$

where

$$\delta_{\eta\xi}(\mathcal{M}, \mathcal{N}) \equiv \sup_{x\in\mathcal{M}} \mathbf{d}_{\eta\xi}(x, \mathcal{N}) \text{ and}$$
$$\mathbf{d}_{\eta\xi}(x, \mathcal{N}) \equiv \inf_{y\in\mathcal{N}} \parallel x - y \parallel_{\eta\xi} .$$

The Hausdorff topology which shall be employed in what follows, denoted by, τ_H , is generated by the family of pseudometrics $\{\rho_{\eta\xi}(.) : \eta\xi \in \mathbb{D}\underline{\otimes}\mathbb{E}\}$.

Definition 1 Let $I \subseteq \mathbb{R}_+$,

(i) A map $X : I \to \tilde{\mathcal{A}}$ is called a stochastic process indexed by I.
(ii) A stochastic process X is called adapted if $X(t) \in \tilde{\mathcal{A}}_t$ for each $t \in I$.
We denote by $Ad(\tilde{\mathcal{A}})$ the set of all adapted stochastic processes indexed by I.
(iii) A member X of $Ad(\tilde{\mathcal{A}})$ is called

(a) weakly absolutely continuous if the map $t \to \langle \eta, X(t)\xi \rangle,\ t \in I$, is absolutely continuous for arbitrary $\eta, \xi \in \mathbb{D}\underline{\otimes}\ \mathbb{E}$. We denote this subset of $Ad(\tilde{\mathcal{A}})$ by $Ad(\tilde{\mathcal{A}})_{wac}$.

(b) locally absolutely p-integrable if $\|X(\cdot)\|^p_{\eta\xi}$ is Lebesgue measurable and integrable on $[t_0, t) \subseteq I$ for each $t \in I$, $p \in (0, \infty)$ and arbitrary $\eta, \xi \in \mathbb{D}\underline{\otimes}\ \mathbb{E}$. We denote this subset of $Ad(\tilde{\mathcal{A}})$ by $L^p_{loc}(\tilde{\mathcal{A}})$.

Stochastic Integrators: Let $B(\gamma)$ denote the Banach space of bounded endomorphisms of γ and let the spaces $L^\infty_{\gamma,loc}(\mathbb{R}_+)$ and $L^\infty_{B(\gamma),loc}(\mathbb{R}_+)$ be defined by: $L^\infty_{\gamma,loc}(\mathbb{R}_+) = \{f : \mathbb{R}_+ \longrightarrow \gamma | f$ is linear, measurable and locally bounded function on $\mathbb{R}_+\}$. $L^\infty_{B(\gamma),loc}(\mathbb{R}_+) = \{\pi : \mathbb{R}_+ \longrightarrow B(\gamma) | \pi$ is linear, measurable and locally bounded function on $\mathbb{R}_+\}$.

For $f \in L^\infty_{\gamma,loc}(\mathbb{R}_+)$ and $\pi \in L^\infty_{B(\gamma),loc}(\mathbb{R}_+)$, we define $\pi f \in L^\infty_{\gamma,loc}(\mathbb{R}_+)$ by $(\pi f)(t) = \pi(t) f(t),\ t \in \mathbb{R}_+$. Also, for $f \in L^2_\gamma(\mathbb{R}_+)$ and $\pi \in L^\infty_{B(\gamma),loc}(\mathbb{R}_+)$, we define the operators $a(f)$, $a^+(f)$ and $\lambda(\pi)$ in $L^+(\mathbb{D}, \Gamma(L^2_\gamma(\mathbb{R}_+)))$ as follows:

$$a(f)e(g) = \langle f, g\rangle L^2_\gamma(\mathbb{R}_+) e(g)$$

$$a^+(f)e(g) = \frac{d}{d\sigma} e(g + \sigma f)|\sigma = 0$$

$$\lambda(\pi)e(g) = \frac{d}{d\sigma} e(e^{\sigma\pi} f)|\sigma = 0$$

for $g \in L^2_\gamma(\mathbb{R}_+)$.

The operators $a(f)$, $a^+(f)$ and $\lambda(\pi)$ for arbitrary $f \in L^\infty_{\gamma,loc}(\mathbb{R}_+)$ and $\pi \in L^\infty_{B(\gamma),loc}(\mathbb{R}_+)$ give rise to the operator-valued maps A_f, A^+_f and Λ_π defined by

$$A_f(t) \equiv a(f\chi[0, t))$$

$$A^+_f(t) \equiv a^+(f\chi[0, t))$$

$$\Lambda_\pi(t) \equiv \lambda(\pi\chi[0, t))$$

$t \in \mathbb{R}_+$ where χ_I denotes the indicator function of the Borel set $I \subseteq \mathbb{R}_+$.

The operators $a(f)$, $a^+(f)$ and $\lambda(\pi)$ are the annihilation, creation and gauge operators of quantum field theory. The maps A_f, A^+_f and Λ_π are stochastic processes, called the annihilation, creation and gauge processes, respectively when their values are identified with their ampliations on $\mathcal{R}\underline{\otimes}\ \Gamma(L^2_\gamma(\mathbb{R}_+))$. These are the stochastic integrators in the Hudson and Parthasarathy [17] formulation of Boson quantum stochastic integration which we adopt in the sequel.

2.1 Quantum stochastic differential inclusion

Definition 2 (1) By a multivalued stochastic process indexed by $I \subseteq R_+$, we mean a multifunction on I with values in clos$(\tilde{\mathcal{A}})$.

(2) If Φ is a multivalued stochastic process indexed by $I \subseteq R_+$, then a selection of Φ is a stochastic process $X : I \to \tilde{\mathcal{A}}$ with the property that $X(t) \in \Phi(t)$ for almost all $t \in I$.
(3) A multivalued stochastic process Φ will be called (i) adapted if $\Phi(t) \subseteq \tilde{\mathcal{A}}_t$ for each $t \in R_+$; (ii) measurable if $t \mapsto d_{\eta\xi}(x, \Phi(t))$ is measurable for arbitrary $x \in \tilde{\mathcal{A}}, \eta, \xi \in D\underline{\otimes}\, \mathbb{E}$; (iii) locally absolutely p-integrable if $t \mapsto \|\Phi(t)\|_{\eta\xi}, t \in R_+$, lies in $L^p_{loc}(I)$ for arbitrary $\xi \in D\underline{\otimes}\, \mathbb{E}$

The set of all locally absolutely p-integrable multivalued stochastic processes will be denoted by $L^p_{loc}(\tilde{\mathcal{A}})_{mvs}$. For $p \in (0, \infty)$ and $I \subseteq R_+$, $L^p_{loc}(I \times \tilde{\mathcal{A}})_{mvs}$ is the set of maps $\Phi : I \times \tilde{\mathcal{A}} \to clos(\tilde{\mathcal{A}})$ such that $t \mapsto \Phi(t, X(t)), t \in I$, lies in $L^p_{loc}(\tilde{\mathcal{A}})_{mvs}$ for every $X \in L^p_{loc}(\tilde{\mathcal{A}})$. If $\Phi \in L^p_{loc}(I \times \tilde{\mathcal{A}})_{mvs}$, then

$$L_p(\Phi) \equiv \{\varphi \in L^p_{loc}(\tilde{\mathcal{A}}) : \varphi \text{ is a selection of } \Phi\}$$

Let $f, g \in L^\infty_{\Upsilon,loc}(R_+), \pi \in L^\infty_{B(\Upsilon),loc}(R_+)$, 1 is the identity map on , $\mathcal{R} \otimes \Gamma(L^2_\gamma(R_+))$, and M is any of the stochastic processes A_f, A^+_g, Λ_π , and $s \mapsto s1, s \in R_+$. We introduce stochastic integral (resp. differential) expressions as follows. If $\Phi \in L^2_{loc}(I \times \tilde{\mathcal{A}})_{mvs}$ and $(t, X) \in I \times L^2_{loc}(I \times \tilde{\mathcal{A}})$, then we make the definition

$$\int_{t0}^{t} \Phi(s, X(s))dM(s) \equiv \left\{\int_{t0}^{t} \varphi(s)\, dM(s) : \varphi \in L_2(\Phi)\right\}$$

This leads to the following definition.

Definition 3 Let $E, F, G, H \in L^2_{loc}(I \times \tilde{\mathcal{A}})_{mvs}$ and (t_0, x_0) be a fixed point of $I \times \tilde{\mathcal{A}}$. Then, a relation of the form

$$\begin{aligned} dX(t) &\in +E(t, X(t)d\Lambda_\pi(t) + F(t, X(t)dA_f(t) \\ &+G(t, X(t))dA^+_g(t) + H(t, X(t))dt, \quad t \in I, \end{aligned} \tag{1}$$

is called quantum stochastic differential inclusions(QSDI) with coefficients in E,F,G, H and initial data (t_0, x_0).

Equation (1) is understood in the integral form:

$$\begin{aligned} X(t) \in x_0 + &\int_{t_0}^{t} (E(s, X(s))d\Lambda_\pi(s) + F(s, X(s))dA_f(s)) \\ &+G(s, X(s))dA^+_g(s) + H(s, X(s))ds), \quad t \in I, \end{aligned}$$

called a stochastic integral inclusion with coefficients E,F,G,H and initial data (t_0, x_0) An equivalent form of (1) was established in [13], Theorem 6.2 as follows: For $\eta, \xi \in D\underline{\otimes}\, \mathbb{E}.\alpha, \beta \in L^2_\gamma(R_+)$ with $\eta = c \otimes e(\alpha)$, $\xi = d \otimes e(\beta)$,define the following complex-valued functions:

$$\mu_{\alpha\beta}, \upsilon_\beta, \sigma_\alpha : I \longrightarrow C, I \subset \mathcal{R}_+, by$$

$$\mu_{\alpha\beta}(t) = \langle \alpha(t), \pi(t)\beta(t)\rangle_{\gamma}$$
$$v_{\beta}(t) = \langle f(t), \beta(t)\rangle_{\gamma}$$
$$\sigma_{\alpha}(t) = \langle \alpha(t), g(t)\rangle_{\gamma}$$

$t \in I$, $f, g \in L^2_{\gamma,loc}(R_+)$, $\pi \in L^{\infty}_{B(\gamma),loc}$. To these functions we associate the maps $\mu E, vF, \sigma G, P$ from $I \times \tilde{A}$ into the set of sesquilinear forms on $D\underline{\otimes}\,\mathbb{E}$ define by

$$\begin{aligned}
(\mu E)(t,x)(\eta,\xi) &= \left\{\langle \eta, \mu_{\alpha\beta}(t)p(t,x)\xi\rangle : p(t,x) \in E(t,x)\right\}\\
(vF)(t,x)(\eta,\xi) &= \left\{\langle \eta, v_{\beta}(t)q(t,x)\xi\rangle : q(t,x) \in F(t,x)\right\}\\
(\sigma G)(t,x)(\eta,\xi) &= \{\langle \eta, \sigma_{\alpha}(t)u(t,x)\xi\rangle : u(t,x) \in G(t,x)\}\\
P_{\alpha\beta}(t,x) &= \mu_{\alpha\beta}(t)E(t,x) + v_{\beta}(t)F(t,x) + \sigma_{\alpha}(t)G(t,x) + H(t,x)\\
\mathbb{P}(t,x)(\eta,\xi) &= \langle \eta, P_{\alpha\beta}(t,x)\xi\rangle, i.e\\
P(t,x)(\eta,\xi) &= (\mu E)(t,x)(\eta,\xi) + (vF)(t,x)(\eta,\xi)\\
&\quad +(\sigma G)(t,x)(\eta,\xi) + H(t,x)(\sigma,\xi)\\
H(t,x)(\eta,\xi) &= \left\{v(t,x)(\eta,\xi) : v(.,X(.)) \text{ is a selection of} H(.,X(.))\forall X \in L^2_{loc}(\tilde{A})\right\}
\end{aligned} \tag{2}$$

Then problem (1) is equivalent to

$$\begin{aligned}
\frac{d}{dt}\langle \eta, X(t)\xi\rangle &\in \mathbb{P}(t, X(t))(\eta,\xi)\\
\langle \eta, X(t_0)\xi\rangle &= \langle \eta, x_0\xi\rangle
\end{aligned} \tag{3}$$

for arbitrary $\eta, \xi \in \mathbb{D}\underline{\otimes}\,\mathbb{E}$ and almost all $t \in I$. Hence the existence of solution of (1) implies the existence of solution of (3) and vice-versa. As explained in [13], the sesquilinear form valued map P:

$$\mathbb{P}(t,x)(\eta,\xi) \neq \tilde{\mathbb{P}}(t, \langle \eta, x\xi\rangle)$$

For some complex-valued multifunction $\tilde{P}$ defined on $I \times C$ for $t \in I$, $x \in \tilde{A}$, $\eta, \xi \in D\underline{\otimes}\,\mathbb{E}$.

Before proceeding to the proof of the main result in this work , we make use of a result in [20] in which the multifunction $(t,x) \longrightarrow \mathbb{P}(t,x)(\eta,\xi)$ is lower semi continuous with respect to the seminorm $\|.\|_{\eta\xi}$, closed and convex.

then there exists a continuous selection, $P : I \times \tilde{A} \longrightarrow sesq(\mathbb{D}\underline{\otimes}\mathbb{E})$ of $\mathbb{P}$ such that

$$\begin{aligned}
\frac{d}{dt}\langle \eta, X(t)\xi\rangle &= P(t, X(t))(\eta,\xi)\\
\langle \eta, X(t_0)\xi\rangle &= \langle \eta, x_0\xi\rangle \ a.e \ \ t \in I
\end{aligned} \tag{4}$$

for arbitrary pair $\eta, \xi \in \mathbb{D}\underline{\otimes}\mathbb{E})$,$(t,x) \longrightarrow P(t,x)(\eta,\xi)$ is continuous.

3 Viability theory

Definition 4 Let $P : I \times \tilde{\mathcal{A}} \longrightarrow sesq(\mathbb{D}\underline{\otimes}\,\mathbb{E})^2$ be a sesqulinear valued funtion, then the subset K of $\tilde{\mathcal{A}}$ is viable with respect to P if for every $(t_0, x_0) \in I \times K$ there exists $T \in I, T > t_0$ such that Eq. (4) have at least one solution K.

Definition 5 Let $K \subseteq \tilde{\mathcal{A}}$, A subset $K \in clos(\tilde{\mathcal{A}})$ is locally closed if $K(\eta, \xi)$ is a closed subset with values in $\mathbb{C}$ then $K(\eta, \xi)$ is locally closed if for each $x_{\eta\xi} \in K(\eta, \xi)$, there exists $\rho > 0$ such that $D(x_{\eta\xi}, \rho) \cap K(\eta, \xi)$ is closed for arbitrary pair $\eta, \xi \in \mathbb{D}\underline{\otimes}\mathbb{E}$.

We now define tangent cone as it applies to our non commutative settings. We make use of Bouligand–Severi tangency concept in [11].

Definition 6 Let $K \subset \tilde{\mathcal{A}}$, $E \subset \tilde{\mathcal{A}}$ such that $K(\eta, \xi) \subset \mathbb{C}$, $E(\eta, \xi) \subset \mathbb{C}$ and $x \in K$ such that $x_{\eta\xi} \in K(\eta, \xi) \subset \mathbb{C}$. Then the set $E(\eta, \xi)$ is tangent to the set $K(\eta, \xi)$ at the point $x_{\eta\xi}$ if

$$\liminf_{h \mapsto 0} \frac{1}{h} \mathbf{d}(x_{\eta\xi} + hE(\eta, \xi); K(\eta, \xi)) = 0$$

We denote by $\mathcal{T}_{K(\eta,\xi)}$ the class of all sets which are tangent to $K(\eta, \xi)$ at the point $x_{\eta\xi}$ for arbitrary $\eta, \xi \in D\underline{\otimes}\,\mathbb{E}$.

Proposition 1 *The set $\mathcal{T}_{K(\eta,\xi)}(x_{\eta\xi})$ of all vectors which are tangent to the set $K(\eta, \xi)$ at the point $x_{\eta\xi}$ is a closed cone.*

Proof Let $(x_{\eta\xi}) \in K(\eta, \xi)$ According to definition 6, $E(\eta, \xi) \in \mathcal{T}_{K(\eta,\xi)}(x_{\eta\xi})$ if

$$\liminf_{t \mapsto 0} \frac{1}{t} \mathbf{d}(x_{\eta\xi} + tE(\eta, \xi); K(\eta, \xi)) = 0$$

Let $s > 0$, we observe that

$$\begin{aligned}
\liminf_{t \mapsto 0} \frac{1}{t} \mathbf{d}(x_{\eta\xi} + tsE(\eta, \xi); K(\eta, \xi)) &= s \liminf_{t \mapsto 0} \frac{1}{ts} \mathbf{d}(x_{\eta\xi} + tsE(\eta, \xi); K(\eta, \xi)) \\
&= s \liminf_{\tau \mapsto 0} \frac{1}{\tau} \mathbf{d}(x_{\eta\xi} + \tau E(\eta, \xi); K(\eta, \xi))
\end{aligned}$$

Hence, $sE(\eta, \xi) \in \mathcal{T}_K(x_{\eta\xi})$ To complete the proof, we need to show that $\mathcal{T}_K(x_{\eta\xi})$ is a closed set.

Let $\mathbb{N}^*$ be the set of strictly positive natural numbers. Let $(E_n(\eta, \xi))_{n \in \mathbb{N}^*}$ be a sequence of elements in $\mathcal{T}_K(x_{\eta\xi})$, convergent to $E(\eta, \xi)$ then we have

$$\begin{aligned}
\frac{1}{t} \mathbf{d}(x_{\eta\xi} + tE(\eta, \xi); K(\eta, \xi)) &\leq \frac{1}{t} |t(E(\eta, \xi) - E_n(\eta, \xi))| \\
&\quad + \frac{1}{t} \mathbf{d}(x_{\eta\xi} + tE_n(\eta, \xi); K(\eta, \xi)) \\
&= |E(\eta, \xi) - E_n(\eta, \xi)| \\
&\quad + \frac{1}{t} \mathbf{d}(x_{\eta\xi} + tE_n(\eta, \xi); K(\eta, \xi))
\end{aligned}$$

for every $n \in \mathbb{N}^*$. So

$$\liminf_{t\mapsto 0} \frac{1}{t}\mathbf{d}\left(x_{\eta\xi} + tE(\eta,\xi); K(\eta,\xi)\right) \leq |E(\eta,\xi) - E_n(\eta,\xi)|$$

for every $n \in \mathbb{N}^*$. Since $\lim_{n\to\infty} |E(\eta,\xi) - E_n(\eta,\xi)| = 0$, it follows that

$$\liminf_{t\mapsto 0} \frac{1}{t}\mathbf{d}(x_{\eta\xi} + tE(\eta,\xi); K(\eta,\xi)) = 0.$$

which shows that the set $\mathcal{T}_{K(\eta,\xi)}(x_{\eta\xi})$ is a closed cone and this achieves the proof. □

Proposition 2 *A set $E(\eta,\xi) \in \mathbb{C}$ belongs to the cone $\mathcal{T}_{K(\eta,\xi)}(x_{\eta\xi})$ if and only if for every $\epsilon > 0$ there exists $h \in (0,\epsilon)$ and $q_{\eta\xi,h} \in D_{\eta\xi}(0,\epsilon)$ with the property*

$$x_{\eta\xi} + h\left(E(\eta,\xi) + q_{\eta\xi,h}\right) \in K(\eta,\xi)$$

Proof We see that $E(\eta,\xi) \in \mathcal{T}_{K(\eta,\xi)}(x_{\eta\xi})$ if and only if for every $\epsilon > 0$ there exists $h \in (0,\epsilon)$ and $p_{\eta\xi,h} \in K(\eta,\xi)$ such that

$$\frac{1}{h}|x_{\eta\xi} + hE(\eta,\xi) - p_{\eta\xi,h}| \leq \epsilon.$$

let

$$q_{\eta\xi,h} = \frac{1}{h}(p_{\eta\xi,h} - x_{\eta\xi} - hE(\eta,\xi)),$$

and we have both $|q_{\eta\xi,h}| \leq \epsilon$ and $x_{\eta\xi} + h(E(\eta,\xi) + q_{\eta\xi,h}) = p_{\eta\xi,h} \in K(\eta,\xi)$. □

3.1 Main result

In this section, we establish the quantum generalization of Nagumo viability result.

Existence of Approximate Solutions : Let $(t_0, x_{0,\eta\xi}) \in I \times K(\eta,\xi)$, then there exists $\rho > 0$, such that $D(x_{0,\eta\xi},\rho) \cap K(\eta,\xi)$ be closed, then there exists $M_{\eta\xi} > 0$, such that

$$|P(t,x)(\eta,\xi)| \leq M_{\eta\xi} \tag{5}$$

for every $t \in [t_0, T]$ and $x \in D_{\eta\xi}(x,\rho) \cap K \subset \tilde{\mathcal{A}}$ and $x_{\eta\xi} \in D(x_{0,\eta\xi},\rho) \cap K(\eta,\xi)$ and

$$(T - t_0)(M_{\eta\xi} + 1) \leq \rho \tag{6}$$

The existence of these three numbers will be made possible because $K(\eta,\xi)$ is locally closed and by the continuity of P which implies its boundedness on

$[t_0, T] \times D(x_{0,\eta\xi}, \rho)$, and so the existence of $M_{\eta\xi} > 0$, and the fact that $T \in I$, $T > t_0$, is chosen very close to t_0. The following lemma concerns the existence of family of approximate solutions for the problem defined on interval $[t_0, c]$.

Lemma 1 *Suppose $K \subset \tilde{\mathcal{A}} \neq \emptyset$ satisfying the following*

(i) K *is locally closed.*
(ii) $P : I \times \tilde{\mathcal{A}} \longrightarrow sesq(\mathbb{D}\underline{\otimes}\mathbb{E})^2$ *is continuous.*
(iii) $P(t_0, x_0)(\eta, \xi) \in \mathcal{T}_{\mathcal{K}(\equiv,\sim)}(x_{0,\eta\xi})$ *for each* $(t_0, x_0) \in I \times K$.

Then, for each $\epsilon \in (0, 1)$, there exist: a non decreasing function

$$\sigma : [t_0, T] \longrightarrow [t_0, T]$$

and two stochastic processes

$$g : [t_0, T] \longrightarrow \tilde{\mathcal{A}} \tag{7}$$

and

$$\varphi : [t_0, T] \longrightarrow \tilde{\mathcal{A}}$$

lying in $Ad(\tilde{\mathcal{A}})_{wac} \cap L^2_{loc}(\tilde{\mathcal{A}})$ such that the corresponding sesquilinear form valued maps associated with any pair of $\eta, \xi \in \mathbb{D}\underline{\otimes}\mathbb{E})$ given by

$$g : [t_0, T] \longrightarrow sesq(\mathbb{D}\underline{\otimes}\mathbb{E})^2$$

such that

$$g(t)(\eta, \xi) = \langle \eta, g(t)\xi \rangle$$

and

$$\varphi : [t_0, T] \longrightarrow sesq(\mathbb{D}\underline{\otimes}\mathbb{E})^2$$

such that

$$\varphi(t)(\eta, \xi) = \langle \eta, \varphi(t)\xi \rangle$$

satisfy the followin

(i) $t - \epsilon \leq \sigma(t) \leq t$ *for every* $t \in [t_0, T]$
(ii) $|g_{\eta\xi}(t)| \leq \epsilon$ *for every* $t \in [t_0, T]$
(iii) $\varphi_{\eta\xi}(\sigma(t)) \in D(x_{0,\eta\xi}, \rho) \cap K(\eta, \xi)$ *for every* $t \in [t_0, T]$ *and* $\varphi_{\eta\xi}(T) \in D(x_{0,\eta\xi}, \rho) \cap K(\eta, \xi)$

(iv) φ *satisfies*

$$\langle \eta, \varphi(t)\xi \rangle = \langle \eta, x_0\xi \rangle + \int_{t_0}^{t} P(\sigma(s), \varphi(\sigma(s))(\eta, \xi)ds + \int_{t_0}^{t} g(s)(\eta, \xi)ds$$

for every $t \in [t_0, T]$

A pair of the triple (σ, g, φ) *as in Eq. (7) with the associated* $(\sigma, g_{\eta\xi}, \varphi_{\eta\xi})$ *satisfying (i), (ii), (iii) and (iv) above is called an* ϵ*- approximate solution to the problem (4) on the interval* $[t_0, T]$.

Proof Let $t_0 \in I$, $x_{0,\eta\xi} \in K(\eta, \xi)$ and let $\rho > 0$, $M > 0$ and $T > t_0$ be as above. Let $\epsilon \in (0, 1)$. We first show the existence of an ϵ- approximate solution on an interval $[t_0, c]$ with $c \in (t_0, T]$.
Since for every $(t_0, x_{0,\eta\xi}) \in I \times K(\eta, \xi)$, $P(t_0, x_0)(\eta, \xi) \in \mathcal{T}_{\mathcal{K}(\equiv,\sim)}(x_{0,\eta\xi})$, from Proposition (2), it follows that there exists $c \in (t_0, T]$, $c - t_0 \leq \epsilon$ and $q_{\eta\xi,h}$ has values in $\mathbb{C}$ with $|\ q_{\eta\xi,h}\ | \leq \epsilon$ such that

$$x_{0,\eta\xi} + (c - t_0)P(t_0, x_0)(\eta, \xi) + (c - t_0)q_{\eta\xi}, h \in K(\eta, \xi)$$

Let $I_c = [t_0, c]$, we now define the functions $\sigma : [t_0, c] \longrightarrow [t_0, c]$, and stochastic processes $g : [t_0, c] \longrightarrow \tilde{\mathcal{A}}$ and $\varphi : [t_0, c] \longrightarrow \tilde{\mathcal{A}}$ lying in $Ad(\tilde{\mathcal{A}})_{wac} \cap L^2_{loc}(\tilde{\mathcal{A}})$ satisfying

$$\begin{cases} \sigma(t) = t_0 & \text{for } t \in [t_0, c], , \\ g_{\eta\xi}(t) = q_{\eta\xi} & \text{for } t \in [t_0, c], \\ \varphi_{\eta\xi}(t) = x_{0,\eta\xi} + (t - t_0)P(t_0, x_0)(\eta, \xi) + (t - t_0)q_{\eta\xi} & \text{for } t \in [t_0, c]. \end{cases}$$

The triple $(\sigma, g_{\eta\xi}, \varphi_{\eta\xi})$ is an ϵ approximate solution to the problem (4) on the interval $[t_0, c]$. This shows that conditions (i), (ii) and (iv) are satisfied, we now show that condition (iii) is also satisfied using (5), (6) and (i). From (i) $\sigma(t) = t_0$ and $\langle \eta, X(t_0)\xi \rangle = \langle \eta, x_0\xi \rangle$,then $\langle \eta, \varphi(\sigma(t))\xi \rangle = \langle \eta, x_0\xi \rangle$, therefore we have $\varphi(\sigma(t))(\eta, \xi) \in D(x_{0,\eta\xi}, \rho) \cap K(\eta, \xi)$ for every $t \in [t_0, c]$. Therefore, $\varphi(c)(\eta, \xi) \in K(\eta, \xi)$. However, by (5) and (6),we have

$$\begin{aligned} |\varphi(c)(\eta, \xi) - \varphi_0(\eta, \xi)| \leq\ & (c - t_0)\,|P(t_0, \varphi_0)(\eta, \xi)| + (c - t_0)\,|q| \\ & \leq (T - t_0)(M_{\eta\xi} + 1) \leq \rho \end{aligned}$$

For every $t \in [t_0, c]$. Thus (iii) is also satisfied.

We now define the $\epsilon-$ approximate solution on the whole interval I. We make use of Brezis–Browder theorem in [8]. Let $\mathcal{S}$ be the set of all ϵ—approximate solutions to the problem (4) defined on the interval $[t_0, c]$ with $c \in (t_0, T]$. On $\mathcal{S}$ we define the relation " $\preceq''$ by $(\sigma_1, g_{1,\eta\xi}, \varphi_{1,\eta\xi}) \preceq (\sigma_2, g_{2,\eta\xi}, \varphi_{2,\eta\xi})$ if the domain of definition $[t_0, c_1]$ of the first triple is included in the domain of definition $[t_0, c_2]$ of the second triple, and the two ϵ—approximate solutions coincide on the common part of the domains. Then, " $\preceq''$is a pre-order relation on $\mathcal{S}$. Firstly, we show that each increasing sequence $((\sigma_m, g_{m,\eta\xi}, \varphi_{m,\eta\xi}))_m$ is bounded from above . Let $((\sigma_m, g_{m,\eta\xi}, \varphi_{m,\eta\xi}))_m$ be an increasing sequence, and let $c^* = lim_m c_m$ where $[t_0, c_m]$ denotes the domain of definition of $(\sigma_m, g_{m,\eta\xi}, \varphi_{m,\eta\xi})$. Then $c^* \in (t_0, T]$.

We will show that there exists at least one element, $(\sigma^*, g^*_{\eta\xi}, \varphi^*_{\eta\xi}) \in \mathcal{S}$, defined on $[t_0, c^*]$ and satisfying $(\sigma_m, g_{m,\eta\xi}, \varphi_m) \preceq (\sigma^*, g^*_{\eta\xi}, x^*_{\eta\xi})$ for each $m \in \mathbb{N}$. In order to do this, we first prove that there exists $\lim_m \varphi_m(c_m)(\eta, \xi)$.

For each $m, n \in \mathbb{N}, m \leq n$ we have $u_m(s) = u_n(s)$ for all $s \in [t_0, c_m]$. Taking into account (iii), (iv) and (5), we have

$$\begin{aligned} | \varphi_m(c_m)(\eta, \xi) - \varphi_n(c_n)(\eta, \xi) | &\leq \int_{c_m}^{c_n} | P(\sigma_n(s), \varphi_n(\sigma_n(s)))(\eta, \xi) | \, ds \\ &\quad + \int_{c_m}^{c_n} | g_n(s)(\eta, \xi) | \, ds \\ &\leq (M_{\eta\xi} + \epsilon) \mid c_n - c_m \mid \end{aligned}$$

for every $m, n \in \mathbb{N}$, which shows that there exists

$$\lim_{m \mapsto \infty} \varphi_m(c_m)(\eta, \xi) \in D(x_{0,\eta\xi}, \rho) \cap K(\eta, \xi)$$

Furthermore, because all the functions in the set $\{\sigma_m : m \in \mathbb{N}\}$ are non decreasing, with values in $[t_0, c^*]$, and satisfy

$\sigma_m(c_m) \leq \sigma_p(c_p)$ for every $m, p \in \mathbb{N}$, there exists $\lim_{m \mapsto \infty} \sigma_m(c_m)$, then the limit exists and belongs to $[t_0, c^*]$. We now define a triple function $(\sigma^*, g^*_{\eta\xi}, \varphi^*_{\eta\xi}) : [t_0, c^*] \longrightarrow [t_0, c^*] \times \mathbb{C} \times \mathbb{C}$ by

$$\sigma^*(t) = \begin{cases} \sigma_m(t) & \text{for } t \in [t_0, c_m], m \in \mathbb{N}, \\ \lim\limits_{m \mapsto \infty} \sigma_m(c_m) & \text{for } t = c^*, \end{cases}$$

$$g^*_{\eta\xi}(t) = \begin{cases} g_m(t)(\eta, \xi) & \text{for } t \in [t_0, c_m], m \in \mathbb{N}, \text{for all } \eta, \xi \in \mathbb{D}\underline{\otimes}\mathbb{E}), \\ 0 & \text{for } t = c^*, \end{cases}$$

$$\varphi^*_{\eta\xi}(t) = \begin{cases} \varphi_m(t)(\eta, \xi) & \text{for } t \in [t_0, c_m], m \in \mathbb{N}, \text{for all } \eta, \xi \in \mathbb{D}\underline{\otimes}\mathbb{E}), \\ \lim\limits_{m \mapsto \infty} \varphi_m(c_m)(\eta, \xi) & \text{for } t = c^*, \end{cases}$$

This shows that $(\sigma^*, g^*_{\eta\xi}, \varphi^*_{\eta\xi})$ is an ϵ- approximate solution which is an upper bound for $((\sigma_m, g_{m,\eta\xi}, \varphi_{m,\eta\xi}))_m$. Applying (ii) of Brezis–Browder theorem, we define the function

$\mathcal{M} : \mathcal{S} \longrightarrow \mathbb{R} \cup \{+\infty\}$. Then, for each $\zeta_0 \in \mathcal{S}$ there exists an $\mathcal{M}$- maximal element $\bar{\zeta} \in \mathcal{S}$ satisfying $\zeta_0 \preceq \bar{\zeta}$. This shows that $\mathcal{M}((\sigma, g_{\eta\xi}, \varphi_{\eta\xi})) = c$ where $[t_0, c]$ is the domain of definition of $(\sigma, g_{\eta\xi}, \varphi_{\eta\xi})$. Then $\mathcal{M}$ satisfies the hypothesis of Brezis–Browder theorem . Then, $\mathcal{S}$ contains at least one $\mathcal{M}$- maximal element $(\bar{\sigma}, \bar{g}_{\eta\xi}, \bar{\varphi}_{\eta\xi})$ defined on $[t_0, \bar{c}]$. In other words, if $(\tilde{\sigma}, \tilde{g}_{\eta\xi}, \tilde{\varphi}_{\eta\xi}) \in \mathcal{S}$, defined on $[t_0, \tilde{c}]$, satisfies $(\bar{\sigma}, \bar{g}_{\eta\xi}, \bar{\varphi}_{\eta\xi}) \preceq (\tilde{\sigma}, \tilde{g}_{\eta\xi}, \tilde{\varphi}_{\eta\xi})$, then we necessarily have $\bar{c} = \tilde{c}$. We will show next that $\bar{c} = T$. we assume by contradiction that $\bar{c} < T$. Then, taking into account the fact that $\bar{\varphi}_{\eta\xi}(\bar{c}) \in D(x_{0,\eta\xi}, \rho) \cap K(\eta, \xi)$, we have

$$
\begin{aligned}
&| \varphi_{\bar{\eta}\xi}(\bar{c}) - x_{0,\eta\xi} | \\
&\leq \int_{t_0}^{\bar{c}} | P(\bar{\sigma}(s), \bar{\varphi}(\bar{\sigma}(s)))(\eta, \xi) | ds + \int_{t_0}^{\bar{c}} | \bar{g}(\eta, \xi)(s) | ds \\
&\leq (\bar{c} - t_0)(M_{\eta\xi} + \epsilon) \\
&\leq (\bar{c} - t_0)(M_{\eta\xi} + 1) < (T - t_0)(M_{\eta\xi} + 1) \leq \rho
\end{aligned}
$$

Then, as $\bar{\varphi}_{\eta\xi}(\bar{c}) \in K(\eta, \xi)$ and $P(\bar{c}, \bar{\varphi}(\bar{c}))(\eta, \xi) \in \mathcal{T}_K(\bar{\varphi}(\bar{c}))(\eta, \xi)$, there exists $\delta(0, T - \bar{c})$, $\delta \leq \epsilon$ and $q_{\eta\xi} \in \mathbb{C}$ such that $| q_{\eta\xi} | \leq \epsilon$ and

$$
\bar{\varphi}_{\eta\xi}(\bar{c}) + \delta P(\bar{c}, \bar{\varphi}(\bar{c}))(\eta, \xi) + \delta q_{\eta\xi} \in K(\eta, \xi)
$$

From the inequality above we have

$$
| \bar{\varphi}(\bar{c})(\eta, \xi) + \delta[P(\bar{c}, \bar{\varphi}(\bar{c}))(\eta, \xi) + q_{\eta\xi}] - \varphi_0(\eta, \xi) | \leq \rho
$$

We now define the functions $\sigma : [t_0, \bar{c} + \delta] \longrightarrow [t_0, \bar{c} + \delta]$ and $g : [t_0, \bar{c} + \delta] \longrightarrow \mathbb{C}$ by

$$
\sigma(t) = \begin{cases} \bar{\sigma}(t) & \text{for } t \in [t_0, \bar{c}], \\ \bar{c} & \text{for } t \in [\bar{c}, \bar{c} + \delta], \end{cases}
$$

$$
g_{\eta\xi}(t) = \begin{cases} \bar{g}_{\eta\xi}(t) & \text{for } t \in [t_0, \bar{c}], \text{and for any } \eta, \xi \in \mathbb{D}\underline{\otimes}\mathbb{E}), \\ q & \text{for } t \in [\bar{c}, \bar{c} + \delta], \end{cases}
$$

so $| g_{\eta\xi}(t) | \leq \epsilon$ for every $t \in [t_0, \bar{c} + \delta]$. In addition, for every $t \in [t_0, \bar{c} + \delta]$, $\sigma(t) \in [t_0, \bar{c}]$ and therefore $\bar{\varphi}(\sigma(t))$ is well-defined and belongs to the set $D(x_{0,\eta\xi}, \rho) \cap K$. Accordingly, we can define $\varphi_{\eta\xi} : [t_0, \bar{c} + \delta] \longrightarrow \mathbb{C}$ by

$$
\langle \eta, \varphi(t)\xi \rangle = \langle \eta, \varphi_0 \xi \rangle + \int_{t_0}^{t} P(\sigma(s), \bar{\varphi}(\sigma(s)))(\eta, \xi) ds + \int_{t_0}^{t} g(\eta, \xi)(s) ds
$$

for every $t \in [t_0, \bar{c} + \delta]$. clearly, $\varphi_{\eta,\xi}$ coincides with $\varphi_{\bar{\eta},\xi}$ on $[t_0, \bar{c}]$ since the domain $[t_0, \bar{c}]$ is included in the domain of $[\bar{c}, \bar{c} + \delta]$ and then it readily follows that $\varphi_{\eta,\xi}$, σ and $g_{\eta,\xi}$ satisfy all the conditions in (i) and (ii). In order to prove (iii) and (iv) we observe that

$$
\varphi_{\eta\xi}(t) = \begin{cases} \bar{\varphi}_{\eta\xi}(t) & \text{for } t \in [t_0, \bar{c}]. \\ \varphi_{\eta\xi}(\bar{c}) + (t - \bar{c}) P(\bar{c}, \bar{\varphi}(\bar{c}))(\eta\xi) + (t - \bar{c}) q & \text{for } t \in [\bar{c}, \bar{c} + \delta] \end{cases}
$$

Then $\varphi_{\eta\xi}$ satisfies the equation in (iv). since

$$
\varphi_{\eta\xi}(\sigma(t)) = \begin{cases} \bar{\varphi}_{\eta\xi}(\bar{\sigma}(t) & \text{for } t \in [t_0, \bar{c}]. \\ \bar{\varphi}_{\eta\xi}(\bar{c}) & \text{for } t \in [t_0, \bar{c} + \delta] \end{cases}
$$

it follows that $\varphi_{\eta\xi}(\sigma(t)) \in D(x_{0,\eta\xi}, \rho) \cap K$.

Furthermore, from the choice of δ and q, we have both $\varphi_{\eta\xi}(\bar{c}+\delta) = \bar{\varphi_{\eta\xi}}(\bar{c})(\eta, \xi) + \delta P(\bar{c}, \bar{\varphi}(\bar{c}))(\eta, \xi) + \delta q \in K(\eta, \xi)$ and

$$\begin{aligned} &| \varphi(\bar{c} + \delta)(\eta, \xi) - x_0(\eta, \xi) | \\ &\quad = | \bar{\varphi}(\bar{c})(\eta, \xi) + \delta P(\bar{c}, \bar{\varphi}(\bar{c}))(\eta, \xi) + \delta q - x_0(\eta, \xi) | \\ &\quad \leq \rho \end{aligned}$$

and consequently, $\varphi_{\eta\xi}$ satisfies (iii). Thus $(\sigma, g_{\eta\xi}, \varphi_{\eta\xi}) \in \mathcal{S}$. Furthermore, since $(\bar{\sigma}, \bar{g}_{\eta\xi}, \bar{\varphi}_{\eta\xi}) \preceq (\sigma, g_{\eta\xi}, \varphi_{\eta\xi})$ and $\bar{c} < \bar{c} + \delta$, it follows that $(\bar{\sigma}, \bar{g}_{\eta\xi}, \bar{\varphi}_{\eta\xi})$ is not a $\mathcal{M}$- maximal element. But this is absurd, we can eliminate this contradiction, only if each maximal element in the set $\mathcal{S}$ is defined on $[t_0, T]$. Hence $\bar{c} = T$.

Theorem 1 *Let $K \subset \tilde{\mathcal{A}}$ Assume that the following conditions hold:*

(i) The map $(t, x) \longrightarrow P(t, x)(\eta, \xi)$ associated with the right hand-side of equation (4) is continuous.
(ii) $K(\eta, \xi)$ is non empty and locally closed
(iii) There exists $M_{\eta\xi} > 0$ such that $|P(t, x)(\eta, \xi)| \leq M_{\eta\xi}$ for every $t \in [t_0, T]$ and $x \in D(x_{0,\eta\xi}, \rho) \cap K(\eta, \xi)$
(iv) $(T - t_0)(M_{\eta\xi} + 1) \leq \rho$

Then $K(\eta, \xi)$ is viable with respect to P if and only if for every $(t_0, x_0) \in I \times K$ we have $P(t_0, x_0)(\eta, \xi) \in \mathcal{T}_{\mathcal{K}(\equiv,\sim)}(x_{0,\eta\xi})$

Proof The proof is divided into two parts; We proceed as follows: **If Part**: Suppose $K(\eta, \xi)$ is viable with respect to P, then there exists a solution φ that satisfy Eq. (4).

Let $(t_0, x_{0,\eta\xi}) \in I \times K(\eta, \xi)$, We prove that

$$\lim_{h \mapsto 0} \frac{1}{h} \mathbf{d}(x_{0,\eta\xi} + hP(t_0, x_0)(\eta, \xi); K(\eta, \xi)) = 0.$$

then, there exists $T \in I, T > t_0$, and a stochastic process $\varphi \in K$ with $\langle \eta, \varphi(t_0)\xi \rangle \in K(\eta, \xi)$ satisfying Eq. (4).

$$\begin{aligned} &\mathbf{d}(x_{0,\eta\xi} + hP(t_0, x_0)(\eta, \xi); K(\eta, \xi)) \\ &\quad \leq \left| x_{0,\eta\xi} + hP(t_0, x_0)(\eta, \xi) - \varphi(t_0)(\eta, \xi) \right| \\ &\quad = \lim_{h \mapsto 0} \frac{1}{h} \left| x_{0,\eta\xi} + hP(t_0, x_0)(\eta, \xi) - \langle \eta, \varphi(t_0 + h)\xi \rangle \right| \\ &\quad = \lim_{h \mapsto 0} \left| P(t_0, \varphi(t_0))(\eta, \xi) - \frac{\langle \eta, (\varphi(t_0 + h) - \varphi(t_0)\xi \rangle}{h} \right| \\ &\quad = \left| P(t_0, \varphi(t_0))(\eta, \xi) - \lim_{h \mapsto 0} \frac{\langle \eta, (\varphi(t_0 + h) - \varphi(t_0)\xi \rangle}{h} \right| \\ &\quad = \left| P(t_0, \varphi(t_0))(\eta, \xi) - \frac{d}{dt} \langle \eta, \varphi(t)\xi \rangle \right|_{t=t_0} \\ &\quad = 0 \end{aligned}$$

This shows that the stochastic process φ is a solution to Eq. (4) and belongs to K.

Only If Part Suppose $P(t_0, x_0)(\eta, \xi) \in \mathcal{T}_{\mathcal{K}(\equiv,\sim)}$ then we prove that P is viable to K.

This concerns the existence and convergence of approximate solutions.

The proof is divided into two steps. The first step is concerned with the proof of existence of a family of "approximate solutions" for the problem defined on interval $[t_0, c]$ with $c \in I$ and later showed that the problem above admits such approximate solutions, all defined on an interval $[t_0, T]$ independent of the "approximate order". The proof of the approximate solution is given by lemma 1 Finally, in the second step, we shall prove the uniform convergence on $[t_0, T]$ of a sequence of such approximate solutions to a solution of the problem (4). □

3.2 Convergence of approximate solutions

Let $(\epsilon_k)_{k\in\mathbb{N}}$ be a sequence from (0, 1) decreasing to 0 and let $(\sigma_k, g_{\eta\xi,k}, \varphi_{\eta\xi,k})_{k\in\mathbb{N}}$ be a sequence of ϵ_k- approximate solutions defined on $[t_0, T]$.
From (i) and (ii), it follows that

$$\lim_{k\mapsto\infty} \sigma_k(t) = t \quad \text{and} \lim_{k\mapsto\infty} g_{\eta\xi,k}(t) = 0 \tag{8}$$

uniformly on $[t_0, T]$. On the other hand, from (iii), (iv) and (6) we have

$$\begin{aligned}
&|\langle\eta, \varphi_k(t)\xi\rangle| \\
&\quad \leq |\langle\eta, (\varphi_k(t) - \varphi_0)\xi\rangle| + |\langle\eta, \varphi_0\xi\rangle| \\
&\quad \leq \int_{t_0}^{T} \mid P(\sigma_k(s), \varphi_k(\sigma_k(s)))(\eta, \xi) \mid ds + \int_{t_0}^{T} |g_k(s)(\eta, \xi)|ds + |\varphi_{0,\eta\xi}| \\
&\quad \leq (T - t_0)(M_{\eta\xi} + 1) + |\varphi_0|_{\eta\xi} \leq \rho + |\varphi_{0,\eta\xi}|
\end{aligned}$$

for every $k \in \mathbb{N}$ and every $t \in [t_0, T]$. Hence, the sequence $(\varphi_k)_{k\in\mathbb{N}}$ is uniformly bounded on $[t_0, T]$. Again from (iv), we have

$$\begin{aligned}
&|\langle\eta, \varphi_k(t) - \varphi_k(s)\xi\rangle| \\
&\quad \leq |\int_s^t \frac{d}{dt}\langle\eta, \varphi_k(t)\xi\rangle dt_0| + |\int_s^t |\langle\eta, g_k(t_0)\xi\rangle|dt_0 \,| \\
&\quad \leq | P(\sigma_k(s), \varphi_k(\sigma_k(s)))(\eta, \xi) \mid ds + |g_k(s)(\eta, \xi)|ds \\
&\quad \leq (M_{\eta\xi} + 1) \, |t - s|
\end{aligned}$$

for every $t, s \in [t_0, T]$. Consequently the sequence $(\varphi_k)_{k\in\mathbb{N}}$ is equicontinuous on $[t_0, T]$. However from Arzela - Ascolis theorem there exists at least a subsequence of $(\varphi_{\eta\xi,k})_{k\in\mathbb{N}}$ that is uniformly convergent to some point $\varphi_{\eta\xi}$. i.e there exists a stochastic process $\varphi : [t_0, T] \longrightarrow \tilde{\mathcal{A}}$ lying in$Ad(\tilde{\mathcal{A}})_{wac} \cap L^2_{loc}(\tilde{\mathcal{A}})$ such that $\varphi_{\eta\xi} = \langle\eta, \varphi\xi\rangle$ and $\varphi_{\eta\xi,k} = \langle\eta, \varphi_k\xi\rangle$ then,

$$\lim_{k\to\infty} \langle\eta, \varphi_k\xi\rangle = \langle\eta, \lim_{k\to\infty} \varphi_k\xi\rangle$$

$$= \langle \eta, \varphi \xi \rangle$$

Now using (iii), (8) and of the fact that $D(x_{0,\eta\xi}, \rho) \cap K(\eta, \xi)$ is closed, we conclude that $\varphi(t)(\eta, \xi) \in D(x_{0,\eta\xi}, \rho) \cap K(\eta, \xi)$ for every $t \in [t_0, T]$.

$$\langle \eta, \varphi_k(t)\xi \rangle = \langle \eta, \varphi_0 \xi \rangle + \int_{t_0}^{t} P(\sigma_k(t_0), \varphi_k(\sigma_k(s)))(\eta, \xi) ds + \int_{t_0}^{t} g_k(s)(\eta, \xi) ds$$

now, taking the limit of the above and using (8), we have that

$$\langle \eta, \varphi(t)\xi \rangle = \langle \eta, \varphi_0 \xi \rangle + \int_{t_0}^{t} P(s, \varphi(s))(\eta\xi) ds$$

for every $t \in [t_0, T]$, which gives the proof of the theorem.

Compliance with ethical standards

Conflict of interest The authors declare that there is no conflict of interest.

References

1. Aubin, J.: Viability Theory. Birkhauser, Basel (1991)
2. Aubin, J., Bayen, A., Pierre, P.: Viability Theory New Directions. Springer, Berlin (2011)
3. Aubin, J.P., Cellina, A.: Differential Inclusions; Set-Valued Maps and Viability Theory, pp. 139–171. Springer, Berlin (1984)
4. Ayoola, E.O.: Lipschitzian quantum stochastic differential equations and the associated Kurzweil equations. Stoch. Anal. Appl. **19**(4), 581–603 (2001)
5. Ayoola, E.O.: On convergence of one-step schemes for weak solutions of quantum stochastic differential equations. Acta Appl. Math. **67**(1), 19–58 (2001)
6. Ayoola, E.O.: Continuous selections of solution sets of Lipschitzian quantum stochastic differential inclusions. Int. J. Theor. Phys. **43**(10), 2041–2059 (2004)
7. Ayoola, E.O.: Topological properties of solution sets of Lipschitzian quantum stochastic differential inclusions. Acta Appl. Math. **100**(1), 15–37 (2008)
8. Ayoola, E.O.: Error estimates for discretized quantum stochastic differential inclusions. Stoch. Anal. Appl. **21**(6), 1215–1230 (2003)
9. Ayoola, E.O.: Exponential formula for the reachable sets of quantum stochastic differential inclusions. Stoch. Anal. Appl. **21**(3), 515–543 (2003)
10. Carja, O., Necula, M., Vrabie, I.I.: Viability, Invariance and Applications, vol. 207. Elsevier, Amsterdam (2007)
11. Cârja, O., Vrabie, I.I.: Differential equations on closed sets. In: Canada, A., Drabek, P., Fonda, A. (eds.) Handbook of Differential Equations, vol. 2, pp. 147–238. Elsevier, North Holland (2005)
12. Ekhaguere, G.: The functional Ito formula in quantum stochastic calculus. J. Math. Phys. **31**(12), 2921–2929 (1990)
13. Ekhaguere, G.: Lipschitzian quantum stochastic differential inclusions. Int. J. Theor. Phys. **31**(11), 2003–2027 (1992)
14. Ekhaguere, G.: Quantum stochastic differential inclusions of hypermaximal monotone type. Int. J. Theor. Phys. **34**(3), 323–353 (1995)
15. Ekhaguere, G.: Quantum stochastic evolutions. Int. J. Theor. Phys. **35**(9), 1909–1946 (1996)
16. Ekhaguere, G.: Topological solutions of noncommutative stochastic differential equations. Stoch. Anal. Appl. **25**(5), 961–993 (2007)
17. Hudson, R.L., Parthasarathy, K.R.: Quantum Ito's formula and stochastic evolutions. Commun. Math. Phys. **93**(3), 301–323 (1984)

18. Marques, M.D.M., et al.: Viability results for nonautonomous differential inclusions. J. Convex Anal. **7**(2), 437–443 (2000)
19. Ogundiran, M.O., Ayoola, E.O.: Upper semicontinuous quantum stochastic differential inclusions via Kakutani-fan fixed point theorem. Dyn. Syst. Appl. **21**, 121–132 (2012)
20. Ogundiran, M.O., Ayoola, E.O.: Lower semicontinuous quantum stochastic differential inclusions. Eur. J. Math. Sci. **2**(1), 1–16 (2013)
21. Smirnov, G.V.: Introduction to the Theory of Differential Inclusions, vol. 41. American Mathematical Society, Providence (2002)
22. Vrabie, I.I.: A nagumo type viability theorem. An. Stiint. Univ. Al. I. Cuza Iasi. Mat. (NS) **51**, 293–308 (2005)

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LIPSCHITZIAN QUANTUM STOCHASTIC DIFFERENTIAL EQUATIONS AND THE ASSOCIATED KURZWEIL EQUATIONS

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ABSTRACT

Kurzweil or generalized differential equations associated with Lipschitzian quantum stochastic differential equations (QSDEs) are introduced and studied. This is accomplished within the framework of the Hudson-Parthasarathy formulations of quantum stochastic calculus. Results concerning the equivalence of these classes of equations satisfying the Caratheodory conditions are presented. It is further shown that the associated Kurzweil equation may be used to obtain a reasonably high accurate approximate solutions of the QSDEs. This generalize analogous results for classical initial value problems to the noncommutative quantum setting involving unbounded linear operators on a Hilbert space. Numerical examples are given.

Key Words: QSDE; Fock spaces; Exponential vectors; Kurzweil equations and integrals; Noncommutative stochastic processes.

AMS Subject Classification (1991): 60H10, 60H20, 65L05, 81S25.

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1. INTRODUCTION

The role of generalized ordinary differential or Kurzweil equations in applying topological dynamics to the study of ordinary differential equations as well as their semigroup properties outlined in Artstein [1] is an interesting motivation for studying this class of equations associated with the weak forms of the Lipschitzian quantum stochastic differential equations.

In the framework of the Hudson-Parthasarathy [13] formulations of quantum stochastic calculus, existence,uniqueness and the equivalent forms of Lipschitzian quantum stochastic differential equations have been established. In the formulations of Ekhaguere [8], the equivalent form is a first order initial value ordinary differential equation of a nonclassical type having a sesquilinear form - valued map as the right hand side (see [2,3,4]).

We remark that the equivalent form of QSDEs facilitates the introduction and study of the associated Kurzweil equations. This is accomplished in the framework of the Kurzweil integral calculus (called the generalized Perron integral calculus in the formulations of [18]). The results obtained here are generalizations of analogous results due to references [1,5,6,7,18] concerning classical initial value problems to the noncommutative quantum setting involving unbounded linear operators on a Hilbert space.

Consequently, the technique of topological dynamics can be applied to QSDEs as outlined in [1] by embedding the equivalent forms of these equations in the space of the associated Kurzweil equations when sufficient analytical properties of these equations have been developed. This question as well as the applications of this concept to quantum fields/systems will be addressed elsewhere.

Finally, since the construction of Kurzweil integrals is a simple extension of the Riemann theory of integration based on Riemann type integral sums,we use this fact to obtain discrete approximations of weak solutions of QSDEs using the associated Kurzweil equations.

Our numerical experiments show that the approximation methods developed in this paper are of a reasonably high level of accuracy than the Euler scheme and some multistep schemes considered in [4]. Moreover, the methods here are applicable to a wider class of equations than the considerations in [4] since we work with pure Caratheodory conditions. The rest of the paper is organised as follows: In section 2, we outline some of the concepts which feature in the subsquent analysis including the Kurzweil integral and some of its properties that are of interest in respect of noncummutative quantum stochastic processes.

The Kurzweil equations associated with quantum stochastic differential equation and some results on approximation of matrix elements of solution of the equation are established in section 3. Sections 4 and 5 contain the major results of this paper. In section 4, we derive a necessary and sufficient condition for a sesquilinear form-valued map to be Kurzweil integrable. We then show that the space of Kurzweil integrable sesquilinear form-valued maps contains sesquilinear form-valued maps that satisfy the Caratheodory conditions. In section 5, we employ our results in the previous section to prove that the weak form of quantum stochastic differential equation and its associated Kurzweil equation are equivalent. We then employ our approximation results of section 3 to generate approximate values of the weak solution of quantum stochastic differential equation formulated in Kurzweil form in section 6. We present some numerical examples.

In what follows, as in [2,3,4,8,9,10] we employ the locally convex topological state space $\tilde{\mathcal{A}}$ of noncommutative stochastic processes and we adopt the definitions and notations of spaces $Ad(\tilde{\mathcal{A}})$, $Ad(\tilde{\mathcal{A}})_{wac}$, $L^p_{loc}(\tilde{\mathcal{A}})$, $L^\infty_{\gamma,loc}(I\!R_+)$, and the integrator processes $\wedge_\pi$, A^+_g, A_f, for $f, g \in L^\infty_{\gamma,loc}(I\!R_+)$, $\pi \in L^\infty_{B(\gamma),loc}(I\!R_+)$. For E, F, G, H lying in $L^2_{loc}[\tilde{\mathcal{A}} \times I]$, we consider the quantum stochastic differential equation in integral form given by

$$X(t) = X_0 + \int_{t_0}^{t} (E(X(s), s)d \wedge_\pi (s) + F(X(s), s)dA_f(s) \\ + G(X(s), s)dA^+_g(s) + H(X(s), s)ds), \quad t \in I, \tag{1.1}$$

where the integral in equation (1.1) is understood in the sense of Hudson and Parthasarathy [13]. However, Ekhaguere [8] has shown that equation (1.1) is equivalent to the following first order initial value nonclassical ordinary differential equation

$$\frac{d}{dt}\langle \eta, X(t)\xi\rangle = P(X(t), t)(\eta, \xi) \\ X(t_0) = X_0, \quad t \in [t_0, T] \tag{1.2}$$

As explained in [2,3,4,8–10], the map P appearing in equation (1.2) has the form

$$P(x, t)(\eta, \xi) = (\mu E)(x, t)(\eta, \xi) + (\gamma F)(x, t)(\eta, \xi) + (\sigma G)(x, t)(\eta, \xi) \\ + H(x, t)(\eta, \xi) \tag{1.3}$$

$\eta, \xi \in I\!D \underline{\otimes} I\!E$, $(x, t) \in \tilde{\mathcal{A}} \times I$ where $H(x, t)(\eta, \xi) := \langle \eta, H(x, t)\xi\rangle$.

The map P may sometimes be written in the form $P(x, t)(\eta, \xi) = \langle \eta, P_{\alpha\beta}(x, t)\xi\rangle$ where $P_{\alpha\beta} : \tilde{\mathcal{A}} \times I \longrightarrow \tilde{\mathcal{A}}$ is given by

$$P_{\alpha\beta}(x, t) = \mu_{\alpha\beta}(t)E(x, t) + \gamma_\beta(t)F(x, t) + \sigma_\alpha(t)G(x, t) + H(x, t)$$

for $(x, t) \in \tilde{\mathcal{A}} \times I$.

Equation (1.2) is known to have a unique weakly absolutely continuous adapted solution $\Phi : I \rightarrow \tilde{\mathcal{A}}$ for the Lipschitzian coefficients E, F, G, H.

2. KURZWEIL INTEGRALS ASSOCIATED WITH QUANTUM STOCHASTIC PROCESSES

Let $\eta, \xi \in \mathbb{D}\underline{\otimes}\mathbb{E}$ be arbitrary. Assume that $U : [t_0, T] \times [t_0, T] \to \tilde{\mathcal{A}}$ is an $\tilde{\mathcal{A}}$-valued map of two variables $\tau, t \in [t_0, T]$. We consider the family of complex valued functions: $U(\tau, t)(\eta, \xi) : = \langle \eta, U(\tau, t)\xi\rangle$ for arbitrary $\eta, \xi \in \mathbb{D}\underline{\otimes}\mathbb{E}$ associated with the map U. We shall use the notation $\int_{t_0}^{T} DU(\tau, t)(\eta, \xi)$ to denote the Kurzweil integral of $U(\tau, t)(\eta, \xi)$ in the sense and notations of Artstein [1] and using the formulations of Schwabik [18] and write

$$S(U, D)(\eta, \xi) = \sum_{j=1}^{k}[U(\tau_j, t_j)(\eta, \xi) - U(\tau_j, t_{j-1})(\eta, \xi)]$$

for the Riemann-Kurzweil sum corresponding to the function $U(\tau, t)(\eta, \xi)$ and partition $D : t_0 < \tau_1 < t_1 < \ldots . < t_k = T$ of $[t_0, T]$.

If $f : [t_0, T] \to \tilde{\mathcal{A}}$ is a stochastic process, then for arbitrary $\eta, \xi \in \mathbb{D}\underline{\otimes}\mathbb{E}$, we set $U(\tau, t)(\eta, \xi) = (\langle \eta, f(\tau)\xi\rangle)t$ for $\tau, t \in [t_0, T]$ and therefore we have

$$\begin{aligned} S(U, D)(\eta, \xi) &= \sum_{j=1}^{k}[U(\tau_j, t_j)(\eta, \xi) - U(\tau_j, t_{j-1})(\eta, \xi)] \\ &= \sum_{j=1}^{k}[\langle \eta, f(\tau_j)\xi\rangle(t_j - t_{j-1})] \end{aligned}$$

representing the classical Riemann sum for the function $f_{\eta\xi}(t) := \langle \eta, f(t)\xi\rangle$ and a given partition D of $[t_0, T]$. In this case, we write

$$\int_{t_0}^{T} \langle \eta, f(s)\xi\rangle ds = \int_{t_0}^{T} D[f_{\eta\xi}(\tau), t]$$

provided that the Kurzweil integral $\int_{t_0}^{T} DU(\tau, t)(\eta, \xi)$ exists in this case. Hence

$$\int_{t_0}^{T} DU(\tau, t)(\eta, \xi) = \int_{t_0}^{T} D[f_{\eta\xi}(\tau), t] = \int_{t_0}^{T} f_{\eta\xi}(s)ds. \tag{2.1}$$

We remark that by Theorem (1.16) (Schwabik [18]) if $U : [t_0, T] \times [t_0, T] \to \mathbb{C}$ be such that U is Kurzweil integrable over $[t_0, T]$, then for $c \in [t_0, T]$, we have

$$\lim_{s\to c}\left[\int_{t_0}^{s} DU(\tau, t) - U(c, s) + U(c, c)\right] = \int_{t_0}^{c} DU(\tau, t) \tag{2.2}$$

For several properties enjoyed by Kurzweil integrals and the existence of at least one ∂-fine partition D of $[t_0, T]$ for a given gauge ∂, we refer to Chapter 1 and Lemma (1.4) in Schwabik [18].

We now introduce the Kurzweil equations associated with equation (1.2).

3. KURZWEIL EQUATIONS ASSOCIATED WITH QUANTUM STOCHASTIC DIFFERENTIAL EQUATIONS

(i) Let the map $P : \tilde{\mathcal{A}} \times [t_0, T] \to sesq(\mathbb{D}\underline{\otimes}\mathbb{E})$ be given by equation (1.3). Then we refer to the equation

$$\frac{d}{d\tau}\langle \eta, X(\tau)\xi\rangle = DP(X(\tau), t)(\eta, \xi) \tag{3.1}$$

as the Kurzweil equation associated with equation (1.2).

(ii) A map $\Phi : [t_0, T] \to \tilde{\mathcal{A}}$ is called a solution of equation (3.1) if

$$\langle \eta, \Phi(s_2)\xi\rangle - \langle \eta, \Phi(s_1)\xi\rangle = \int_{s_1}^{s_2} DP(\Phi(\tau), t)(\eta, \xi) \tag{3.2}$$

holds for every $s_1, s_2 \in [t_0, T]$ identically.

The integral on the right hand side of equation (3.2) is the Kurzweil integral introduced in section 2. Equation (3.1) is understood in integral form (3.2) via its solution.

We have the following results as immediate consequences of our definitions.

Proposition 3.1. *If a map $\Phi : [t_0, T] \to \tilde{\mathcal{A}}$ is a solution of the Kurzweil equation* (3.1) *on $[t_0, T]$, then for every $u \in [t_0, T]$, we have*

$$\langle \eta, \Phi(s)\xi\rangle = \langle \eta, \Phi(u)\xi\rangle + \int_{u}^{s} DP(\Phi(\tau), t)(\eta, \xi), \ \ s \in [t_0, T] \tag{3.3}$$

Conversely if a map $\Phi : [t_0, T] \to \tilde{\mathcal{A}}$ satisfies the integral equation (3.3) *for some $u \in [t_0, T]$ and all $s \in [t_0, T]$ then Φ is a solution of equation* (3.1).

Proof: The first statement follows directly from the definition of a solution of (3.1) when we put $s_1 = u$ and $s_2 = s$. Conversely, if $\Phi : [t_0, T] \to \tilde{\mathcal{A}}$ satisfies the integral equation (3.3) then by the additivity of the integral, equation (3.2) follows.

Proposition 3.2. *If $\Phi : [t_0, T] \to \tilde{\mathcal{A}}$ is a solution of equation* (3.1) *on $[t_0, T]$ then*

$$\lim_{s\to\sigma}[\langle \eta, \Phi(s)\xi\rangle - P(\Phi(\sigma), s)(\eta, \xi) + P(\Phi(\sigma), \sigma)(\eta, \xi)] \\ = \langle \eta, \Phi(\sigma)\xi\rangle \tag{3.4}$$

Proof: Let $\sigma \in [t_0, T]$ be fixed. Then by Proposition (3.1) we have

$$\langle \eta, \Phi(s)\xi\rangle - \int_{\sigma}^{s} DP(\Phi(\tau), t)(\eta, \xi) = \langle \eta, \Phi(\sigma)\xi\rangle$$

therefore

$$\langle\eta,\Phi(s)\xi\rangle - P(\Phi(\sigma),s)(\eta,\xi) + P(\Phi(\sigma),\sigma)(\eta,\xi) - \int_\sigma^s DP(\Phi(\tau),t)(\eta,\xi) + P(\Phi(\sigma),s)(\eta,\xi) - P(\Phi(\sigma),\sigma)(\eta,\xi) - \langle\eta,\Phi(\sigma)\xi\rangle = 0 \tag{3.5}$$

for every $s \in [t_0, T]$.

By equation (2.2)

$$\lim_{s\to\sigma}\left[\int_\sigma^s DP(\Phi(\tau),t)(\eta,\xi) - P(\Phi(\sigma),s)(\eta,\xi) + P(\Phi(\sigma),\sigma)(\eta,\xi)\right] = 0 \tag{3.6}$$

Equation (3.6) and (3.5) yield the existence of the limit given by

$$\lim_{s\to\sigma}[\langle\eta,\Phi(s)\xi\rangle - P(\Phi(\sigma),s)(\eta,\xi) + P(\Phi(\sigma),\sigma)(\eta,\xi) - \langle\eta,\Phi(\sigma)\xi\rangle]$$

as well as the relation

$$\lim_{s\to\sigma}[\langle\eta,\Phi(s)\xi\rangle - P(\Phi(\sigma),s)(\eta,\xi) + P(\Phi(\sigma),\sigma)(\eta,\xi) - \langle\eta,\Phi(\sigma)\xi\rangle] = 0$$

which gives (3.4).

Remark 3.3. By virtue of Proposition (3.2), the following approximation holds:- If $\Phi : [t_0, T] \to \tilde{\mathcal{A}}$ is a solution of equation (3.1), then for every $\sigma \in [t_0, T]$ and for arbitrary $\eta, \xi \in \mathbb{D}\underline{\otimes}\mathbb{E}$, the matrix element

$$\langle\eta,\Phi(s)\xi\rangle \;\cong\; \langle\eta,\Phi(\sigma)\xi\rangle + P(\Phi(\sigma),s)(\eta,\xi) - P(\Phi(\sigma),\sigma)(\eta,\xi),$$

provided that s in $[t_0, T]$ is sufficiently close to σ.

We now introduce a class of sesquilinear form - valued maps $P : \tilde{\mathcal{A}} \times [t_0, T] \longrightarrow sesq(\mathbb{D}\underline{\otimes}\mathbb{E})$, which are Kurzweil integrable.

4. A CLASS OF KURZWEIL INTEGRABLE SESQUILINEAR FORM - VALUED MAPS

In what follows, we adopt some notations and terminologies employed in [18, Chapter 1]. For each $\eta, \xi \in \mathbb{D}\underline{\otimes}\mathbb{E}$, let $h_{\eta\xi} : [t_0, T] \to \mathbb{R}$ be a family of nondecreasing functions defined on $[t_0, T]$ and $W : [0, \infty) \to I\!R$ be a continuous and increasing function such that $W(0) = 0$. Then we say that the map $P : \tilde{\mathcal{A}} \times [t_0, T] \longrightarrow sesq(I\!D\underline{\otimes}I\!E)$ belongs to the class $I\!F(\tilde{\mathcal{A}} \times [t_0, T], h_{\eta\xi}, W)$ for each $\eta, \xi \in I\!D\underline{\otimes}I\!E$ if for all $x, y \in \tilde{\mathcal{A}}, t_1, t_2 \in [t_0, T]$

(i) $|P(x, t_2)(\eta, \xi) - P(x, t_1)(\eta, \xi)| \le |h_{\eta\xi}(t_2) - h_{\eta\xi}(t_1)|$ (4.1.)

(ii) $|P(x, t_2)(\eta, \xi) - P(x, t_1)(\eta, \xi) - P(y, t_2)(\eta, \xi) + P(y, t_1)(\eta, \xi)| \le W(\|x - y\|_{\eta\xi})|h_{\eta\xi}(t_2) - h_{\eta\xi}(t_1)|$ (4.2.)

We now present a number of results which are simple extensions of similar results in Schwabik [18] to the present noncommutative quantum setting. The next Theorem is an extension of the convergence results of Corollary 1.31 in [18]. The proof follows exact arguments as in [18].

Theorem 4.1. *Assume that the following conditions hold* :

(i) *the maps* $U, U_m : [t_0, T] \times [t_0, T] \to \tilde{\mathcal{A}}$ *are such that* $(\tau, t) \to U_m(\tau, t)(\eta, \xi)$ *are real valued and Kurzweil integrable over* $[t_0, T]$ *for each* $\eta, \xi \in I\!D\underline{\otimes}I\!E$, $\forall\, m = 1, 2, \ldots$

(ii) *there is a gauge* w *on* $[t_0, T]$ *such that for every* $\epsilon > 0$, *there exists a map* $p : [t_0, T] \to I\!N$ *and a family of positive superadditive interval functions* $\Phi_{\eta\xi}$ *on* $[t_0, T]$ *defined for closed intervals* $J \subset [t_0, T]$ *with* $\Phi_{\eta\xi}([t_0, T]) < \epsilon$ *such that for every* $\tau \in [t_0, T]$

$$|U_m(\tau, J)(\eta, \xi) - U(\tau, J)(\eta, \xi)| < \Phi_{\eta\xi}(J)$$

provided that $m > p(\tau)$, *and* (τ, J) *is an* w-*fine tagged interval with* $\tau \in J \subseteq [t_0, T]$.

(iii) *there exist real valued Kurzweil integrable functions*

$$V_{\eta\xi}, W_{\eta\xi} : [t_0, T] \times [t_0, T] \to I\!R$$

and a gauge θ *on* $[t_0, T]$ *such that for all* $m \in I\!N$, $\tau \in [t_0, T]$,

$$V_{\eta\xi}(\tau, J) \le U_m(\tau, J)(\eta, \xi) \le W_{\eta\xi}(\tau, J).$$

for any θ-*fine interval* (τ, J), $\forall\, \eta, \xi \in I\!D\underline{\otimes}I\!E$. *Then the map* $(\tau, t) \to U(\tau, t))(\eta, \xi)$ *is Kurzweil integrable over* $[t_0, T]$ *and that*

$$\lim_{m\to\infty} \int_{t_0}^{T} DU_m(\tau, t)(\tau, t)(\eta, \xi) = \int_{t_0}^{T} DU(\tau, t)(\eta, \xi).$$

The next Theorem concerns some fundamental properties of Kurzweil integrals in the framework of [18].

Theorem 4.2.

(i) *Let* $U : [t_0, T] \times [t_0, T] \longrightarrow \tilde{\mathcal{A}}$ *be such that* $(\tau, t) \to U(\tau, t)(\eta, \xi)$ *is Kurzweil integrable over* $[t_0, c]$ *for* $c \in [t_0, T]$ *and that the limit*

$$\lim_{c \to T-} \left[\int_{t_0}^{c} DU(\tau, t)(\eta, \xi) - U(T, c)(\eta, \xi) + U(T, T)(\eta, \xi) \right] = I. \tag{4.3}$$

exists for all $\eta, \xi \in \mathbb{D}\underline{\otimes}\mathbb{E}$. *Then* $\int_{t_0}^{T} DU(\tau, t)(\eta, \xi)$ *exists and equals* I.

(ii) *Let* $U : [t_0, T] \times [t_0, T] \longrightarrow \tilde{\mathcal{A}}$ *be such that* $(\tau, t) \to U(\tau, t)(\eta, \xi)$ *is Kurzweil integrable over* $[c, T]$ *for every* $c \in (t_0, T]$ *and that the limit*

$$\lim_{c \to t_0+} \left[\int_{c}^{T} DU(\tau, t)(\eta, \xi) + U(t_0, c)(\eta, \xi) - U(t_0, t_0)(\eta, \xi) \right] = I \tag{4.4}$$

exists for all $\eta, \xi \in \mathbb{D}\underline{\otimes}\mathbb{E}$. *Then* $\int_{t_0}^{T} DU(\tau, t)(\eta, \xi)$ *exists and equals* I.

(iii) *Let* $U : [t_0, T] \times [t_0, T] \longrightarrow \tilde{\mathcal{A}}$ *be such that* $(\tau, t) \to U(\tau, t)(\eta, \xi)$ *is Kurzweil integrable over* $[t_0, T]$. *Then for* $c \in [t_0, T]$

$$\lim_{s \to c} \left[\int_{t_0}^{s} DU(\tau, t)(\eta, \xi) - U(c, s)(\eta, \xi) + U(c, c)(\eta, \xi) \right] = \int_{t_0}^{c} DU(\tau, t)(\eta, \xi) \tag{4.5}$$

for all $\eta, \xi \in \mathbb{D}\underline{\otimes}\mathbb{E}$.

Proof: The proofs are simple adaptation of arguements employed in Theorem 1.14, Remark 1.15 and Theorem 1.16 in [18] to the present noncommutative quantum setting.

Next, we present some results concerning the existence of the integral involved in the definition of the solution of the Kurzweil equation (3.1).

Theorem 4.3. *Assume that the map* $(x, t) \to P(x, t)(\eta, \xi)$ *belongs to* $\mathbb{F}.(\tilde{\mathcal{A}} \times [t_0, T], h_{\eta\xi}, W)$, *and* $X : [a, b] \to \tilde{\mathcal{A}}$, $[a, b] \subseteq [t_0, T]$ *is the limit of a sequence* $\{X_k\}_{k \in \mathbb{N}}$ *of processes* $X_k : [a, b] \to \tilde{\mathcal{A}}$ *such that* $\int_a^b DP(X_k(\tau), t)(\eta, \xi)$ *exists for every* $k \in \mathbb{N}$. *Then the integral* $\int_a^b DP(X(\tau), t)(\eta, \xi)$ *exists and*

$$\int_a^b DP(X(\tau), t)(\eta, \xi) = \lim_{k \to \infty} \int_a^b DP(X_k(\tau), t)(\eta, \xi)$$

Proof: Since the complex field $\mathbb{C} \cong \mathbb{R}^2$, we assume without any lost of generality that the map $P(X(\tau), t)(\eta, \xi)$ is real valued. Let $\epsilon > 0$ be given, then by (4.2), we have

$$\begin{aligned} |P(X_k(\tau), t_2)(\eta, \xi) - P(X_k(\tau), t_1)(\eta, \xi) - P(X(\tau), t_2)(\eta, \xi) \\ + P(X(\tau), t_1)(\eta, \xi)| \leq W(\|X_k(\tau) - X(\tau)\|_{\eta\xi})|h_{\eta\xi}(t_2) - h_{\eta\xi}(t_1)| \end{aligned} \tag{4.6}$$

for every $\tau \in [a, b]$, $t_1 < \tau < t_2$, and $[t_1, t_2] \subset [a, b]$.
If we set

$$U_{\eta\xi}(t) = \frac{\epsilon}{h_{\eta\xi}(b) - h_{\eta\xi}(a) + 1} h_{\eta\xi}(t),$$

for $t \in [a, b]$ then the function $U_{\eta\xi} : [a, b] \longrightarrow \mathbb{R}$ is nondecreasing and

$$(U_{\eta\xi}(b) - U_{\eta\xi}(a)) < \epsilon.$$

Since $\lim_{k\to\infty} X_k(\tau) = X(\tau)$ in $\tilde{\mathcal{A}}$ for every $\tau \in [a, b]$ and the function W is continuous at 0, then there is a $p = p(\tau) \in \mathbb{N}$ such that for $k > p(\tau)$,

$$W(\|X_k(\tau) - X(\tau)\|_{\eta\xi}) \leq \frac{\epsilon}{h_{\eta\xi}(b) - h_{\eta\xi}(a) + 1}$$

i.e. for $k \geq p(\tau)$, the inequality (4.2) can be written as

$$\begin{aligned} |P(X_k(\tau), t_2)(\eta, \xi) - P(X_k(\tau), t_1)(\eta, \xi) - P(X(\tau), t_2)(\eta, \xi) \\ + P(X(\tau), t_1)(\eta, \xi) \leq U_{\eta\xi}(t_2) - U_{\eta\xi}(t_1) \end{aligned}$$

By inequality (4.1)

$$|P(X_k(\tau), t_2)(\eta, \xi) - P(X_k(\tau), t_1)(\eta, \xi)| \leq |h_{\eta\xi}(t_2) - h_{\eta\xi}(t_1)|$$

for every $t \in [a, b]$, $k \in \mathbb{N}$, $t_1 \leq \tau \leq t_2$ and $[t_1, t_2] \subseteq [a, b]$.
Hence the last inequality implies that

$$\begin{aligned} -h_{\eta\xi}(t_2) + h_{\eta\xi}(t_1) \leq P(X_k(\tau), t_2)(\eta, \xi) - P(X_k(\tau), t_1)(\eta, \xi) \\ \leq h_{\eta\xi}(t_2) - h_{\eta\xi}(t_1) \end{aligned}$$

but the integrals

$$\int_a^b D(h_{\eta\xi}(t)) = h_{\eta\xi}(b) - h_{\eta\xi}(a)$$

and

$$\int_a^b D(-h_{\eta\xi}(t)) = h_{\eta\xi}(a) - h_{\eta\xi}(b)$$

exist. We conclude that the integral $\int_a^b DP(X(\tau), t)(\eta, \xi)$ exists and the conclusion of the theorem holds by Theorem (4.1) above .

Theorem 4.4. *Assume that the map* $(x, t) \to P(x, t)(\eta, \xi)$ *belongs to* $\mathbb{F}(\tilde{\mathcal{A}} \times [t_0, T], h_{\eta\xi}, W)$ *and that* $X : [a, b] \longrightarrow \tilde{\mathcal{A}}$ *is the limit of a sequence of simple processes. Then the integral* $\int_a^b DP(X(\tau), t)(\eta, \xi)$ *exists for arbitrary* $\eta, \xi \in \mathbb{D}\underline{\otimes}\mathbb{E}$.

Proof: By Theorem (4.3), it is sufficient to prove that the integral $\int_a^b DP(\phi(\tau), t)(\eta, \xi)$ exists for every simple processes $\phi : [a, b] \to \tilde{\mathcal{A}}$. If ϕ is a simple process then there is a partition $a = s_0 < s_1 < s_2 < \cdots < s_k = b$ of $[a, b]$ such that $\phi(s) = c_j \in \tilde{\mathcal{A}}$ for $s \in (s_{j-1}, s_j)$, $j = 1, 2, \ldots, k$ where c_j, $j = 1, \ldots, k$ are finite number of elements of $\tilde{\mathcal{A}}$.

By the definition of the Kurzweil integral, if $s_{j-1} < \sigma_1 < \sigma_2 < \sigma_j$, then we have the existence of the integral

$$\int_{\sigma 1}^{\sigma_2} DP(\phi(\tau), t)(\eta, \xi) = P(c_j, \sigma_2)(\eta, \xi) - P(c_j, \sigma_1)(\eta, \xi)$$

Assume that $\sigma_0 \in (s_{j-1}, s_j)$ is given, we have

$$\begin{aligned} &\lim_{s \to s_{j-1}+} \left[\int_s^{\sigma_0} DP(\phi(\tau), t)(\eta, \xi) + P(\phi(s_{j-1}), s)(\eta, \xi) \right. \\ &\qquad \left. - P(\phi(s_{j-1}), s_{j-1})(\eta, \xi) \right] = \lim_{s \to s_{j-1}+} [P(c_j, \sigma_0)(\eta, \xi) \\ &\qquad - P(c_j, s)(\eta, \xi) + P(\phi(s_{j-1}), s)(\eta, \xi) - P(\phi(s_{j-1}), s_{j-1})(\eta, \xi)] \\ &\quad = P(c_j, \sigma_0)(\eta, \xi) - P(c_j, s_{j-1+})(\eta, \xi) + P(\phi(s_{j-1}), s_{j-1+})(\eta, \xi) \\ &\qquad - P(\phi(s_{j-1}), s_{j-1})(\eta, \xi) \end{aligned} \tag{4.7}$$

Hence by Theorem (4.2) (ii), the integral $\int_{s_{j-1}}^{\sigma_0} DP(\phi(\tau), t)(\eta, \xi)$ exists and equals the computed limit given by (4.7). Similarly, it can be shown that the integral $\int_{\sigma_0}^{s_j} DP(\phi(\tau), t)(\eta, \xi)$ exists and the following equality holds.

$$\begin{aligned} \int_{\sigma_0}^{s_j} DP(\phi(\tau), t)(\eta, \xi) &= P(c_j, s_j-)(\eta, \xi) - P(c_j, \sigma_0)(\eta, \xi) \\ &\quad - P(\phi(s_j), s_j-)(\eta, \xi) + P(\phi(s_j), s_j)(\eta, \xi) \end{aligned} \tag{4.8}$$

by Theorem (4.2)(i).
Hence by additivity of the integral, we obtain

$$\begin{aligned} &\int_{s_{j-1}}^{s_j} DP(\phi(\tau), t)(\eta, \xi) \\ &\quad = \int_{s_{j-1}}^{\sigma_0} DP(\phi(\tau), t)(\eta, \xi) + \int_{\sigma_0}^{s_j} DP(\phi(\tau), t)(\eta, \xi) \end{aligned}$$

which equals the sum of expressions in (4.7) and (4.8) over the subinterval $[s_{j-1}, s_j]$ of the partition.

Repeating this argument for every interval $[s_{j-1}, s_j]$, $j = 1, 2, \dots, k$ and using the additivity of the integral, we obtain the existence of the integral $\int_a^b DP(\phi(\tau), t)(\eta, \xi)$ and the identity

$$\begin{aligned}\int_a^b & DP(\phi(\tau), t)(\eta, \xi) \\ &= \sum_{j=1}^{k}[P(c_j, s_j-)(\eta, \xi) - P(c_j, s_{j-1+})] + \sum_{j=1}^{k}[P(\phi(s_{j-1+}), s_{j-1}+)(\eta, \xi) \\ &\quad - P(\phi(s_{j-1}), s_{j-1})(\eta, \xi) - P(\phi(s_j), s_{j-})(\eta, \xi) + P(\phi(s_j, s_j)(\eta, \xi)]\end{aligned} \tag{4.9}$$

Theorem 4.5. *Assume that the map* $(x, t) \rightarrow P(x,t)(\eta,\xi)$ *is of class* $I\!F(\tilde{\mathcal{A}} \times [t_0, T], h_{\eta\xi}, W)$ *and* $X : [a, b] \rightarrow \tilde{\mathcal{A}}$, $[a, b] \subseteq [t_0, T]$ *is of bounded variation, then the integral* $\int_a^b DP(X(\tau), t)(\eta, \xi)$, *exists.*

Proof: The result follows from Theorem (4.4) because every process $X : [a, b] \rightarrow \tilde{\mathcal{A}}$ in $L^2_{loc}(\tilde{\mathcal{A}})$ of bounded variation is the uniform limit of finite simple processes (cf [9,10,13]).

Next, we denote by $C(\tilde{\mathcal{A}} \times [t_0, T], W)$, the class of sesquilinear form-valued maps which are Lipschitzian and satisfy the Caratheodory conditions. We then give a result that connects this class with the class $I\!F(\tilde{\mathcal{A}} \times [t_0, T], h_{\eta\xi}, W)$.

Definition 4.6. A map $P : \tilde{\mathcal{A}} \times [t_0, T] \longrightarrow Sesq(I\!D\underline{\otimes}I\!E)$ belongs to the class $C(\tilde{\mathcal{A}} \times [t_0, T], W)$ if for arbitrary $\eta, \xi \in I\!D\underline{\otimes}I\!E$,

(i) $P(x, \cdot)(\eta, \xi)$ is measurable for each $x \in \tilde{\mathcal{A}}$.

(ii) There exists a family of measurable functions $M_{\eta\xi} : [t_0, T] \rightarrow I\!R_+$ such that $\int_{t_0}^T M_{\eta\xi}(s)ds < \infty$ and

$$|P(x, s)(\eta, \xi)| \leq M_{\eta\xi}(s), \quad (x, s) \in \tilde{\mathcal{A}} \times [t_0, T] \tag{4.10}$$

(iii) There exists measurable functions $K^p_{\eta\xi} : [t_0, T] \rightarrow I\!R_+$ such that for each $t \in [t_0, T]$, $\int_{t_0}^t K_{\eta\xi}(s)ds < \infty$, and

$$|P(x, s)(\eta, \xi) - P(y, s)(\eta, \xi)| \leq K^p_{\eta\xi}(s)W(\|x - y\|_{\eta\xi}) \tag{4.11}$$

for $(x, s), (y, s) \in \tilde{\mathcal{A}} \times [t_0, T]$ and where in (i) - (iii) we take $W(t) = t$.

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Definition 4.7. For $(x, t) \in \tilde{\mathcal{A}} \times [t_0, T]$ and P belonging to $C(\tilde{\mathcal{A}} \times [t_0, T], W)$, we define for arbitrary $\eta, \xi \in \mathbb{D}\underline{\otimes}\mathbb{E}$,

$$F(x,t)(\eta,\xi) = \int_{t_0}^{t} P(x,s)(\eta,\xi)ds \tag{4.12}$$

We have the following results that connect the two classes of maps.

Theorem 4.8. *Assume that for arbitrary $\eta, \xi \in \mathbb{D}\underline{\otimes}\mathbb{E}$, the map $P : \tilde{\mathcal{A}} \times [t_0, T] \to sesq(\mathbb{D}\underline{\otimes}\mathbb{E})$ is of class $C(\tilde{\mathcal{A}} \times [t_0, T], W)$. Then for every $x, \ y \in \tilde{\mathcal{A}}, t_1, t_2 \in [t_0, T]$, $F(x,t)(\eta,\xi)$ defined by (4.12) satisfies*

(i) $|F(x,t_2)(\eta,\xi) - F(x,t_1)(\eta,\xi)| \leq \int_{t_1}^{t_2} M_{\eta\xi}(s)ds$

(ii) $|F(x,t_2)(\eta,\xi) - F(x,t_1)(\eta,\xi) - F(y,t_2)(\eta,\xi) + F(y,t_1)(\eta,\xi)|$
$\leq W(\|x-y\|_{\eta\xi}) \int_{t_1}^{t_2} K_{\eta\xi}^{p}(s)ds$

(iii) *The map $F(x,t)(\eta,\xi)$ belong to the class $\mathbb{F}(\tilde{\mathcal{A}} \times [t_0, T], h_{\eta\xi}, W)$ for each $\eta, \xi \in \mathbb{D}\underline{\otimes}\mathbb{E}$, where*

$$h_{\eta\xi}(t) = \int_{t_0}^{t} M_{\eta\xi}(s)ds + \int_{t_0}^{t} K_{\eta\xi}^{p}(s)ds$$

Proof: (i) Since (4.10) holds we have by (4.12) and for all $x \in \tilde{\mathcal{A}}$, $t_1, t_2 \in [t_0, T]$.

$$\begin{aligned}|F(x,t_2)(\eta,\xi) - F(x,t_1)(\eta,\xi)| &= \left|\int_{t_1}^{t_2} P(x,s)(\eta,\xi)ds\right| \\ &\leq \int_{t_1}^{t_2} |P(x,s)(\eta,\xi)|ds \\ &\leq \int_{t_1}^{t_2} M_{\eta,\xi}(s)ds\end{aligned}$$

(ii) Again by (4.12) and (4.11)

$$\begin{aligned}&|F(x,t_2)(\eta,\xi) - F(x,t_1)(\eta,\xi) - F(y,t_2)(\eta,\xi) + F(y,t_1)(\eta,\xi)| \\ &\quad = \left|\int_{t_1}^{t_2} [P(x,s)(\eta,\xi) - P(y,s)(\eta,\xi)]ds\right| \\ &\quad \leq \int_{t_1}^{t_2} |P(x,s)(\eta,\xi) - P(y,s)(\eta,\xi)|ds \leq W(\|x-y\|_{\eta\xi}) \int_{t_1}^{t_2} K_{\eta\xi}^{p}(s)ds\end{aligned}$$

for every $x, y \in \tilde{\mathcal{A}}$ and $t_1, t_2 \in [t_0, T]$.

By (i) above

$$|F(x,t_2)(\eta,\xi) - F(x,t_1)(\eta,\xi)| \le \int_{t_1}^{t_2} M_{\eta\xi}(s)ds \le |h_{\eta\xi}(t_2) - h_{\eta\xi}(t_1)|$$

$\forall x \in \tilde{\mathcal{A}}$, $t_1, t_2 \in [t_0, T]$ satisfying inequality (4.1).
Again by (ii) above

$$\begin{aligned}
&|F(x,t_2)(\eta,\xi) - F(x,t_1)(\eta,\xi) - F(y,t_2)(\eta,\xi) + F(y,t_1)(\eta,\xi)| \\
&\quad \le W(\|x-y\|_{\eta\xi}) \int_{t_1}^{t_2} K^p_{\eta\xi}(s)ds \\
&\quad \le W(\|x-y\|_{\eta\xi})|h_{\eta\xi}(t_2) - h_{\eta\xi}(t_1)|
\end{aligned}$$

for every $x, y \in \tilde{\mathcal{A}}$, $t_1, t_2 \in [t_0, T]$ satisfying inequality (4.2) .

In the next section, we prove that the Kurzweil integral of $F(x,t)(\eta,\xi)$ equals the Lebesque integral of $P(x,t)(\eta,\xi)$. This facilitates the proof of the equivalence of equation (1.2) and the associated Kurzweil equation.

5. EQUIVALENCE OF QUANTUM STOCHASTIC DIFFERENTIAL EQUATION AND THE ASSOCIATED KURZWEIL EQUATION

In connection with subsequent results, we assume that the map $P : \tilde{\mathcal{A}} \times [t_0, T] \to Sesq(\mathbb{D}\underline{\otimes}\mathbb{E})$ given by equation (1.3) is of class $C(\tilde{\mathcal{A}} \times [t_0, T], W)$ and that $F(x,t)(\eta,\xi)$ is given by (4.12).

Theorem 5.1. *If $x : [a,b] \to \tilde{\mathcal{A}}$, $[a,b] \subseteq [t_0, T]$ is the limit of simple processes then*

$$\int_a^b DF(x(\tau),t)(\eta,\xi) = \int_a^b P(x(s),s)(\eta,\xi)ds$$

Proof: By Theorem (4.8)(iii) the map $(x,t) \to F(x,t)(\eta,\xi)$ belongs to $\mathbb{F}(\tilde{\mathcal{A}} \times [t_0, T], h_{\eta\xi}, W)$. Therefore the existence of the integral $\int_a^b DF(x(\tau),t)(\eta,\xi)$ is guaranteed by Theorem (4.4). Also by Theorem (4.4), for every simple process $\phi : [a,b] \to \tilde{\mathcal{A}}$ the integral $\int_a^b P(\phi(s),s)(\eta,\xi)ds$ exists and equals $\int_a^b DF(\phi(\tau),t)(\eta,\xi)$.
Assume now that $\phi_k : [a,b] \to \tilde{\mathcal{A}}$, $k \in \mathbb{N}$ is a sequence of simple processes such that

$$\lim_{k\to\infty} \phi_k(s) = x(s), \quad s \in [a,b]$$

Then by (4.11),

$$\lim_{k\to\infty} P(\phi_k(s),s)(\eta,\xi) = P(x(s),s)(\eta,\xi)$$

and inequality (4.10) enables us to use the Lebesgue dominated convergence theorem for showing that $\int_a^b P(x(s), s)(\eta, \xi)ds$ exists and by Theorem (4.3)

$$\int_a^b DF(x(\tau), t)(\eta, \xi) = \lim_{k\to\infty} \int_a^b DF(\phi_k(\tau), t)(\eta, \xi)$$
$$= \lim_{k\to\infty} \int_a^b P(\phi_k(s), s)(\eta, \xi)ds = \int_a^b P(x(s), s)(\eta, \xi)ds$$

Remark 5.2.

(i) The results given above will be used for the representation of equation (1.2) within the framework of the Kurzweil integral calculus. This is accomplished based on the construction of the map $F(x, t)(\eta, \xi)$ for a given sesquilinear form - valued map $P : \tilde{\mathcal{A}} \times [t_0, T] \to Sesq(\mathbb{D}\underline{\otimes}\mathbb{E})$ of class $C(\tilde{\mathcal{A}} \times [t_0, T], W)$ and for arbitrary $\eta, \xi \in \mathbb{D}\underline{\otimes}\mathbb{E}$.

(ii) Let the quantum stochastic differential equation (1.2) be given. The Caratheodory concept of a solution of (1.2) is equivalent to the requirement that for every $s_1, s_2 \in [t_0, T]$ we have a weakly absolutely continuous map $X : [t_0, T] \to \tilde{\mathcal{A}}$ satisfying

$$\langle \eta, x(s_2)\xi\rangle - \langle \eta, x(s_1)\xi\rangle = \int_{s_1}^{s_2} P(x(s), s)(\eta, \xi)ds \tag{5.1}$$

(iii) The solution X of equation (1.2) lies in $L^2_{loc}(\tilde{\mathcal{A}})$ and is therefore the limit of simple processes in $Ad(\tilde{\mathcal{A}})_{wac}$, see [2,8,13]. Consequently the hypothesis of the last theorem remain true.

We now present our major result in this section.

Theorem 5.3. *A stochastic process $X : [t_0, T] \to \tilde{\mathcal{A}}$ is a solution of equation* (1.2) *if and only if X is a solution of the Kurzweil equation*

$$\frac{d}{d\tau}\langle \eta, X(\tau)\xi\rangle = DF(X(\tau), t)(\eta, \xi) \tag{5.2}$$

on $[t_0, T]$, $t \in [t_0, T]$, *and for arbitrary* $\eta, \xi \in \mathbb{D}\underline{\otimes}\mathbb{E}$.

Proof: Assume that $X : [t_0, T] \to \tilde{\mathcal{A}}$ is a solution of (1.2). By Theorem (4.8), the integral $\int_{s_1}^{s_2} DF(X(\tau), t)(\eta, \xi)$ exists and

$$\langle \eta, X(s_2)\xi\rangle - \langle \eta, X(s_1)\xi\rangle = \int_{s_1}^{s_2} P(X(s), s)(\eta, \xi)ds$$
$$= \int_{s_1}^{s_2} DF(X(\tau), t)(\eta, \xi)$$

for all $s_1, s_2 \in [t_0, t]$. Hence X is a solution of (5.2).

If conversely X is a solution of (5.2), then again Theorem (4.8) shows that X satisfies equation (5.1). Since $F(X,t)(\eta,\xi)$ belongs to $\mathbb{F}(\tilde{\mathcal{A}}\times[t_0,T],h_{\eta\xi},W)$, we have

$$\begin{aligned}|\langle\eta,X(s_2)\xi\rangle-\langle\eta,X(s_1)\xi\rangle| &= \left|\int_{s_1}^{s_2} DF(X(\tau),t)(\eta,\xi)\right| \\ &\leq |h_{\eta\xi}(s_2)-h_{\eta\xi}(s_1)|.\end{aligned}$$

Hence the map $t\to\langle\eta,X(t)\xi\rangle$ is absolutely continuous on $[t_0,T]$ since $h_{\eta\xi}(t)$ is absolutely continuous for each $\eta,\xi\in\mathbb{D}\underline{\otimes}\mathbb{E}$. Hence X is weakly absolutely continuous.

Remark 5.4. Owning to several properties of the sesquilinear form -valued map P given by equation (1.3) as outlined in Ekhaguere [8], it is enough for P to be Lipschitzian and for inequality (4.10) to be satisfied for all $(X,t)\in\tilde{\mathcal{A}}\times[t_0,T]$ for P to be of class $C(\tilde{\mathcal{A}}\times[t_o,T],W)$ where $W(t)=t$. Consequently, $F(X,t)(\eta,\xi)$ defined by equation (4.12) is of class $\mathbb{F}(\tilde{\mathcal{A}}\times[t_0,T],h_{\eta\xi},W)$ and so by Theorem (5.1)

$$\int_{t_0}^{t} DF(X(\tau),t)(\eta,\xi)=\int_{t_0}^{t} P(X(s),s)(\eta,\xi)ds,\quad t\in[t_0,T]. \tag{5.3}$$

Again, Theorem (5.3) asserts that X satisfies equation (5.2) if and only if

$$\begin{aligned}\langle\eta,X(t)\xi\rangle-\langle\eta,X(t_0)\xi\rangle &= \int_{t_0}^{t} DF(X(\tau),t)(\eta,\xi)\\ &= \int_{t_0}^{t} P(X(s),s)(\eta,\xi)ds\end{aligned}$$

by equation (5.3). This follows if and only if

$$\begin{aligned}&\frac{d}{dt}\langle\eta,X(t)\xi\rangle = P(X(t),t)(\eta,\xi)\\ &\langle\eta,X(t_0)\xi\rangle=\langle\eta,X_0\xi\rangle\end{aligned}$$

Hence equations (5.2) and (1.2) are equivalent.

As a consequence of the above results, we now describe a procedure for obtaining approximate solutions of equation (1.2) as follows. We assume hypothesis of Theorems (4.8), (5.1) and (5.2).

The initial value problem (1.2) is equivalent to the integral equation

$$\langle\eta,X(s)\xi\rangle=\langle\eta,X(t_0)\xi\rangle+\int_{t_0}^{s} P(X(u),u)(\eta,\xi)du \tag{5.4}$$

with the Lebesgue integral on the right hand side. If X is a solution of (1.2) on $[t_0,T]$, then by the existence and uniqueness results, X is adapted and weakly absolutely continuous and lie in $L^2_{loc}(\tilde{\mathcal{A}})$. Consequently the matrix elements of the solution can be approximated by matrix elements $\langle \eta, X_l(t)\xi \rangle$ of a simple process $X_l(t) \in \mathrm{Ad}(\tilde{\mathcal{A}})_{wac}$ which is constant on intervals of the form (t_{j-1}, t_j) where $t_0 < t_1 < \cdots < t_{kl} = t$ and which on (t_{j-1}, t_j) assumes the value $\langle \eta, X(\tau_j)\xi \rangle$ where $t_{j-1} \leq \tau_j \leq t_j, \quad j = 1, 2. \ldots, k_l$ such that

$$\lim_{l\to\infty} \langle \eta, X_l(s)\xi \rangle = \langle \eta, X(s)\xi \rangle \tag{5.5}$$

i.e $\lim_{l\to\infty} X_l(s) = X(s)$ uniformly on $[t_0, T]$.
Since

$$P(X, t)(\eta, \xi) \text{ is of class } C(\tilde{\mathcal{A}} \times [t_0, T], h_{\eta\xi}, W)$$

then

$$\lim_{l\to\infty} P(X_l(s), s)(\eta, \xi) = P(X(s), s)(\eta, \xi)$$

on $[t_0, T]$ by inequality (4.11).
Assuming that the sequence

$$P(X_l(s), s)(\eta, \xi), \quad l = 1, 2, \ldots$$

satisfies (4.10) then by the Lebesgue dominated convergence theorem it can be concluded that

$$\lim_{l\to\infty} \int_{t_0}^{t} P(X_l(s), s)(\eta, \xi)ds = \int_{t_0}^{t} P(X(s), s)(\eta, \xi)ds. \tag{5.6}$$

However, for a fixed $l \in \mathbb{N}$, we have

$$\begin{aligned}\int_{t_0}^{t} P(X_l(s), s)(\eta, \xi)ds &= \sum_{j=1}^{k_l} \int_{t_{j-1}}^{t_j} P(X(\tau_j), s)(\eta, \xi)ds \\ &= \sum_{j=1}^{k_l} [F(X(\tau_j), t_j)(\eta, \xi) - F(X(\tau_j), t_{j-1})(\eta, \xi)],\end{aligned}$$

which shows that the integral $\int_{t_0}^{t} P(X(s), s)(\eta, \xi)ds$ appearing in (5.6) can be approximated by the Kurzweil integral sums of the form

$$\sum_{j=1}^{k_l} [F(X(\tau_j), t_j)(\eta, \xi) - F(X(\tau_j), t_{j-1})(\eta, \xi)].$$

Finally, using (5.4) the matrix element $\langle \eta, X(t)\xi \rangle$ of the solution X can be approximated by the sum

$$\begin{aligned}\langle \eta, X(t)\xi \rangle \cong \langle \eta, X_0\xi \rangle + \sum_{j=1}^{k_l} [F(X(\tau_j), t_j)(\eta, \xi) \\ - F(X(\tau_j), t_{j-1})(\eta, \xi)]\end{aligned} \tag{5.7}$$

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provided that a sufficiently fine division $t_0 < t_1 < t_2 < \cdots < t_k = t$ is constructed and the choice of $\tau_j \in [t_{j-1}, t_j]$, $j = 1, 2, \ldots, k$ is fixed in order to obtain the uniform convergence (5.5).

6. NUMERICAL EXAMPLES

In the notation of section 1, we consider the simple Fock space $\Gamma(L^2_\gamma(\mathbb{R}_+))$ where $\gamma = \mathcal{R} = \mathbb{C}$, $f = g \equiv 1$, and its $L^2(\Omega, \mathcal{F}, W)$ realization where $(\Omega, \mathcal{F}, W)$ is a Wiener space. Each random variable X is identified with the operator of multiplication by X so that $Q(t) = A(t) + A^+(t) = w(t)$ is the evaluation of the Brownian path w at time t. In this case, it has been shown that quantum stochastic integrals of adapted Brownian functional F such that $\int_{t_0}^t E[F(s,\cdot)^2]ds < \infty$ exists (see [2,4]). Here E is the expected value function.

For exponential vectors $\eta = e(\alpha)$ and $\xi = e(\beta)$ where α, β are purely imaginary- valued functions in $L^2_{\mathbb{C}}(\mathbb{R}_+)$, the equivalent form (1.2) of the quantum analogue of the classical Ito stochastic differential equation

$$dX(t,w) = -\frac{1}{2}X(t,w)dt - \sqrt{1 - X^2(t,w)}dW(t)$$
$$X(t_0) = X_0, \quad t \in [0, T] \tag{6.1}$$

is given by

$$\frac{d}{dt}E(X(t,w)z(w)) = E(-\beta(t)\sqrt{1 - X^2(t,w)}z(w))$$
$$+ E(-\bar{\alpha}(t)\sqrt{1 - X^2(t,w)}z(w)) + E\left(-\frac{1}{2}X(t,w)z(w)\right)$$
$$X(t_0) = X_0, \quad t \in [t_0, T] \tag{6.2}$$

where

$$z(w) = \exp\left\{\int_0^\infty (-\alpha(s) + \beta(s))dw(s) - \frac{1}{2}\int_0^\infty (\alpha^2(s) + \beta^2(s))ds\right\} \tag{6.3}$$

(see [2,4] for details).

With $X_0(w) = 1$, $\alpha(t) = \beta(t) = i$, and the interval $[0, T] = [0, 1]$, then we have by equation (6.3), $z(w) = e$ and $E(X_0(w)z(w)) = E(z(w)) = e$.
The exact solution of equation (6.2) is then given by

$$E(X(t)z(w)) = e^{1 - \frac{1}{2}t} \tag{6.4}$$

We now apply our approximation procedures to discretize equation (6.2)

For $t \in [0, 1]$, $X \in \tilde{\mathcal{A}}$ the map $P_{\alpha\beta}$ defined in section 1 is

$$P_{\alpha\beta}(t, X) = -\beta(t)\sqrt{1 - X^2(t)} - \bar{\alpha}(t)\sqrt{1 - X^2(t)} - \frac{1}{2}X(t) = -\frac{1}{2}X(t)$$

and

$$P(X, t)(\eta, \xi) = \left\langle \eta, \left(-\frac{1}{2}X(t)\right)\xi \right\rangle$$

for arbitrary $\eta, \xi \in \mathbb{D}\underline{\otimes}\mathbb{E}$. By equation (4.12)

$$F(X(\tau), t)(\eta, \xi) = \left(\left\langle \eta, \left(-\frac{1}{2}X(\tau)\right)\xi \right\rangle\right)t = -\frac{1}{2}tE(X(\tau)z(w))$$

Equation (6.2) is equivalent to (5.2) by Theorem (5.3). Thus we use Proposition (3.2) which leads to the approximation

$$\langle \eta, X(s)\xi\rangle \cong \langle \eta, X(\sigma)\xi\rangle + F(X(\sigma), s)(\eta, \xi) - F(X(\sigma), \sigma)(\eta, \xi) \tag{6.5}$$

for every $\sigma \in [0, 1]$ provided that $s \in [0, 1]$ is sufficiently close to σ. Thus from equation (6.5),

$$\begin{aligned} E(X(s)z(w)) &\cong E(X(\sigma)z(w)) - \frac{1}{2}sE(X(\sigma)z(w)) + \frac{1}{2}\sigma E(X(\sigma)z(w)) \\ &= \left(1 - \frac{1}{2}(s - \sigma)\right)E(X(\sigma)z(w)) \end{aligned} \tag{6.6}$$

Again by equation (5.7), we have

$$\langle \eta, X(t)\xi\rangle \cong \langle \eta, X_0\xi\rangle + \sum_{j=1}^{k_l}\int_{t_{j-1}}^{t_j} P(X(\tau_j), s)(\eta, \xi)ds$$

i.e

$$E(X(t)z(w)) \cong E(X_0z(w)) - \frac{1}{2}\sum_{j=1}^{k_l} E(X(\tau_j)z(w))(t_j - t_{j-1}) \tag{6.7}$$

where $t_o < t_1 < t_2 < \cdots < t_{k_l} = t$ and $\tau_j \in [t_{j-1}, t_j]$.

If we fix $\tau_j = t_{j-1}$ for each $j = 1, 2..$ and a constant steplength h, then we have from equation (6.7)

$$E(X(t_j)z(w)) = \left(1 - \frac{1}{2}h\right)E(X(t_{j-1})z(w)), \quad j = 1, 2..N \tag{6.8}$$

Again, fixing $\tau_j = \frac{1}{2}(t_j + t_{j-1})$, $j = 1, 2..$ then equation (6.7) gives

$$E(X(t_j)z(w)) = E(X(t_{j-1})z(w)) - \frac{1}{2}hE(X(\tau_j)z(w)) \tag{6.9}$$

Table 1. Numerical Values with $\tau_j = t_{j-1}$ and $\alpha(t) = \beta(t) = i$

N	h	Approximate Values $E(X(t_N)z(w))$	Exact Values $e^{1-\frac{1}{2}t_N}$	Absolute Errors $\lvert E(X(t_N)z(w)) - e^{1-\frac{1}{2}t_N}\rvert$
8	2^{-3}	1.622051700	1.648721300	0.02666960000
16	2^{-4}	1.6356182000	1.6487212710	0.01310307000

where the intermediate values $E(X(\tau_j)z(w))$ are calculated by setting $\sigma = t_{j-1}$ and $s = \tau_j$ in (6.6) to give

$$E(X(\tau_j)(z(w)) = \left[1 - \frac{1}{2}(\tau_j - t_{j-1})\right]E(X(t_{j-1})z(w)) \tag{6.10}$$

By putting $h = 2^{-3}$, $h = 2^{-4}$ and fixing τ_j as above ,we generate the following tables of values for the case $\alpha = \beta = i$. Equations (6.8),(6.9),(6.10) are used to generate the following values at the final time $t = 1$ in Tables 1 and 2 below.

In order to compare the accuracy of the method of this paper ,we now apply the method to generate approximate values for the equivalent form of Ito equation

$$\begin{aligned} dX(t) &= \frac{3}{2}X(t)dt + X(t)dW(t) \\ X(t_0) &= 1, \quad t \in [0, 1] \end{aligned} \tag{6.11}$$

given by

$$\frac{d}{dt}E(X(t)z(w)) = \frac{3}{2}E(X(t, w)) \tag{6.12}$$

where $z(w) = e, t \in [0, 1]$, for $\alpha(t) = \beta(t) = i$ with exact solution $E(X)t, w)z(w) = e^{1+\frac{3}{2}t}$

Equation (6.12) had been discretized in [4] using the Euler and a 2-step scheme. We compare the results with those of the present scheme.

Table 2. Numerical Values with $\tau_j = \frac{1}{2}(t_j + t_{j-1})$ and $\alpha(t) = \beta(t) = i$

N	h	τ_N	$E(X(\tau_N)z(w))$	$E(X(t_N)z(w))$	Exact value $= e^{1-\frac{1}{2}t_N}$	Absolute Error $\lvert E(X(t_N)z(w)) - e^{1-\frac{1}{2}t_N}\rvert$
8	2^{-3}	0.9375	1.700716800	1.649283900	1.648721300	0.000562600
16	2^{-4}	0.96875	1.67460900	1.648858600	1.648721271	0.000137329

Table 3. Numerical values with $\tau_j = t_{j-1}$ and $\alpha(t) = \beta(t) = i$

N	h	Approximate Values $E(X(t_N)z(w))$	Exact values $e^{1-\frac{1}{2}t_N}$	Absolute Errors $\lvert E(X(t_N)z(w)) - e^{1-\frac{1}{2}t_N}\rvert$
8	2^{-3}	10.74888529	12.18249396	1.433608668
16	2^{-4}	11.402065680	12.18249396	0.78042828

For equation (6.11),we have the followings. From equation (6.12) and using (4.12),

$$F(X(\tau), t) = \frac{3}{2} t E(X(\tau)z(w))$$

and from (6.5)

$$E(X(s)z(w)) \cong \left(1 + \frac{3}{2}(s - \sigma)\right) E(X(\sigma)z(w)). \tag{6.13}$$

From (5.7),

$$E(X(t)z(w)) = E(X_0 z(w)) + \frac{3}{2} \sum_{j=1}^{k_l} E(X(\tau_j)z(w))(t_j - t_{j-1}). \tag{6.14}$$

Fixing $\tau_j = t_{j-1}$ for each $j = 1, 2 \cdots$ and $h = (t_j - t_{j-1})$, then we have from (6.14)

$$E(X(t_j)z(w)) = \left(1 + \frac{3}{2}h\right) E(X(t_{j-1})z(w)) \tag{6.15}$$

Again fixing $\tau_j = \frac{1}{2}(t_j + t_{j-1})$, we have from (6.14)

$$E(X(t_j)z(w) = E(X(t_{j-1})z(w)) + \frac{3}{2} h E(X(\tau_j)z(w)), \tag{6.16}$$

with intermediate values

$$E(X(\tau_j)z(w)) = \left[1 + \frac{3}{2}(\tau_j - t_{j-1})\right] E(X(t_{j-1})z(w)). \tag{6.17}$$

Our numerical experiments yield the following results at the final time $t = 1$. We use equation (6.15) for Table 3.

Equations (6.16) and (6.17) are used to generate Table 4.

6.1 Conclusion

(i) It is discovered that the schemes (6.8) and (6.15) when τ_j is fixed at the starting point of each subinterval of the partition points generate exactly the same values as Euler scheme considered in [4]. This is confirmed

Table 4. Numerical values with $\tau_j = \frac{1}{2}(t_j + t_{j-1})$ and $\alpha(t) = \beta(t) = i$

N	h	τ_N	$E(X(\tau_N)z(w))$	$E(X(t_N)z(w))$	Exact value $= e^{1-\frac{1}{2}t_N}$	Absolute Error $\lvert E(X(t_N)z(w)) - e^{1-\frac{1}{2}t_N}\rvert$
8	2^{-3}	0.9375	10.97032998	12.08924593	12.18249396	0.093248032
16	2^{-4}	0.96875	11.58995788	12.15756309	12.18249396	0.024930868

by Tables 1 and 3 . The Tables also show that the approximate schemes produce better results with finer gridpoints.

(ii) However, Tables 2 and 4 show a more superior convergence rate when τ_j is fixed at the midpoint of each subintervals of partition. In Table 2, with constant steplengths $h = 2^{-3}$ and $h = 2^{-4}$,we have convergence to at least three decimal places at each of the gridpoint with cummulative absolute errors at the end point $t = 1$ being 0.000562600 and 0.000137329 respectively. This experiment shows that the approximate scheme (5.7) has a superior convergence rate when τ_j is taken as midpoints of each subinterval and expression (6.5) is used to compute intermediate values. This level of accuracy is comparable to that of a 2- stage Runge-Kutta scheme reported in [2] applied to problem (6.12).

In comparison with the Euler and the 2-step method applied in [4] to problem (6.11), Table 4 shows that the method of this paper is more accurate than those two schemes when τ_j is taken as the midpoint of each of the partition subinterval. In particular, for a steplength of $h = 2^{-3}$, the global accumulated error at the final time $t = 1$ is 0.093248032 compared with the global errors of 0.39983038 and 0.11070989 with $h = 2^{-3}$ and 2^{-4} respectively for the 2-step scheme (see [4]). We remark that equation (5.7) permits a change of steplenghts at any point during computation and that this method is suitable for equation (1.2) where the map $(t, x) \mapsto P(t, x)(\eta, \xi)$ is not necessarily continuous jointly in t and x and the matrix elements are not necessarily differentiable more than one time.

In particular, the methods developed in this paper provide a simple approach for computations of expectations of functionals of Ito processes when the quantum equations are considered only in the simple Fock spaces. Applications of the methods to problems in quantum fields/systems will be considered elsewhere.

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REFERENCES

1. Z. Artstein, Topological Dynamics of Ordinary Differential Equations And Kurzweil Equations, *Journal of Differential Equations* 23, (1977), 224–243.
2. E.O. Ayoola, On Numerical Procedures For Solving Lipschitzian Quantum Stochastic Differential Equations. *Ph.D Thesis (*1998*), University of Ibadan, Nigeria.*
3. E.O. Ayoola, Solutions of Lipschizian Quantum Stochastic Differential Equations In A Locally Convex Topological Space. *To Appear In Journal Of Science Research,* Vol. 4, no. 1 (1999).
4. E.O. Ayoola, Converging Multistep Methods For Weak Solutions Of Quantum Stochastic Differential Equations, *Stochastic Analysis And Applications*, Vol.18, no. 4 (2000) pp. 525–554.
5. P.S. Bullen, A survey on integration by parts for Perron integrals, *J. Austral. Math. Soc. (Series A)*, 40, (1986), 343–363.
6. T.S. Chew, On Kurzweil generalized ordinary differential equations, *J. Differential Equations*, 76, no 2 (1988), 286–293.
7. T.S. Chew, F. Flordeliza, On $X' = f(t, X)$ and Henstock - Kurzweil integrals, *Differential and Integral Equations*, 4 (1991) 861–868.
8. G.O.S. Ekhaguere, Lipschitzian quantum stochastic differential inclusions, *Internat. J. Theoret. Phys.* Vol. 31, No. 11, (1992) 2003–2034.
9. G.O.S. Ekhaguere, Quantum stochastic differential inclusions of hypermaximal monotone type, *Internat. J. Theoret. Phys.* Vol. 34, no. 3, (1995), 323–353.
10. G.O.S. Ekhaguere, Quantum stochastic evolutions. *Internat. J. Theoret. Phys.* Vol. 35, no. 9, (1996), 1909–1946.
11. R.L. Hudson, *Quantum Stochastic Calculus and Some of its Applications.* Contemporary Stochastic Analysis 31–70, edited by G.O.S. Ekhaguere, World Scientific (1991).
12. R.L. Hudson, and K.R. Parthasarathy, *Construction of quantum diffusion.* Lecture Notes in Math. 1055, 173–198. Springer (1982).
13. R.L Hudson, and K.R. Parthasarathy, Quantum Ito's formulae and stochastic evolutions, *Commun. Math. Phys. 93,* (1984) 301–324
14. P.E. Kloeden, and E. Platen, *Numerical solutions of stochastic differential equations.* Springer-Verlag (1992).
15. P. Meyer, Quantum Probability for Probabilists. *Lecture Notes in Math. 1538* Springer-Verlag, (1993).

16. K.R. Parthasarathy, *An Introduction to Quantum Stochastic Calculus,* Monographs in Mathematics, Vol. 85, Birkhauser Verlag (1992).
17. K.R. Parthasarathy, Boson Stochastic Calculus, *Pramana J. Phys.* 25 (1985) 457–465.
18. S. Schwabik, *Generalized Ordinary Differential Equations.* World Scientific, (1992).

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Topological Properties of Solution Sets of Lipschitzian Quantum Stochastic Differential Inclusions

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Abstract We establish a continuous mapping of the space of the matrix elements of an arbitrary nonempty set of quasi solutions of Lipschitzian quantum stochastic differential inclusion (QSDI) into the space of the matrix elements of its solutions. As a corollary, we furnish a generalization of a previous selection result. In particular, when the coefficients of the inclusion are integrably bounded, we show that the space of the matrix elements of solutions is an absolute retract, contractible, locally and integrally connected in an arbitrary dimension.

Keywords Lipschitzian QSDI · Solution sets · Continuous selection · Topological properties · Matrix elements · Quantum stochastic processes

Mathematics Subject Classification (2000) 81S25

1 Introduction

Investigation of the diverse features of solution sets of differential inclusions defined on finite dimensional Euclidean spaces has been a major preoccupation of classical analysis. Indeed, many authors (see, for example, [1, 9–12, 16, 25–27]) have studied topological properties of solution and reachable sets of such inclusions to a large extent. In [26], the space of solutions of classical Lipschitz differential inclusions has been shown to be an absolute retract and as a consequence, the space enjoys the topological properties of some kinds of connectivity and contractibility. By contrast, in the context of quantum stochastic differential inclusions (QSDI), matters are somewhat different. The analysis of QSDI concerns quantum stochastic processes as operator valued processes that live in certain infinite

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dimensional locally convex spaces. An extensive study of various features of the solution set of QSDI is yet to be undertaken although some efforts [3, 5–7] have been made concerning the establishment of continuous selections of solution sets and the construction of approximate reachable sets and their continuous representations.

The objective of this paper is to establish further topological properties of solution sets of QSDI within the framework of the Hudson and Parthasarathy [18] formulation of Boson quantum stochastic calculus. To this end, we first establish a continuous mapping of the space of the matrix elements of an arbitrary nonempty set of quasi solutions of QSDI into the space of the matrix elements of its solutions satisfying certain conditions. This mapping consequently facilitates the establishment of the space of matrix elements of solutions as an absolute retract. This ultimately leads us to showing that the space is connected and contractible in some sense. These properties of the solution space are needed in our ongoing study of optimization problems for QSDI. In addition, the result here generalizes our previous selection result in [7] by removing the requirement of compactness of the domain of the selection map.

We remark here that a very strong motivation for studying QSDI among others, is the need for sufficient information and knowledge about the dynamics and fluctuation of the systems described by discontinuous quantum stochastic differential equations which may be reformulated as regularized QSDI. Various analytical and numerical features of solutions of such equations need to be well understood. Continuous and discontinuous quantum stochastic differential equations often arise as mathematical models which describe among other things, quantum dynamical systems and several physical problems in quantum stochastic control theory and quantum stochastic evolutions. Also, in the Hudson-Parthasarathy [18] formulation of quantum stochastic calculus, quantum stochastic differential equations are known to sufficiently generalize classical stochastic differential equations driven by martingales (see, for example [2, 4–6, 13–15, 23, 24] and the references they contain). We mention that in the work of [8], a class of absolute retracts in spaces of integrable functions was established. This class has been shown to contain decomposable sets and sets of solutions to classical Lipschitzian differential inclusions by using some previously established selection results.

The rest of the paper is organized as follows: In Sect. 2, we outline some fundamental definitions, notations and auxiliary results for the establishment of the main results in Sect. 3. Section 3 is devoted to the establishment of the main results of the paper and the associated corollaries.

2 Fundamental Concepts, Structures and Preliminary Results

As established in [13], we outline in this section, some fundamental concepts, structures and preliminary results that are employed in what follows.

Let D be an inner product space and H, the completion of D. We denote by $L^+(D, H)$ the set $\{X : D \to H : X$ is a linear map such that $\mathrm{Dom} X^* \supseteq D$, where X^* is the adjoint of $X\}$. We remark that $L^+(D, H)$ is a linear space under the usual notions of addition and scalar multiplication of operators.

If H is a Hilbert space, we denote by $\Gamma(H)$ the Boson Fock space determined by H. For $f \in H$, $e(f)$ denotes the exponential vector in $\Gamma(H)$ corresponding to f. We remark here that the subspace E of $\Gamma(H)$ generated by the set of exponential vectors in $\Gamma(H)$ is dense in $\Gamma(H)$. Since the exponential vectors are linearly independent, an operator with domain E is well defined by specifying its action on $e(f)$, $f \in H$. For other properties enjoyed by the exponential vectors and Boson Fock space, we refer the reader to [17–20, 23, 24].

In what follows, $\mathbb{D}$ is some inner product space with $\mathcal{R}$ as its completion, and γ is some fixed Hilbert space.

(i) For each $t \in \mathbb{R}_+ \equiv [0, \infty)$, we write $L^2_\gamma(\mathbb{R}_+)$ (resp. $L^2_\gamma([0,t))$; resp. $L^2_\gamma([t,\infty))$), for the Hilbert spaces of square integrable, γ-valued maps on $\mathbb{R}_+$ (resp. $[0,t)$; resp $[t,\infty)$).

(ii) The non-commutative stochastic processes which we shall discuss are densely-defined linear operators on $\mathcal{R} \otimes \Gamma(L^2_\gamma(\mathbb{R}_+))$; the inner product of this complex Hilbert space will be denoted by $\langle \cdot, \cdot \rangle$ and its norm by $\| \cdot \|$.

(iii) Let $\mathbb{E}, \mathbb{E}_t$, and $\mathbb{E}^t$, $t > 0$, be the linear spaces generated by the exponential vectors in $\Gamma(L^2_\gamma(\mathbb{R}_+)), \Gamma(L^2_\gamma([0,t)))$ and $\Gamma(L^2_\gamma([t,\infty)))$, respectively, then we adopt the following spaces as in [13].

(a) $\mathcal{A} \equiv L^+(\mathbb{D} \underline{\otimes} \mathbb{E}, \mathcal{R} \otimes \Gamma(L^2_\gamma(\mathbb{R}_+)))$,

(b) $\mathcal{A}_t \equiv L^+(\mathbb{D} \underline{\otimes} \mathbb{E}_t, \mathcal{R} \otimes \Gamma(L^2_\gamma([0,t)))) \otimes 1^t$,

(c) $\mathcal{A}^t \equiv 1_t \otimes L^+(\mathbb{D} \underline{\otimes} \mathbb{E}^t, \mathcal{R} \otimes \Gamma(L^2_\gamma([t,\infty))))$, $t > 0$,

where $\underline{\otimes}$ denotes algebraic tensor product and $\mathbf{1}_t$ (resp. $\mathbf{1}^t$) denotes the identity map on $\mathcal{R} \otimes \Gamma(L^2_\gamma([0,t)))$ (resp. $\Gamma(L^2_\gamma([t,\infty)))$), $t > 0$. We note that $\mathcal{A}^t$ and $\mathcal{A}_t$, $t > 0$, may be naturally identified with subspaces of $\mathcal{A}$. For $\eta, \xi \in \mathbb{D} \underline{\otimes} \mathbb{E}$, we define $\| \cdot \|_{\eta\xi}$ on $\mathcal{A}$ by $\|x\|_{\eta\xi} = |\langle \eta, x\xi \rangle|$, $x \in \mathcal{A}$. Then $\{x \to \|x\|_{\eta\xi},\ \eta, \xi \in \mathbb{D} \underline{\otimes} \mathbb{E}\}$ is a family of seminorms on $\mathcal{A}$; we write τ_W for the locally convex Hausdorff topology on $\mathcal{A}$ determined by this family. We denote by $\tilde{\mathcal{A}}$, $\tilde{\mathcal{A}}_t$ and $\tilde{\mathcal{A}}^t$ the completions of the locally convex spaces $(\mathcal{A}, \tau_W)$, $(\mathcal{A}_t, \tau_W)$ and $(\mathcal{A}^t, \tau_W)$, $t > 0$, respectively. We remark that the net $\{\tilde{\mathcal{A}}_t : t \in \mathbb{R}_+\}$ furnishes a filtration of $\tilde{\mathcal{A}}$.

Hausdorff Topology If A is a topological space, then $clos(A)$ [resp. $comp(A)$] denotes the collection of all nonvoid closed (resp. compact) subsets of A. We shall employ the Hausdorff topology on $clos(\tilde{\mathcal{A}})$. This is defined as follows: For $x \in \tilde{\mathcal{A}}$, $\mathcal{M}, \mathcal{N} \in clos(\tilde{\mathcal{A}})$, and $\eta, \xi \in \mathbb{D} \underline{\otimes} \mathbb{E}$, set

$$\mathbf{d}_{\eta\xi}(x, \mathcal{N}) \equiv \inf_{y \in \mathcal{N}} \|x - y\|_{\eta\xi},$$

$$\delta_{\eta\xi}(\mathcal{M}, \mathcal{N}) \equiv \sup_{x \in \mathcal{M}} \mathbf{d}_{\eta\xi}(x, \mathcal{N})$$

and

$$\rho_{\eta\xi}(\mathcal{M}, \mathcal{N}) \equiv \max(\delta_{\eta\xi}(\mathcal{M}, \mathcal{N}), \delta_{\eta\xi}(\mathcal{N}, \mathcal{M})).$$

Then by this definition, $\{\rho_{\eta\xi}(\cdot, \cdot) : \eta, \xi \in \mathbb{D} \underline{\otimes} \mathbb{E}\}$ is a family of pseudometrics (see [13–15]) which determines a Hausdorff topology on $clos(\tilde{\mathcal{A}})$. We denote this topology by τ_H in what follows. If $\mathcal{M} \in clos(\tilde{\mathcal{A}})$, then $\|\mathcal{M}\|_{\eta\xi}$ is defined by

$$\|\mathcal{M}\|_{\eta\xi} \equiv \rho_{\eta\xi}(\mathcal{M}, \{0\})$$

for arbitrary $\eta, \xi \in \mathbb{D} \underline{\otimes} \mathbb{E}$.

Similarly, for $A, B \in clos(\mathbb{C})$ and $x \in \mathbb{C}$, the complex numbers, we let

$$\mathbf{d}(x, A) \equiv \inf_{y \in A} |x - y|,$$

$$h(A, B) \equiv \sup_{x \in A} \mathbf{d}(x, B)$$

and

$$\rho(A, B) \equiv \max(h(A, B), h(B, A)).$$

Then ρ induces a metric topology on $clos(\mathbb{C})$.

Sets: We employ the usual set-theoretic operations such as

$$A + B = \{a + b : a \in A \text{ and } b \in B\},$$
$$a + B = \{a + b : b \in B\}$$

for sets A, B and a point a.

Boson Quantum Stochastic Integration We first present here, a number of important notations and definitions.

Let $I \subseteq \mathbb{R}_+$,

(i) A map $X : I \to \tilde{\mathcal{A}}$ is called a stochastic process indexed by I.
(ii) A stochastic process X is called adapted if $X(t) \in \tilde{\mathcal{A}}_t$ for each $t \in I$. We denote by $Ad(\tilde{\mathcal{A}})$ the set of all adapted stochastic processes indexed by I.
(iii) A member X of $Ad(\tilde{\mathcal{A}})$ is called
  (a) weakly absolutely continuous if the map $t \to \langle \eta, X(t)\xi \rangle, t \in I$, is absolutely continuous for arbitrary $\eta, \xi \in \mathbb{D}\underline{\otimes}\mathbb{E}$. We denote this subset of $Ad(\tilde{\mathcal{A}})$ by $Ad(\tilde{\mathcal{A}})_{wac}$.
  (b) locally absolutely p-integrable if $\|X(\cdot)\|^p_{\eta\xi}$ is Lebesgue measurable and integrable on $[t_0, t) \subseteq I$ for each $t \in I$, $p \in (0, \infty)$ and arbitrary $\eta, \xi \in \mathbb{D}\underline{\otimes}\mathbb{E}$. We denote this subset of $Ad(\tilde{\mathcal{A}})$ by $L^p_{loc}(\tilde{\mathcal{A}})$.
(iv) We denote by $wac(\tilde{\mathcal{A}})$ the completion of the locally convex space $(Ad(\tilde{\mathcal{A}})_{wac}, \tau^{wac})$ where the topology τ^{wac} is generated by the family of seminorms $\{\Phi \to |\Phi|_{\eta\xi} : \eta, \xi \in \mathbb{D}\underline{\otimes}\mathbb{E}, \Phi \in Ad(\tilde{\mathcal{A}})_{wac}\}$ by

$$|\Phi|_{\eta\xi} = \|\Phi(t_0)\|_{\eta\xi} + \int_{t_0}^{T} \left| \frac{d}{ds}\langle \eta, \Phi(s)\xi \rangle \right| ds.$$

Associated with $wac(\tilde{\mathcal{A}})$, we define for arbitrary $\eta, \xi \in \mathbb{D}\underline{\otimes}\mathbb{E}$, the space of complex valued functions

$$wac(\tilde{\mathcal{A}})(\eta, \xi) = \{\langle \eta, \Phi(\cdot)\xi \rangle : \Phi \in wac(\tilde{\mathcal{A}})\}.$$

For $\Phi \in wac(\tilde{\mathcal{A}})$, $M \in clos(wac(\tilde{\mathcal{A}}))$, we define

$$d_{\eta\xi}(\Phi, M) := \inf_{u \in M} |\Phi - u|_{\eta\xi}.$$

We shall frequently utilize the function space $AC([t_0, T], \mathbb{C})$, which is the space of absolutely continuous functions on $[t_0, T]$ with values in the complex field $\mathbb{C}$ endowed with the norm defined by

$$|y|_{AC} := |y(t_0)| + \int_{t_0}^{T} \left| \frac{dy}{dt}(t) \right| dt, \quad y \in AC([t_0, T], \mathbb{C}).$$

For any nonempty set $Q \in clos(AC([t_0, T], \mathbb{C}))$, we define the point-set distance by:

$$d_{AC}(y, Q) := \inf_{z \in Q} |y - z|_{AC}.$$

In what follows, we denote by I, the interval $[t_0, T]$ and the characteristic function of a subset E of I is denoted by χ_E.

Stochastic Integrators Let $B(\gamma)$ denote the Banach space of bounded endomorphisms of γ and let the spaces $L^{\infty}_{\gamma,loc}(\mathbb{R}_+)$ (resp. $L^{\infty}_{B(\gamma),loc}(\mathbb{R}_+)$) be the linear space of all measurable, locally bounded functions from $\mathbb{R}_+$ to γ (resp. to $B(\gamma)$). If $f \in L^{\infty}_{\gamma,loc}(\mathbb{R}_+)$ and $\pi \in L^{\infty}_{B(\gamma),loc}(\mathbb{R}_+)$, then πf is the member of $L^{\infty}_{\gamma,loc}(\mathbb{R}_+)$ given by $(\pi f)(t) = \pi(t) f(t),\ t \in \mathbb{R}_+$. For $f \in L^2_{\gamma}(\mathbb{R}_+)$ and $\pi \in L^{\infty}_{B(\gamma),loc}(\mathbb{R}_+)$, we define the operators $a(f)$, $a^+(f)$ and $\lambda(\pi)$ in $L^+(\mathbb{D}, \Gamma(L^2_{\gamma}(\mathbb{R}_+)))$ as follows: $a(f)e(g) = \langle f, g\rangle_{L^2_{\gamma}(\mathbb{R}_+)} e(g)$, $a^+(f)e(g) = \frac{d}{d\sigma} e(g + \sigma f)|_{\sigma=0}$, $\lambda(\pi)e(g) = \frac{d}{d\sigma} e(e^{\sigma\pi} f)|_{\sigma=0}$ for $g \in L^2_{\gamma}(\mathbb{R}_+)$. The operators $a(f), a^+(f)$ and $\lambda(\pi)$ for arbitrary $f \in L^{\infty}_{\gamma,loc}(\mathbb{R}_+)$ and $\pi \in L^{\infty}_{B(\gamma),loc}(\mathbb{R}_+)$ give rise to the operator-valued stochastic processes A_f, A^+_f and $\wedge_{\pi}$ defined by $A_f(t) \equiv a(f\chi_{[0,t)})$, $A^+_f(t) \equiv a^+(f\chi_{[0,t)})$, $\wedge_{\pi}(t) \equiv \lambda(\pi\chi_{[0,t)})$, $t \in \mathbb{R}_+$, where χ_I denotes the indicator function of the Borel set $I \subseteq \mathbb{R}_+$. These are called the annihilation, creation and gauge processes, respectively, when their values are identified with their ampliations on $\mathcal{R} \underline{\otimes} \Gamma(L^2_{\gamma}(\mathbb{R}_+))$. They are the stochastic integrators in the formulations of the Hudson and Parthasarathy [18] quantum stochastic integration which we adopt in this work. Accordingly, if $p, q, u, v \in L^2_{loc}(\tilde{\mathcal{A}})$, $f, g \in L^{\infty}_{\gamma,loc}(\mathbb{R}_+)$ and $\pi \in L^{\infty}_{B(\gamma),loc}(\mathbb{R}_+)$, then we interpret the integral

$$\int_{t_0}^{t} \big(p(s)d\wedge_{\pi}(s) + q(s)dA_f(s) + u(s)dA^+_g(s) + v(s)ds\big), \quad t_0, t \in \mathbb{R}_+$$

as in Hudson and Parthasarathy [18] (see [2, 4] for details).

Quantum Stochastic Differential Inclusions In our present framework, we shall outline some fundamental concepts concerning quantum stochastic differential inclusions involving multivalued stochastic processes in this subsection. We present the following definitions.

(a) By a multivalued stochastic process indexed by $I \subseteq \mathbb{R}_+$, we mean a multifunction on I with values in $clos(\tilde{\mathcal{A}})$.
(b) A selection of a multivalued stochastic process Φ indexed by I is a stochastic process $X : I \to \tilde{\mathcal{A}}$ with the property that $X(t) \in \Phi(t)$ for almost all $t \in I$.
(c) A multivalued stochastic process Φ will be called (i) adapted if $\Phi(t) \subseteq \tilde{\mathcal{A}}_t$ for each $t \in \mathbb{R}_+$; (ii) measurable if the map $t \to \mathbf{d}_{\eta\xi}(x, \Phi(t))$ is measurable for arbitrary $x \in \tilde{\mathcal{A}}$, $\eta, \xi \in \mathbb{D}\underline{\otimes}\mathbb{E}$; (iii) locally absolutely p-integrable if $t \to \|\Phi(t)\|_{\eta\xi}, t \in \mathbb{R}_+$ lies in $L^p_{loc}(I)$ for arbitrary $\eta, \xi \in \mathbb{D}\underline{\otimes}\mathbb{E}$.

Further notations:

(1) The set of all locally absolutely p-integrable multivalued stochastic processes is denoted by $L^p_{loc}(\tilde{\mathcal{A}})_{mvs}$.
(2) For $p \in (0, \infty)$ and $I \subseteq \mathbb{R}_+$, $L^p_{loc}(I \times \tilde{\mathcal{A}})_{mvs}$ is the set of maps $\Phi : I \times \tilde{\mathcal{A}} \to clos(\tilde{\mathcal{A}})$ such that $t \to \Phi(t, X(t)), t \in I$ lies in $L^p_{loc}(\tilde{\mathcal{A}})_{mvs}$ for every $X \in L^p_{loc}(\tilde{\mathcal{A}})$.
(3) If $\Phi \in L^p_{loc}(I \times \tilde{\mathcal{A}})_{mvs}$ then we define the set

$$L_p(\Phi) \equiv \{\phi \in L^p_{loc}(\tilde{\mathcal{A}}) : \phi \text{ is a selection of } \Phi\}.$$

(4) In what follows, $f, g \in L^{\infty}_{\gamma,loc}(\mathbb{R}_+), \pi \in L^{\infty}_{B(\gamma),loc}(\mathbb{R}_+)$, $\mathbf{1}$ is the identity map on $\mathcal{R} \otimes \Gamma(L^2_{\gamma}(\mathbb{R}_+))$, and M is any of the stochastic processes A_f, A^+_g, $\wedge_{\pi}$, and $s \to s\mathbf{1}, s \in \mathbb{R}_+$. Accordingly, if $\Phi \in L^2_{loc}(I \times \tilde{\mathcal{A}})_{mvs}$ and $(t, X) \in I \times L^2_{loc}(\tilde{\mathcal{A}})$, then we define multivalued stochastic integral:

$$\int_{t_0}^{t} \Phi(s, X(s))dM(s) \equiv \left\{\int_{t_0}^{t} \phi(s)dM(s) : \phi \in L_2(\Phi)\right\}.$$

Thus, for coefficients $E, F, G, H \in L^2_{loc}(I \times \tilde{\mathcal{A}})_{mvs}$, this paper is concerned with the establishment of some topological properties of solution sets of quantum stochastic differential inclusion in the integral form given by

$$X(t) \in a + \int_{t_0}^{t} (E(s, X(s))d \wedge_\pi (s) + F(s, X(s))dA_f(s) \\ + G(s, X(s))dA_g^+(s) + H(s, X(s))ds), \quad \text{almost all } t \in [t_0, T]. \tag{2.1}$$

As in [3, 5–7], corresponding to any pair of elements $\eta, \xi \in \mathbb{D}\underline{\otimes}\mathbb{E}$ such that $\eta = c \otimes e(\alpha), \xi = d \otimes e(\beta), \alpha, \beta \in L^2_\gamma(\mathbb{R}_+)$ we shall employ the equivalent form of (2.1) as established in [13] given by the nonclassical ordinary differential inclusion:

$$\frac{d}{dt}\langle \eta, X(t)\xi \rangle \in P(t, X(t))(\eta, \xi), \\ X(0) = a, \quad \text{almost all } t \in [t_0, T]. \tag{2.2}$$

The multivalued map P appearing in (2.2) is of the form

$$P(t, x)(\eta, \xi) = \langle \eta, P_{\alpha\beta}(t, x)\xi \rangle.$$

The map $P_{\alpha\beta} : [t_0, T] \times \tilde{\mathcal{A}} \to 2^{\tilde{\mathcal{A}}}$ has the explicit form:

$$P_{\alpha\beta}(t, x) = \mu_{\alpha\beta}(t)E(t, x) + \nu_\beta(t)F(t, x) + \sigma_\alpha(t)G(t, x) + H(t, x).$$

The complex valued functions $\mu_{\alpha\beta}, \nu_\beta, \sigma_\alpha : [t_0, T] \to \mathbb{C}$ are defined by

$$\mu_{\alpha\beta}(t) = \langle \alpha(t), \pi(t)\beta(t) \rangle_\gamma, \qquad \nu_\beta(t) = \langle f(t), \beta(t) \rangle_\gamma, \\ \sigma_\alpha(t) = \langle \alpha(t), g(t) \rangle_\gamma, \quad t \in [t_0, T]$$

for all $(t, x) \in [t_0, T] \times \tilde{\mathcal{A}}$ and the coefficients E, F, G, H belong to the space $L^2_{loc}([t_0, T] \times \tilde{\mathcal{A}})_{mvs}$ of multivalued stochastic processes with closed values.

As explained in [13], the map P cannot in general be written in the form:

$$P(t, x)(\eta, \xi) = \tilde{P}(t, \langle \eta, x\xi \rangle)$$

for some complex valued multifunction $\tilde{P}$ defined on $[t_0, T] \times \mathbb{C}$, for $t \in [t_0, T]$, $x \in \tilde{\mathcal{A}}$, $\eta, \xi \in \mathbb{D}\underline{\otimes}\mathbb{E}$.

A solution of QSDI (2.1) is a stochastic process $\Phi \in Ad(\tilde{\mathcal{A}})_{wac} \cap L^2_{loc}(\tilde{\mathcal{A}})$ satisfying QSDI (2.1). Since solutions of QSDI (2.1) corresponding to an initial value $a \in \tilde{\mathcal{A}}$ are not unique, the symbol $S^{(T)}(a)$ denotes the set of such solutions. Associated with $S^{(T)}(a)$, we define the space of absolutely continuous complex valued functions on the interval $[t_0, T]$ corresponding to each pair $\eta, \xi \in \mathbb{D}\underline{\otimes}\mathbb{E}$ as follows:

$$S^{(T)}(a)(\eta, \xi) = \{\langle \eta, \Phi(\cdot)\xi \rangle : \Phi \in S^{(T)}(a)\}.$$

We remark that in view of the equivalence of QSDI (2.1) and (2.2) which has already been established in [13], we can study the diverse features of the solutions of QSDI (2.1) by equivalently study the features of the solutions of the nonclassical ordinary differential inclusion (2.2). This has been the situation in our previous works mentioned earlier.

In what follows, we assume that the coefficients E, F, G, H in (2.1) and the map $(t, x) \to P(t, x)(\eta, \xi)$ appearing in (2.2) satisfy the following conditions.

$\mathcal{S}_{(i)}$ The values of the map $(t, x) \to P(t, x)(\eta, \xi)$ are nonempty closed, subsets of the field $\mathbb{C}$ of complex numbers for each point $(t, x) \in I \times \tilde{\mathcal{A}}$ and $\eta, \xi \in \mathbb{D}\underline{\otimes}\mathbb{E}$.

$\mathcal{S}_{(ii)}$ The map $t \to P(t, x)(\eta, \xi)$ is measurable.

$\mathcal{S}_{(iii)}$ The coefficients E, F, G, H are continuous from $I \times \tilde{\mathcal{A}}$ to $(clos(\tilde{\mathcal{A}}), \tau_H)$.

$\mathcal{S}_{(iv)}$ There exists a map $K_{\eta\xi} : [t_0, T] \to \mathbb{R}_+$ lying in $L^1_{loc}([t_0, T])$ such that

$$\rho(P(t, x)(\eta, \xi), P(t, y)(\eta, \xi)) \leq K_{\eta\xi}(t)\|x - y\|_{\eta\xi}$$

for $t \in [t_0, T]$, and for each pair $x, y \in \tilde{\mathcal{A}}$.

Subject to the conditions $\mathcal{S}_{(i)}$–$\mathcal{S}_{(iv)}$ and corresponding to each quasi solution process $Z \in Ad(\tilde{\mathcal{A}})_{wac}$ of QSDI (2.2), where

$$d\left(\frac{d}{dt}\langle\eta, Z(t)\xi\rangle, P(t, Z(t))(\eta, \xi)\right) \leq \rho_{\eta\xi}(t), \quad t \in [t_0, T],$$

$\rho_{\eta\xi} \in L^1_{loc}([t_0, T])$, the following existence result has been established in [13].

Theorem 2.1 ([13]) *Suppose that $\mathcal{S}_{(i)}$–$\mathcal{S}_{(iv)}$ holds. Then there exists a solution $R(Z) \in S^{(T)}(Z(t_0))$ of QSDI (2.1) such that*

$$\|Z(t) - R(Z)(t)\|_{\eta\xi} \leq \mathbb{E}_{\eta\xi}(t), \quad t \in [t_0, T], \tag{2.3}$$

$$\left|\frac{d}{dt}\langle\eta, Z(t)\xi\rangle - \frac{d}{dt}\langle\eta, R(Z)(t)\xi\rangle\right| \leq K_{\eta\xi}(t)\mathbb{E}_{\eta\xi}(t) + \rho_{\eta\xi}(t), \quad \textit{almost all } t \in [t_0, T], \tag{2.4}$$

where

$$\mathbb{E}_{\eta\xi}(t) = \int_{t_0}^{t} \exp(M_{\eta\xi}(t) - M_{\eta\xi}(s))\rho_{\eta\xi}(s)ds, \tag{2.5}$$

$$M_{\eta\xi}(t) = \int_{t_0}^{t} K_{\eta\xi}(s)ds. \tag{2.6}$$

We remark that Theorem 2.1, which is a generalization of the Filippov existence theorem to the present noncommutative setting played a central role in our previous works in [3, 5–7]. These works concerned the establishment of a continuous selection of some multifunctions associated with the solution sets and some continuous representations of the reachable sets. The result has also facilitated the derivation of some exponential formulae for the reachable sets, construction of their approximate sets and the establishment of error estimates for discretized versions of QSDI (2.2).

Define the space of solutions of QSDI (2.2) by

$$S^{(T)}(P) := \bigcup_{a \in \tilde{\mathcal{A}}} S^{(T)}(a)$$

and the associated space of absolutely continuous convex valued functions corresponding to each pair of $\eta, \xi \in \mathbb{D}\underline{\otimes}\mathbb{E}$ is defined by:

$$S^{(T)}(P)(\eta, \xi) := \{\langle\eta, \Phi(\cdot)\xi\rangle : \Phi \in S^{(T)}(P)\}.$$

For arbitrary pair of elements $\eta, \xi \in \mathbb{D}\underline{\otimes}\mathbb{E}$, and for any process $y \in wac(\tilde{\mathcal{A}})$ and a family of processes $\{y_j\}_{j\geq 0} \subseteq wac(\tilde{\mathcal{A}})$, we employ the respective notations: $y_{\eta\xi}(\cdot) := \langle \eta, y(\cdot)\xi\rangle$ and $y_{\eta\xi,j}(\cdot) := \langle \eta, y_j(\cdot)\xi\rangle$, $j = 0, 1, 2, \ldots$. For any non empty subset $M \subseteq wac(\tilde{\mathcal{A}})$ we define the function space $M(\eta, \xi) := \{\Phi_{\eta\xi}(\cdot) := \langle \eta, \Phi(\cdot)\xi\rangle : \Phi \in M\}$ and for any element $y_{\eta\xi}(\cdot) \in M(\eta, \xi)$, we define the number

$$\nu(y_{\eta\xi}) := d_{AC}(y_{\eta\xi}(\cdot), S(P)(\eta, \xi)).$$

Our main result in this paper is the construction of a pair of mappings $\Upsilon : wac(\tilde{\mathcal{A}}) \to wac(\tilde{\mathcal{A}})$ and $R : wac(\tilde{\mathcal{A}})(\eta, \xi) \to wac(\tilde{\mathcal{A}})(\eta, \xi)$ with respective domains M and $M(\eta, \xi)$ of general types such that the map R is continuous on its domain. The construction of the maps leads to the establishment of some topological properties of the function space $S^{(T)}(P)(\eta, \xi)$ associated with the solution space $S^{(T)}(P)$ of QSDI (2.2). To prove the main result in the next section, we present here some further notations and results. If $\mathcal{B} \subset [t_0, T]$ is a measurable set, we write $\mu(\mathcal{B})$ for its Lebesgue measure and $\chi_{\mathcal{B}}(t)$ for its characteristic function. We shall employ the notion of a continuous decomposition (or partition) of the interval $[t_0, T]$ depending on a parameter $y \in Y$ where Y is a separable metric space. We recall the following definition from [26].

Definition 2.2 Let Y be a separable metric space. A family of subsets $\{T_i(y)\}_{i\geq 1}$ of the interval $[t_0, T]$ depending on the parameter $y \in Y$ will be called a continuous decomposition of $[t_0, T]$ if the following conditions hold.

(1) $T_i(y)$ is a measurable set for all $y \in Y$ and $i \geq 1$.
(2) $T_i(y) \cap T_j(y) = \emptyset$ for $y \in Y, i \neq j$ and $\bigcup_{i\geq 1} T_i(y) = [t_0, T]$.
(3) If $y_0 \in Y$ then for any $y \in Y$, there exists a neighbourhood $V(y_0)$ and a finite index set $\Omega(y_0)$ such that when i does not belong to $\Omega(y_0)$ and $y \in V(y_0)$ the relation $\mu(T_i(y)) = 0$ holds.
(4) $\mu(T_i(y) \triangle T_i(y_0)) \to 0$ for $y \to y_0$ when $y_0 \in Y$ and for $i \geq 1$, where $A \triangle B = A \cup B \setminus A \cap B$ for any two subsets A, B of $[t_0, T]$.

Next, we recall the only lemma established in [26, p. 159] in a form that is suitable for the setting of this paper. The result remains valid for the present framework.

Lemma 2.3 *Let Y be a nonempty subset of $wac(\tilde{\mathcal{A}})$ so that $Y(\eta, \xi) \subset wac(\tilde{\mathcal{A}})(\eta, \xi)$ corresponding to an arbitrary pair $\eta, \xi \in \mathbb{D}\underline{\otimes}\mathbb{E}$. Let $\nu(y_{\eta\xi}(\cdot)) > 0$ for $y_{\eta\xi} \in Y(\eta, \xi)$. Suppose that $\{T^0_m(y_{\eta\xi})\}_{m\geq 1}$ is a continuous decomposition of the interval $[t_0, T]$. Then for $\sigma > 0$ there exists a continuous decomposition $\{T^1_j(y_{\eta\xi})\}_{j\geq 1}$ of $[t_0, T]$ and a set of processes $\{y_j\}_{j\geq 1} \subset Y$ with the corresponding set of complex valued functions $\{y_{\eta\xi,j}\}_{j\geq 1} \subset Y(\eta, \xi)$ satisfying the following*:

(1) *If* $\mu(T^1_j(y_{\eta\xi})) > 0,$ *then*

$$\sum_{m\geq 1} \mu(T^0_m(y_{\eta\xi}) \triangle T^0_m(y_{\eta\xi,j})) \leq \min\{\sigma, 2\nu(y_{\eta\xi})\},$$

(2) $$\sum_{m,j\geq 1} \mu\left(T^1_j(y_{\eta\xi}) \cap (T^0_m(y_{\eta\xi}) \triangle T^0_m(y_{\eta\xi,j}))\right) \leq \min\{\sigma, 2\nu(y_{\eta\xi})\}, \quad y_{\eta\xi} \in Y(\eta, \xi).$$

Remark 2.4 In the proof of the only lemma in [26], a continuous decomposition of the interval $[t_0, T]$ was constructed with parameters that are elements of the space $AC([t_0, T], \mathbb{R}^n)$.

This construction remains valid in our case for parameters lying in the space $AC([t_0, T], \mathbb{C})$. By using suitable notations, we outline here a similar construction that would be needed in the proof of our main result.

By the definition of a continuous decomposition of the interval $[t_0, T]$, we can choose for each $y_0 \in Y \subseteq wac(\tilde{\mathcal{A}})$ with $y_{\eta\xi,0}(\cdot) \in Y(\eta, \xi)$ a number $\delta(y_{\eta\xi,0}) > 0$ satisfying

$$\delta(y_{\eta\xi,0}) < \frac{1}{2} \min\{\sigma, \nu(y_{\eta\xi,0})\}$$

such that

$$\sum_{i \geq 1} \mu\left(T_i^0(y_{\eta\xi}) \triangle T_i^0(y_{\eta\xi,0})\right) \leq \frac{1}{2} \min\{\sigma, \nu(y_{\eta\xi,0})\},$$

when $y_{\eta\xi} \in Y(\eta, \xi)$ satisfying $|y_{\eta\xi}(\cdot) - y_{\eta\xi,0}(\cdot)|_{AC} < \delta(y_{\eta\xi,0})$.

We construct a ball $B(y_{\eta\xi}, \delta(y_{\eta\xi}))$ of radius $\delta(y_{\eta\xi})$ about each function $y_{\eta\xi}(\cdot) \in Y(\eta, \xi)$. These balls form an open covering of the separable metric space $Y(\eta, \xi) \subseteq AC([t_0, T], \mathbb{C})$. The covering contains a locally open countable covering $\{U_j\}_{j \geq 1}$ depending on the pair $\eta, \xi \in \mathbb{D} \underline{\otimes} \mathbb{E}$. Furthermore, there is a locally open covering $\{V_j\}_{j \geq 1}$ such that $\overline{V}_j \subset U_j$, $j \geq 1$, where $\overline{V}_j$ denotes the closure of the set V_j (see [21] and [26, p. 159]).

Let $\{P_j(\cdot)\}_{j \geq 1}$ be a continuous partition of unity subordinate to the covering $\{V_j\}_{j \geq 1}$ and let $\{q_j(\cdot)\}_{j \geq 1}$ be the set of continuous functions such that

$$q_j(y_{\eta\xi}) = \begin{cases} 0, & \text{if } y_{\eta\xi} \in Y(\eta, \xi) \setminus U_j, \\ 1, & \text{if } y_{\eta\xi} \in V_j. \end{cases}$$

Suppose that $A \subset [t_0, T]$ is a measurable set and let

$$\begin{aligned} w_k(y_{\eta\xi}) &:= \inf\{w \in [t_0, T] : \mu(A \cap [t_0, w]) \geq \mu(A) q_k(y_{\eta\xi})\}, \\ \Phi_k^0(y_{\eta\xi})[A] &:= A \cap [t_0, w_k(y_{\eta\xi})), \\ \Phi_k^1(y_{\eta\xi})[A] &:= [t_0, T] \setminus \Phi_k^0(y_{\eta\xi})[A]. \end{aligned}$$

The number $c(A)$ and the sets $c^0[A], c^1[A]$ are defined as follows:

$$\begin{aligned} c(A) &= \inf\{c \in [t_0, T] : \mu(A \cap [t_0, c]) \geq \mu(A)\}, \\ c^0[A] &= A \cap [t_0, c(A)), \qquad c^1[A] = [t_0, T] \setminus c^0[A]. \end{aligned}$$

Next we define

$$\begin{aligned} \tau_0(y_{\eta\xi}) &= t_0, \\ \tau_j(y_{\eta\xi}) &= \sup\{\tau \in [t_0, T] : \mu(A \cap [\tau_{j-1}(y_{\eta\xi}), \tau]) \leq \mu(A) P_j(y_{\eta\xi})\}, \quad j \geq 1, \\ \Psi_1(y_{\eta\xi})[A] &= A \cap [t_0, \tau_1(y_{\eta\xi})], \end{aligned}$$

and

$$\Psi_j(y_{\eta\xi})[A] = A \cap (\tau_{j-1}(y_{\eta\xi}), \tau_j(y_{\eta\xi})], \quad j > 1.$$

We fix arbitrary functions $y_{\eta\xi,j}(\cdot) \in V_j$, where $y_{\eta\xi,j}(\cdot) = \langle \eta, y_j(\cdot)\xi \rangle$, $j \geq 1$ for some $y_j \in Y$ and let Θ denotes the family of sets $\tilde{\theta} = \{(\vartheta_{i,k}^0, \vartheta_i^1)\}_{i,k \geq 1}$ formed of number pairs $(\vartheta_{i,k}^0, \vartheta_i^1)$

in which the numbers take the value 0 or 1. The sets $T_j^1(y_{\eta\xi})$, $j \geq 1$ are defined as follows:

$$T_j^1(y_{\eta\xi}) = \bigcup_{\tilde{\theta}\in\Theta} \Psi_j(y_{\eta\xi})\left[\bigcap_{k,i\geq 1}(\Phi_k^{\vartheta_{i,k}^0}(y_{\eta\xi})[T_i^0(y_{\eta\xi,k})] \cap c^{\vartheta_i^1}[T_i^0(y_{\eta\xi})])\right].$$

The family of sets $T_j^1(y_{\eta\xi})$ thus defined, is a continuous decomposition of the interval $[t_0, T]$. This can be shown exactly by the same way as in the proof of the only lemma in [26, p. 159] satisfying the conclusion of Lemma 2.3 above.

3 Main Results

We shall establish the main results of this paper in this section. To this end, the proof of the main theorem below adapts the technique employed in [26] in a way that is suitable for the analysis of the present quantum stochastic differential inclusion. In what follows as before, η, ξ is a pair of arbitrary elements in $\mathbb{D}\underline{\otimes}\mathbb{E}$.

Theorem 3.1 *Assume that the map $(t, x) \to P(t, x)(\eta, \xi)$ satisfies conditions $\mathcal{S}_{(i)}$–$\mathcal{S}_{(iii)}$. Assume further that a non empty set $M \subset wac(\tilde{\mathcal{A}})$ is given and there exist positive functions $\rho_{\eta\xi}, N_{\eta\xi} : [t_0, T] \to \mathbb{R}_+$ lying in $L^1_{loc}([t_0, T], \mathbb{R}_+)$ such that*

$$d\left(\frac{d}{dt}\langle\eta, Y(t)\xi\rangle, P(t, Y(t))(\eta, \xi)\right) \leq \rho_{\eta\xi}(t),$$

and

$$\left|\frac{d}{dt}\langle\eta, Y(t)\xi\rangle\right| \leq N_{\eta\xi}(t), \quad t \in [t_0, T],\ Y \in M.$$

Then for arbitrary $\epsilon > 0$, there exists a map $R : M(\eta, \xi) \to S^{(T)}(P)(\eta, \xi)$ continuous in the norm topology of the space $AC([t_0, T], \mathbb{C})$ such that:

(1) $R(y_{\eta\xi}(\cdot))(t_0) = y_{\eta\xi}(t_0)$.
(2) $R(y_{\eta\xi}(\cdot)) = y_{\eta\xi}(\cdot)$, *for* $y_{\eta\xi} \in M(\eta, \xi) \cap S^{(T)}(P)(\eta, \xi)$.
(3) $|R(y_{\eta\xi}(\cdot))(t) - y_{\eta\xi}(t)| \leq \mathbb{E}_{\eta\xi}(t) + \epsilon$, $t \in [t_0, T]$, $y_{\eta\xi} \in M(\eta, \xi) \setminus S^{(T)}(P)(\eta, \xi)$.

Proof Let $\epsilon > 0$ be given. We define a sequence of real positive numbers that depend on η, ξ by:

$$\alpha_k = \frac{\epsilon}{2^{k+2}\exp[2M_{\eta\xi}(T)]}, \quad k = 0, 1, 2, \ldots. \tag{3.1}$$

Then there exists a number $\sigma_0 = \sigma_0(\eta, \xi)$ depending on η, ξ such that

$$\sigma_0 \in \left(0, \frac{\alpha_0}{3 + 4M_{\eta\xi}(T)}\right)$$

and

$$\int_A N_{\eta\xi}(t)dt < \alpha_0 \tag{3.2}$$

for every measurable set $A \subseteq [t_0, T]$ satisfying $\mu(A) < \sigma_0$.

As highlighted in Remark 2.4, we shall construct locally finite open coverings $\{U_j\}$, $\{V_j\}$ and function systems $\{P_j(\cdot)\}$, $\{q_j(\cdot)\}$ and $\{y_{\eta\xi,j}(\cdot)\}$ for $j \geq 1$. We first obtain a partition of the interval $[t_0, T]$ into L equidistant points $t_0 = \tau_0 < \tau_1 < \cdots < \tau_L = T$ where $L > \frac{T-t_0}{\sigma_0}$.

By employing the notion of simple processes as defined in [18], we let W_j be simple processes in $L^1_{loc}(\tilde{\mathcal{A}})$ having a countable number of values $W_j(t)$ such that

$$\|W_j(t)\|_{\eta\xi} := |W_{\eta\xi,j}(t)| < N_{\eta\xi}(t), \quad t \in [t_0, T]$$

and

$$|\dot{y}_{\eta\xi,j}(\cdot) - W_{\eta\xi,j}(\cdot)|_{L^1} < \frac{\delta(y_{\eta\xi,j})}{2^j L}, \tag{3.3}$$

where $\delta(y_{\eta\xi})$ is defined under Remark 2.4, $\dot{y}_{\eta\xi,j}(t) = \frac{d}{dt} y_{\eta\xi,j}(t)$, and $|\cdot|_{L^1}$ is the norm in the space $L^1([t_0, T], \mathbb{C})$ of integrable complex valued functions on the interval $[t_0, T]$.

Let $T_{i,k}$ be measurable subsets of $[t_0, T]$ such that

$$W_k(t) \equiv W_k^i, \quad W_{\eta\xi,k}(t) \equiv W^i_{\eta\xi,k}, \quad t \in T_{i,k}, \; T_{i,k} \cap T_{i',k} = \emptyset, \; i \neq i',$$

and $\bigcup_{i\geq 1} T_{i,k} = [t_0, T]$. We put $T^l_{i,k} = [\tau_l, \tau_{l+1}] \cap T_{i,k}$ and follow a similar argument as in the proof of the lemma in [26]. Employing the procedure under Remark 2.4, we construct, for each fixed l, $y_{\eta\xi}(\cdot) \in M(\eta, \xi)$, the sets $\Psi_j(y_{\eta\xi})[\cdot]$, $\Phi^0_k(y_{\eta\xi})[\cdot]$, interchanging $[t_0, T]$ and $[\tau_l, \tau_{l+1}]$. Next define

$$T_{j,l}(y_{\eta\xi}) = \bigcup_{\tilde{\theta}\in\Theta} \Psi_j(y_{\eta\xi}) \Bigg[\bigcap_{i,k\geq 1} \Phi_k^{\vartheta^0_{i,k}}(y_{\eta\xi})[T^l_{i,k}] \Bigg], \tag{3.4}$$

and

$$T^0_j(y_{\eta\xi}) = \bigcup_{l=1}^{L} T_{j,l}(y_{\eta\xi}). \tag{3.5}$$

As shown in [26], the family of sets $\{T^0_j(y_{\eta\xi}(\cdot))\}$ is a continuous decomposition of the interval $[t_0, T]$ depending in this case on the matrix element $y_{\eta\xi}(\cdot)$ of the process $y \in M$.

If $y \in S^{(T)}(P)$, then $y \in S^{(T)}(a)$ for some $a \in \tilde{\mathcal{A}}$. Thus, by the properties of such solutions as established in [13] (see also [7]), there exists a stochastic process $V : [t_0, T] \to \tilde{\mathcal{A}}$ lying in $Ad(\tilde{\mathcal{A}})_{wac} \cap L^1_{loc}(\tilde{\mathcal{A}})$ such that:

$$y(t) = a + \int_{t_0}^{t} V(s)ds, \quad \text{a.e. } t \in [t_0, T] \tag{3.6}$$

and for any $y_k \in S^{(T)}(a)$, we write

$$y_k(t) = a + \int_{t_0}^{t} V_k(s)ds, \quad \text{a.e. } t \in [t_0, T],$$

for some stochastic process

$$V_k \in Ad(\tilde{\mathcal{A}})_{wac} \cap L^1_{loc}(\tilde{\mathcal{A}}).$$

Next, we consider the following pair of mappings: $g_0 : M \to L^1_{loc}(\tilde{\mathcal{A}})$ and $G_0 : M(\eta,\xi) \to L^1_{loc}([t_0,T],\mathbb{C})$ defined by:

$$g_0(y)(t) = \begin{cases} V(t), & y \in M \cap S^{(T)}(P), \\ \sum_{j\geq 1} \chi_{T^0_j(y_{\eta\xi})}(t) W_j(t), & y \in M \setminus S^{(T)}(P), \end{cases} \tag{3.7}$$

where W_j and V are given respectively by (3.3) and (3.6) above and

$$G_0(y_{\eta\xi})(t) = \begin{cases} \frac{d}{dt}\langle \eta, y(t)\xi\rangle, & y_{\eta\xi} \in M(\eta,\xi) \cap S^{(T)}(P)(\eta,\xi), \\ \sum_{j\geq 1} \chi_{T^0_j(y_{\eta\xi})}(t) W_{\eta\xi,j}(t), & y_{\eta\xi} \in M(\eta,\xi) \setminus S^{(T)}(P)(\eta,\xi). \end{cases} \tag{3.8}$$

Notice that by definition, $G_0(y_{\eta\xi})(t) = \langle \eta, g_0(y)(t)\xi\rangle$ where

$$\frac{d}{dt}\langle \eta, y(t)\xi\rangle = \langle \eta, V(t)\xi\rangle, \quad t \in [t_0,T].$$

We prove that the mapping G_0 is continuous in the norm of $AC([t_0,T],\mathbb{C})$ defined on $M(\eta,\xi)$. If $y_0 \in M \setminus S^{(T)}(P)$, then

$$\begin{aligned} |G_0(y_{\eta\xi}) - G_0(y_{\eta\xi,0})|_{L^1} &\leq \int_{t_0}^{T} \sum_{j\geq 1} \chi_{T^0_j(y_{\eta\xi}) \triangle T^0_j(y_{\eta\xi,0})}(t) N_{\eta\xi}(t)\,dt \\ &\to 0, \quad \text{as } y_{\eta\xi}(\cdot) \to y_{\eta\xi,0}(\cdot). \end{aligned} \tag{3.9}$$

Suppose that $y_0 \in S^{(T)}(P)$ and $y \in M \setminus S^{(T)}(P)$, then $y_{\eta\xi,0} \in S^{(T)}(P)(\eta,\xi)$ and $y_{\eta\xi} \in M(\eta,\xi) \setminus S^{(T)}(P)(\eta,\xi)$. By the property of the partition of unity $P_j(\cdot)$, $P_j(y_{\eta\xi}) > 0$ implies that $y_{\eta\xi} \in V_j$. By the construction of the covering $\{V_j\}$, there is a ball centred at $\tilde{y}_{\eta\xi} \in M(\eta,\xi) \setminus S^{(T)}(P)(\eta,\xi)$ with radius

$$\delta(\tilde{y}_{\eta\xi}) \leq \frac{1}{2}\nu(\tilde{y}_{\eta\xi}) \tag{3.10}$$

containing an element of V_j. Hence, for $y_{\eta\xi} \in V_j$, $|y_{\eta\xi} - \tilde{y}_{\eta\xi}|_{AC} \leq \frac{1}{2}\nu(\tilde{y}_{\eta\xi})$. We conclude that

$$\begin{aligned} \nu(\tilde{y}_{\eta\xi}) &\leq \nu(y_{\eta\xi}) + |y_{\eta\xi} - \tilde{y}_{\eta\xi}|_{AC} \\ &\leq \nu(y_{\eta\xi}) + \frac{1}{2}\nu(\tilde{y}_{\eta\xi}). \end{aligned}$$

Hence

$$\nu(\tilde{y}_{\eta\xi}) \leq 2\nu(y_{\eta\xi}). \tag{3.11}$$

Estimating the distance between $y_{\eta\xi}$ and $y_{\eta\xi,j} \in V_j$, we have

$$\begin{aligned} |y_{\eta\xi} - y_{\eta\xi,j}|_{AC} &\leq |y_{\eta\xi} - \tilde{y}_{\eta\xi}|_{AC} + |y_{\eta\xi,j} - \tilde{y}_{\eta\xi}|_{AC} \\ &\leq 2\delta(\tilde{y}_{\eta\xi}) \\ &\leq \nu(\tilde{y}_{\eta\xi}) \quad \text{by (3.10)} \end{aligned}$$

$$\leq 2\nu(y_{\eta\xi}) \quad \text{by (3.11)}$$
$$\leq 2|y_{\eta\xi,0} - y_{\eta\xi}|_{AC}. \tag{3.12}$$

Inequality (3.12) holds since $y_{\eta\xi,0} \in S^{(T)}(P)(\eta,\xi)$. Thus

$$\begin{aligned}|y_{\eta\xi,0} - y_{\eta\xi,j}|_{AC} &\leq |y_{\eta\xi,0} - y_{\eta\xi}|_{AC} + |y_{\eta\xi} - y_{\eta\xi,j}|_{AC} \\ &\leq 3|y_{\eta\xi,0} - y_{\eta\xi}|_{AC}\end{aligned} \tag{3.13}$$

by applying (3.12).

Assuming that $z_k \in M \setminus S^{(T)}(P)$ and $z_{\eta\xi,k} \to y_{\eta\xi,0}$ in $AC([t_0,T],\mathbb{C})$ for $k \to \infty$, we estimate the norm:

$$|G_0(y_{\eta\xi,0}) - G_0(z_{\eta\xi,k})|_{L^1} = \int_{t_0}^{T} \left| \frac{d}{dt}\langle \eta, y_0(t)\xi\rangle - \sum_{j\geq 1} \chi_{T_j^0(Z_{\eta\xi,k})}(t) W_{\eta\xi,j}(t) \right| dt. \tag{3.14}$$

Let j^* be such that $\mu(T_{j^*}^0(z_{\eta\xi,k})) > 0$, then from (3.14), we have

$$\begin{aligned}
&\int_{t_0}^{T} \left| \frac{d}{dt}\langle \eta, y(t)\xi\rangle - \sum_{j\geq 1} \chi_{T_j^0(z_{\eta\xi,k})}(t) W_{\eta\xi,j}(t) \right| dt \\
&\leq \int_{t_0}^{T} \left(\left| \frac{d}{dt}\langle \eta, y_0(t)\xi\rangle - \frac{d}{dt}\langle \eta, y_{j^*}(t)\xi\rangle \right| \right. \\
&\quad \left. + \left| \frac{d}{dt}\langle \eta, y_{j^*}(t)\xi\rangle - W_{\eta\xi,j^*}(t) \right| \right) dt + \sum_{j\geq 1} \int_{T_j^0(z_{\eta\xi,k})} |W_{\eta\xi,j^*}(t) - W_{\eta\xi,j}(t)| dt \\
&\leq 5|y_{\eta\xi,0}(\cdot) - z_{\eta\xi,k}(\cdot)|_{AC} + \sum_{j\geq 1} P_j(z_{\eta\xi,k}) |W_{\eta\xi,j^*}(\cdot) - W_{\eta\xi,j}(\cdot)|_{L^1} \\
&\leq 12|y_{\eta\xi,0}(\cdot) - z_{\eta\xi,k}(\cdot)|_{AC}
\end{aligned} \tag{3.15}$$

by applying estimates (3.12) and (3.13).

Consequently,

$$|G_0(y_{\eta\xi,0}) - G_0(z_{\eta\xi,k})|_{L^1} \leq 12|y_{\eta\xi,0}(\cdot) - z_{\eta\xi,k}(\cdot)|_{AC}. \tag{3.16}$$

Thus, the map $G_0 : M(\eta,\xi) \to L^1([t_0,T],\mathbb{C})$ is continuous.

Furthermore for $y \in M \setminus S^{(T)}(P)$; $y_{\eta\xi} \in M(\eta,\xi) \setminus S^{(T)}(P)(\eta,\xi)$ and by denoting the unit ball in the complex field by S_2, we have

$$\begin{aligned}
\int_{\tau_l}^{\tau_{l+1}} G_0(y_{\eta\xi})(t)dt &= \sum_{i\geq 1} \int_{T_{j,l}(y_{\eta\xi})} W_{\eta\xi,j}(t)dt \\
&= \sum_{j\geq 1} P_j(y_{\eta\xi}) \int_{\tau_l}^{\tau_{l+1}} W_{\eta\xi,j}(t)dt \\
&\in \sum_{j\geq 1} P_j(y_{\eta\xi})(y_{\eta\xi,j}(\tau_{l+1}) - y_{\eta\xi,j}(\tau_l)) + \frac{\sigma_0}{2L} S_2.
\end{aligned}$$

Hence,

$$\begin{aligned}\int_{t_0}^{\tau_{l+1}} G_0(y_{\eta\xi})(t)dt &\in \sum_{j\geq 1} P_j(y_{\eta\xi})(y_{\eta\xi,j}(\tau_{l+1}) - y_{\eta\xi,j}(t_0)) + \frac{\sigma_0}{2}S_2\\ &\subseteq \sum_{j\geq 1} P_j(y_{\eta\xi})(y_{\eta\xi}(\tau_{l+1}) - y_{\eta\xi}(t_0)) + \sigma_0 S_2\\ &= y_{\eta\xi}(\tau_{l+1}) - y_{\eta\xi}(t_0) + \sigma_0 S_2.\end{aligned} \tag{3.17}$$

Equation (3.17) holds since the functions $y_{\eta\xi,j}$ lie in some neighbourhood of the function $y_{\eta\xi}(\cdot)$ and by using the relation $\sum_{j\geq 1} P_j(y_{\eta\xi}) = 1$. Therefore for $t \in [t_0, T]$, we have from (3.17),

$$\begin{aligned}&\left|y_{\eta\xi}(t_0) + \int_{t_0}^{t} G_0(y_{\eta\xi})(s)ds - y_{\eta\xi}(t)\right|\\ &\leq \left|y_{\eta\xi}(t_0) + \int_{t_0}^{\tau_l} G_0(y_{\eta\xi})(s)ds - y_{\eta\xi}(\tau_l)\right| + \left|y_{\eta\xi}(\tau_l) + \int_{\tau_l}^{t} G_0(y_{\eta\xi})(s)ds - y_{\eta\xi}(t)\right|\\ &\leq 3\sigma_0 \leq \alpha_0.\end{aligned} \tag{3.18}$$

Now assume that $\mu(T_j^0(y_{\eta\xi})) > 0$, and $t \in T_j^0(y_{\eta\xi})$, then by employing (3.18) we have:

$$\begin{aligned}&d\left(G_0(y_{\eta\xi})(t), P\left(t, y(t_0) + \int_{t_0}^{t} g_0(y)(s)ds\right)(\eta,\xi)\right)\\ &= d\left(W_{\eta\xi,j}(t), P\left(t, y(t_0) + \int_{t_0}^{t} g_0(y)(s)ds\right)(\eta,\xi)\right)\\ &\leq d\left(\frac{d}{dt}\langle\eta, y_j(t)\xi\rangle + W_{\eta\xi,j}(t) - \frac{d}{dt}\langle\eta, y_j(t)\xi\rangle, P(t, y_j(t))(\eta,\xi)\right)\\ &\quad + \rho\left(P(t, y_j(t))(\eta,\xi), P\left(t, y(t_0) + \int_{t_0}^{t} g_0(y)(s)ds\right)(\eta,\xi)\right)\\ &\leq d\left(\frac{d}{dt}\langle\eta, y_j(t)\xi\rangle, P(t, y_j(t))(\eta,\xi)\right) + \left|W_{\eta\xi,j}(t) - \frac{d}{dt}\langle\eta, y_j(t)\xi\rangle\right|\\ &\quad + K_{\eta\xi}(t)\left\|y_j(t) - y(t) + y(t) - \left(y(t_0) + \int_{t_0}^{t} g_0(y)(s)ds\right)\right\|_{\eta\xi}\\ &= d\left(\frac{d}{dt}\langle\eta, y_j(t)\xi\rangle, P(t, y_j(t))(\eta,\xi)\right) + \left|W_{\eta\xi,j}(t) - \frac{d}{dt}\langle\eta, y_j(t)\xi\rangle\right|\\ &\quad + K_{\eta\xi}(t)|y_{\eta\xi,j}(t) - y_{\eta\xi}(t)| + K_{\eta\xi}(t)\left|y_{\eta\xi}(t) - y_{\eta\xi}(t_0) - \int_{t_0}^{t} G_0(y_{\eta\xi})(s)ds\right|\\ &\leq d\left(\frac{d}{dt}\langle\eta, y_j(t)\xi\rangle, P(t, y_j(t))(\eta,\xi)\right) + \left|W_{\eta\xi,j}(t) - \frac{d}{dt}\langle\eta, y_j(t)\xi\rangle\right|\\ &\quad + K_{\eta\xi}(t)|y_{\eta\xi,j}(t) - y_{\eta\xi}(t)| + 3\sigma_0 K_{\eta\xi}(t) \quad \text{(by (3.18))}\end{aligned} \tag{3.19}$$

$$\leq d\left(\frac{d}{dt}\langle\eta, y_j(t)\xi\rangle, P(t, y_j(t))(\eta,\xi)\right) + \beta_{\eta\xi}(t), \tag{3.20}$$

where

$$\beta_{\eta\xi}(t) = 4\sigma_0 K_{\eta\xi}(t) + \sum_{j\geq 1}\left|\frac{d}{dt}\langle \eta, y_j(t)\xi\rangle - W_{\eta\xi,j}(t)\right|. \tag{3.21}$$

The estimate given by (3.20) holds by applying the inequality $|y_{\eta\xi,j}(t) - y_{\eta\xi}(t)| < \sigma_0$ that follows from (3.17). Again, by applying (3.2) and (3.3) to the expression in (3.21), we conclude that

$$\int_{t_0}^{T} \beta_{\eta\xi}(t)dt := |\beta_{\eta\xi}(\cdot)|_{L^1} < \alpha_0. \tag{3.22}$$

Next, we shall apply mathematical induction principles to establish a sequence of continuous decomposition $\{T_j^k(y_{\eta\xi})\}_{j\geq 1},\ k \geq 0$ of the interval $[t_0, T]$ corresponding to a stochastic process $y \in M \setminus S^{(T)}(P)$ with the associated matrix element $y_{\eta\xi} \in M(\eta,\xi) \setminus S^{(T)}(P)(\eta,\xi)$. In addition, we shall establish a sequence of mappings $g_k : M \to L^1_{loc}(\tilde{\mathcal{A}})$ with the associated continuous sequence of maps $G_k : M(\eta,\xi) \to L^1_{loc}([t_0,T],\mathbb{C})$, and sequences of functions $\zeta_{\eta\xi,k}(\cdot), N_{\eta\xi,k}(\cdot) \in L^1([t_0,T],\mathbb{R}_+)$ corresponding to arbitrary pair of elements $\eta, \xi, \in \mathbb{D}\underline{\otimes}\mathbb{E}$.

Notice that the decomposition $\{T_j^0(y_{\eta\xi})\}$, the mappings g_0, G_0 and the functions $\zeta_{\eta\xi,0}(\cdot) = \rho_{\eta\xi}(\cdot) + \beta_{\eta\xi}(\cdot)$ and $N_{\eta\xi,0}(\cdot) = N_{\eta\xi}(\cdot)$ hold as defined above.

By induction hypothesis, suppose that the decomposition $\{T_j^k(y_{\eta\xi})\}$, the mappings g_k, G_k and the functions $\zeta_{\eta\xi,k}(\cdot)$ and $N_{\eta\xi,k}(\cdot)$ have been constructed and let

$$I_k(y)(t) = y(t_0) + \int_{t_0}^{t} g_k(y)(s)ds, \quad y \in M,\ k \geq 0$$

and

$$J_k(y_{\eta\xi})(t) = y_{\eta\xi}(t_0) + \int_{t_0}^{t} G_k(y_{\eta\xi})(s)ds, \quad y_{\eta\xi} \in M(\eta,\xi),\ k \geq 0 \tag{3.23}$$

satisfy the following inequalities:

$$d(G_k(y_{\eta\xi})(t), P(t, I_k(y)(t))(\eta,\xi)) \leq \zeta_{\eta\xi,k}(t) \tag{3.24}$$

and

$$|G_k(y_{\eta\xi})(t)| \leq N_{\eta\xi,k}(t), \tag{3.25}$$

when $t \in [t_0, T]$, $y \in M$, $y_{\eta\xi} \in M(\eta,\xi)$. Notice that (3.24) and (3.25) hold for $k = 0$. Moreover, the maps $I_k(y)$ and $J_k(y_{\eta\xi})$ respectively lie in the spaces $wac(\tilde{\mathcal{A}})$ and $wac(\tilde{\mathcal{A}})(\eta,\xi)$ and notice also the validity of the relation:

$$\langle \eta, I_k(y)(t)\xi\rangle = J_k(y_{\eta\xi})(t).$$

Then there exists a number $\sigma_{k+1} \in (0, \frac{\alpha_{k+1}}{6M_{\eta\xi}(T)})$ such that

$$\int_A N_{\eta\xi}(t)dt < \frac{\alpha_{k+1}}{6M_{\eta\xi}(T) + 4},$$

for all measurable sets $A \subseteq [t_0, T]$, for which $\mu(A) < \sigma_{k+1}$.

By applying Lemma 2.3 to the set $M(\eta,\xi) \setminus S^{(T)}(P)(\eta,\xi)$, the continuous decomposition $\{T_j^k(y_{\eta\xi})\}$ and the number σ_{k+1}, we construct a continuous decomposition $\{T_j^{k+1}(y_{\eta\xi})\}$ and the set of functions $\{y_{\eta\xi,j}(\cdot)\} \subset M(\eta,\xi) \setminus S^{(T)}(P)(\eta,\xi)$. Such functions are of the form $y_{\eta\xi,j}(t) = \langle \eta, y_j(t)\xi\rangle$, $t \in [t_0, T]$ for some stochastic processes $y_j \in wac(\tilde{\mathcal{A}})$.

Since $t \to P(t,x)(\eta,\xi)$ is measurable with closed values (by assumptions $\mathcal{S}_{(i)}$–$\mathcal{S}_{(ii)}$), then we can choose $U_{\eta\xi,j}(\cdot)$, $j \geq 1$, to be measurable selection from $P(t, I_k(y_j)(t))(\eta,\xi)$ such that

$$|U_{\eta\xi,j}(t) - G_k(y_{\eta\xi,j})(t)| = d(G_k(y_{\eta\xi,j})(t), P(t, I_k(y_j)(t))(\eta,\xi)), \quad t \in [t_0, T].$$

The existence of such selection follows from the classical Filippov lemma (see, for example, Theorem 1.14.2 of [1]).

As the map $(\eta,\xi) \to U_{\eta\xi,j}(t)$ is a sesquilinear form and by adaptedness of the process y_j, there exists an adapted stochastic process $U_j : [t_0, T] \to \tilde{\mathcal{A}}$ such that $U_{\eta,\xi,j}(t) = \langle \eta, U_j(t)\xi\rangle$, $t \in [t_0, T]$.

Again since

$$|U_{\eta\xi,j}(t)| \leq |U_{\eta\xi,j}(t) - G_k(y_{\eta\xi,j})(t)| + |G_k(y_{\eta\xi,j})(t)| \leq \zeta_{\eta\xi,k}(t) + N_{\eta\xi,k}(t), \tag{3.26}$$

we conclude that $U_{\eta\xi,j} \in L^1_{loc}([t_0, T], \mathbb{C})$. Hence, $U_j \in Ad(\tilde{\mathcal{A}})_{wac} \cap L^1_{loc}(\tilde{\mathcal{A}})$. Moreover, we notice that the right hand side of (3.26) is independent of j. The inequality therefore holds for all $j \geq 1$.

We now put $N_{\eta\xi,k+1}(t) = \zeta_{\eta\xi,k}(t) + N_{\eta\xi,k}(t)$ and define the following maps:

$$g_{k+1}(y)(t) = \begin{cases} V(t), & y \in M \cap S^{(T)}(P), \\ \sum_{j\geq 1} \chi_{T_j^{k+1}(y_{\eta\xi})}(t) U_j(t), & y \in M \setminus S^{(T)}(P), \end{cases} \tag{3.27}$$

where the process $V(t)$ is given by (3.6).

Associated with (3.27), we define the following map:

$$G_{k+1}(y_{\eta\xi})(t) = \begin{cases} \frac{d}{dt}\langle \eta, y(t)\xi\rangle, & y_{\eta\xi} \in M(\eta,\xi) \cap S^{(T)}(P)(\eta,\xi), \\ \sum_{j\geq 1} \chi_{T_j^{k+1}(y_{\eta\xi})}(t) U_{\eta\xi,j}(t), & y_{\eta\xi} \in M(\eta,\xi) \setminus S^{(T)}(P)(\eta,\xi). \end{cases} \tag{3.28}$$

Next, we establish some bounds: Let $\mu(T_j^{k+1}(y_{\eta\xi})) > 0$, then

$$|y_{\eta\xi}(\cdot) - y_{\eta\xi,j}(\cdot)|_{AC} < \min\{\sigma_{k+1}, \nu(y_{\eta\xi})\}.$$

Employing (3.23) and using the form of G_k, we have

$$\begin{aligned} &|J_k(y_{\eta\xi,j})(t) - J_k(y_{\eta\xi})(t)| \\ &\quad \leq |y_{\eta\xi,j}(\cdot) - y_{\eta\xi}(\cdot)|_{AC} + \int_{t_0}^{t} \sum_{j\geq 1} \chi_{T_i^k(y_{\eta\xi,j}) \triangle T_i^k(y_{\eta\xi})}(s) N_{\eta\xi,k}(s)ds. \end{aligned} \tag{3.29}$$

For $t \in T_j^{k+1}(y_{\eta\xi})$ and by using (3.28), we have:

$$\begin{aligned} &|G_k(y_{\eta\xi})(t) - G_{k+1}(y_{\eta\xi})(t)| \\ &\quad = |G_k(y_{\eta\xi})(t) - U_{\eta\xi,j}(t)| \\ &\quad \leq |U_{\eta\xi,j}(t) - G_k(y_{\eta\xi,j})(t)| + |G_k(y_{\eta\xi,j})(t) - G_k(y_{\eta\xi})(t)| \\ &\quad = d(G_k(y_{\eta\xi,j})(t), P(t, I_k(y_j)(t))(\eta,\xi)) + |G_k(y_{\eta\xi,j})(t) - G_k(y_{\eta\xi})(t)| \end{aligned}$$

$$\begin{aligned}
&\leq d(G_k(y_{\eta\xi,j})(t), P(t, I_k(y)(t))(\eta,\xi)) + \rho(P(t, I_k(y)(t))(\eta,\xi), P(t, I_k(y_j)(t))(\eta,\xi))\\
&\quad + |G_k(y_{\eta\xi,j})(t) - G_k(y_{\eta\xi})(t)|\\
&\leq d(G_k(y_{\eta\xi})(t), P(t, I_k(y)(t))(\eta,\xi)) + \rho(P(t, I_k(y)(t))(\eta,\xi), P(t, I_k(y_j)(t))(\eta,\xi))\\
&\quad + 2|G_k(y_{\eta\xi,j})(t) - G_k(y_{\eta\xi})(t)|\\
&\leq d(G_k(y_{\eta\xi})(t), P(t, I_k(y)(t))(\eta,\xi)) + K_{\eta\xi}(t)\|I_k(y)(t) - I_k(y_j)(t)\|_{\eta\xi}\\
&\quad + 2|G_k(y_{\eta\xi,j})(t) - G_k(y_{\eta\xi})(t)|\\
&\leq d(G_k(y_{\eta\xi})(t), P(t, I_k(y)(t))(\eta,\xi)) + K_{\eta\xi}(t)|J_k(y_{\eta\xi})(t) - J_k(y_{\eta\xi,j})(t)|\\
&\quad + 2\sum_{j\geq 1} \chi_{T_j^{k+1}(y_{\eta\xi})\cap(T_i^k(y_{\eta\xi,j})\,\triangle\, T_i^k(y_{\eta\xi}))}(t) N_{\eta\xi,k}(t). \qquad (3.30)
\end{aligned}$$

Furthermore, we have

$$\begin{aligned}
&d(G_{k+1}(y_{\eta\xi})(t), P(t, I_k(y)(t))(\eta,\xi))\\
&\quad = d(U_{\eta\xi,j}(t), P(t, I_k(y)(t))(\eta,\xi))\\
&\quad \leq \rho(P(t, I_k(y_j)(t))(\eta,\xi), P(t, I_k(y)(t))(\eta,\xi))\\
&\quad \leq K_{\eta\xi}(t)|J_k(y_{\eta\xi,j})(t) - J_k(y_{\eta\xi})(t)|. \qquad (3.31)
\end{aligned}$$

Employing (3.20), (3.29) and (3.30), it follows that

$$\begin{aligned}
&\int_{t_0}^t |G_1(y_{\eta\xi})(s) - G_0(y_{\eta\xi})(s)|ds\\
&\quad \leq \int_{t_0}^t d(G_0(y_{\eta\xi})(s), P(t, I_0(y)(s))(\eta,\xi))ds + \alpha_1\\
&\quad \leq \int_{t_0}^t \rho_{\eta\xi}(s)ds + \alpha_0 + \alpha_1. \qquad (3.32)
\end{aligned}$$

Again, by employing (3.29), (3.30) and (3.31), we have

$$\begin{aligned}
&\int_{t_0}^t |G_{k+1}(y_{\eta\xi})(s) - G_k(y_{\eta\xi})(s)|ds\\
&\quad \leq \int_{t_0}^t d(G_k(y_{\eta\xi})(s), P(s, I_k(y)(s))(\eta,\xi))ds + \alpha_{k+1}\left(\frac{1}{6} + \frac{M_{\eta\xi}(T)+2}{6M_{\eta\xi}(T)+4}\right)\\
&\quad \leq \int_{t_0}^t d(G_k(y_{\eta\xi})(s), P(s, I_{k-1}(y)(s))(\eta,\xi))ds\\
&\qquad + \int_{t_0}^t \rho(P(s, I_k(y)(s))(\eta,\xi), P(s, I_{k-1}(y)(s))(\eta,\xi))ds\\
&\qquad + \alpha_{k+1}\left(\frac{1}{6} + \frac{M_{\eta\xi}(T)+2}{6M_{\eta\xi}(T)+4}\right)\\
&\quad \leq \int_{t_0}^t d(G_k(y_{\eta\xi})(s), P(s, I_{k-1}(y)(s))(\eta,\xi))ds
\end{aligned}$$

$$
+\int_{t_0}^{t} K_{\eta\xi}(s)|J_k(y_{\eta\xi})(s)-J_{k-1}(y_{\eta\xi})(s)|ds+\alpha_{k+1}\left(\frac{1}{6}+\frac{M_{\eta\xi}(T)+2}{6M_{\eta\xi}(T)+1}\right)
$$

$$
\leq \int_{t_0}^{t} K_{\eta\xi}(s)|J_k(y_{\eta\xi})(s)-J_{k-1}(y_{\eta\xi})(s)|ds+\alpha_{k+1}. \tag{3.33}
$$

Continuing the iteration in (3.33) and using (3.32) and the relation

$$
|J_k(y_{\eta\xi})(s)-J_{k-1}(y_{\eta\xi})(s)| \leq \int_{t_0}^{s} |G_k(y_{\eta\xi})(u)-G_{k-1}(y_{\eta\xi})(u)|du
$$

by induction, we have

$$
\begin{aligned}
&\int_{t_0}^{t} |G_{k+1}(y_{\eta\xi})(s)-G_k(y_{\eta\xi})(s)|ds \\
&\quad \leq \int_{t_0}^{t} \frac{(M_{\eta\xi}(t)-M_{\eta\xi}(s))^k}{k!}\rho_{\eta\xi}(s)ds+\alpha_{k+1} \\
&\qquad + M_{\eta\xi}(T)\alpha_k+\cdots+\frac{(M_{\eta\xi}(t))^k}{k!}\alpha_1+\frac{(M_{\eta\xi}(t))^k}{k!}\alpha_0 \\
&\quad \leq \int_{t_0}^{t} \frac{(M_{\eta\xi}(t)-M_{\eta\xi}(s))^k}{k!}\rho_{\eta\xi}(s)ds+\frac{\epsilon}{2^{k+3}}+\frac{(M_{\eta\xi}(t))^k}{k!}\alpha_0.
\end{aligned} \tag{3.34}
$$

Applying (3.29), (3.31), (3.34) and the Lipschitz condition, we have:

$$
\begin{aligned}
&d(G_{k+1}(y_{\eta\xi})(t), P(t, I_{k+1}(y)(t))(\eta,\xi)) \\
&\quad \leq d(G_{k+1}(y_{\eta\xi})(t), P(t, I_k(y)(t))(\eta,\xi))+K_{\eta\xi}(t)|J_{k+1}(y_{\eta\xi})(t)-J_k(y_{\eta\xi})(t)| \\
&\quad \leq K_{\eta\xi}(t)\left(\frac{\alpha_{k+1}}{3M_{\eta\xi}(T)}+\int_{t_0}^{t} \frac{(M_{\eta\xi}(t)-M_{\eta\xi}(s))^k}{k!}\rho_{\eta\xi}(s)ds+\frac{\epsilon}{2^{k+3}}+\frac{(M_{\eta\xi}(t))^k}{k!}\alpha_0\right) \\
&\quad := \zeta_{\eta\xi,k+1}(t).
\end{aligned} \tag{3.35}
$$

It is easy to see that the map $\zeta_{\eta\xi,k+1}(\cdot)$ lies in $L^1([t_0,T],\mathbb{R}_+)$.

Next we show that the map $G_{k+1}: M(\eta,\xi)\to L^1([t_0,T],\mathbb{C})$ is continuous. At a point $y_{\eta\xi,0}\in M(\eta,\xi)\setminus S^{(T)}(P)(\eta,\xi)$ where $y_{\eta\xi,0}(\cdot)=\langle\eta, y_0(\cdot)\xi\rangle$ for some $y_0\in M$, estimate of the form (3.9) holds showing that the map is continuous at the point. Suppose now that $y_{\eta\xi,0}\in M(\eta,\xi)\cap S^{(T)}(P)(\eta,\xi)$, $z_{\eta\xi,m}\in M(\eta,\xi)\setminus S^{(T)}(P)(\eta,\xi)$ and $z_{\eta\xi,m}\to y_{\eta\xi,0}$ as $m\to\infty$, then for $P_j(z_{\eta\xi,m})>0$ estimate of the form (3.13) holds, i.e.

$$
|y_{\eta\xi,0}(\cdot)-y_{\eta\xi,j}(\cdot)|_{AC}\leq 3|y_{\eta\xi,0}-z_{\eta\xi,m}|_{AC}.
$$

Thus, by employing (3.29) and (3.30), conclusions of Lemma 2.3, continuity of the map G_k and by putting $\dot{y}_{\eta\xi,0}=\frac{d}{dt}y_{\eta\xi,0}(t)$, we have

$$
\begin{aligned}
|\dot{y}_{\eta\xi,0}-G_{k+1}(z_{\eta\xi,m})|_{L^1} &= |G_{k+1}(y_{\eta\xi,0})-G_{k+1}(z_{\eta\xi,m})|_{L^1} \\
&\leq |\dot{y}_{\eta\xi,0}-G_k(z_{\eta\xi,m})|_{L^1}+|G_k(z_{\eta\xi,m})-G_{k+1}(z_{\eta\xi,m})|_{L^1} \\
&\to 0 \quad \text{as } m\to\infty.
\end{aligned}
$$

The map G_{k+1} is continuous. Therefore G_k is continuous for all $k\geq 0$.

Let $y_{\eta\xi} \in M(\eta, \xi)$. Summing of the inequalities in (3.34) and applying the definition of α_0, we have:

$$
\begin{aligned}
&\int_{t_0}^{t} |G_{k+1}(y_{\eta\xi})(s) - G_0(y_{\eta\xi})(s)|ds \\
&\quad \le \int_{t_0}^{t} \exp(M_{\eta\xi}(t) - M_{\eta\xi}(s))\rho_{\eta\xi}(s)ds + \sum_{\nu=0}^{k} \frac{\epsilon}{2^{\nu+3}} + \exp(M_{\eta\xi}(t))\alpha_0 \\
&\quad \le \int_{t_0}^{t} \exp(M_{\eta\xi}(t) - M_{\eta\xi}(s))\rho_{\eta\xi}(s)ds + \frac{2\epsilon}{2^3} + \frac{\epsilon}{4\exp(M_{\eta\xi}(T))} \\
&\quad \le \int_{t_0}^{t} \exp(M_{\eta\xi}(t) - M_{\eta\xi}(s))\rho_{\eta\xi}(s)ds + \frac{\epsilon}{2}.
\end{aligned} \tag{3.36}
$$

By employing (3.36) and (3.18), we obtain:

$$
\begin{aligned}
&|J_{k+1}(y_{\eta\xi})(t) - y_{\eta\xi}(t)| \\
&\quad \le |J_{k+1}(y_{\eta\xi})(t) - J_0(y_{\eta\xi})(t)| + |J_0(y_{\eta\xi})(t) - y_{\eta\xi}(t)| \\
&\quad \le \int_{t_0}^{t} \exp(M_{\eta\xi}(t) - M_{\eta\xi}(s))\rho_{\eta\xi}(s)ds + \frac{\epsilon}{2} + \frac{\epsilon}{4\exp(2M_{\eta\xi}(T))} \\
&\quad \le \mathbb{E}_{\eta\xi}(t) + \epsilon.
\end{aligned} \tag{3.37}
$$

Since

$$
\begin{aligned}
&|J_{k+1}(y_{\eta\xi}) - J_k(y_{\eta\xi})|_{AC} \\
&\quad = |J_{k+1}(y_{\eta\xi})(t_0) - J_k(y_{\eta\xi})(t_0)| + \int_{t_0}^{T} |g_{k+1}(y_{\eta\xi})(s) - g_k(y_{\eta\xi})(s)|ds, \quad y_{\eta\xi} \in M(\eta, \xi),
\end{aligned}
$$

then from (3.34), we conclude that the sequence of functions $\{J_k(y_{\eta\xi})\}$ converges in $AC([t_0, T], \mathbb{C})$ to a function $R(y_{\eta\xi}) \in AC([t_0, T], \mathbb{C})$. Again by definition of the sequence $\{I_k(y)\}$ in $wac(\tilde{\mathcal{A}})$,

$$
\langle \eta, I_k(y)(t)\xi \rangle = J_k(y_{\eta\xi})(t), \quad y \in M
$$

and

$$
\begin{aligned}
|I_k(y)|_{\eta\xi} &= \|I_k(y)(t_0)\|_{\eta\xi} + \int_{t_0}^{T} \left| \frac{d}{dt} \langle \eta, I_k(y)(t)\xi \rangle \right| dt \\
&= |J_k(y_{\eta\xi})(t_0)| + \int_{t_0}^{T} \left| \frac{d}{dt} J_k(y_{\eta\xi})(t) \right| dt.
\end{aligned}
$$

Thus, by (3.34), the sequence $\{I_k(y)\}$ is a Cauchy sequence in $wac(\tilde{\mathcal{A}})$ which converges to a map $\Upsilon(y) \in wac(\tilde{\mathcal{A}})$. Also $\langle \eta, \Upsilon(y)(t)\xi \rangle = R(y_{\eta\xi})(t)$. Hence $R(y_{\eta\xi}) \in wac(\tilde{\mathcal{A}})(\eta, \xi)$. The map $R : M(\eta, \xi) \to wac(\tilde{\mathcal{A}})(\eta, \xi)$ is continuous by the continuity of the sequence of maps G_k. Also

$$
\begin{aligned}
&R(y_{\eta\xi})(t_0) = y_{\eta\xi}(t_0), \\
&R(y_{\eta\xi})(\cdot) = y_{\eta\xi}(\cdot), \quad y_{\eta\xi} \in M(\eta, \xi) \cap S^{(T)}(P)(\eta, \xi).
\end{aligned}
$$

Taking the limit in (3.37), we have

$$|R(y_{\eta\xi})(t) - y_{\eta\xi}(t)| \le \mathbb{E}_{\eta\xi}(t) + \epsilon, \quad t \in [t_0, T].$$

Next, we show that $\Upsilon(y) \in S^{(T)}(P)$. First we have:

$$\begin{aligned}
d&\left(\frac{d}{dt}\langle \eta, \Upsilon(y)(t)\xi\rangle, P(t, \Upsilon(y)(t))(\eta, \xi)\right) \\
&\le d(G_{k+1}(y_{\eta\xi})(t), P(t, I_k(y)(t))(\eta, \xi)) + \left|\frac{d}{dt}\langle \eta, \Upsilon(y)(t)\xi\rangle - G_{k+1}(y_{\eta\xi})(t)\right| \\
&\quad + K_{\eta\xi}(t)|\langle \eta, \Upsilon(y)(t)\xi\rangle - \langle \eta, I_k(y)(t)\xi\rangle|. \qquad (3.38)
\end{aligned}$$

Applying (3.31) and (3.29) to the first term at the right hand side of (3.38) and integrate both sides we have:

$$\begin{aligned}
\int_{t_0}^{T} & d\left(\frac{d}{dt}\langle \eta, \Upsilon(y)(t)\xi\rangle, P(t, \Upsilon(y)(t))(\eta, \xi)\right) \\
&\le \int_{t_0}^{T} K_{\eta\xi}(t)\left[|y_{\eta\xi,j} - y_{\eta\xi}|_{AC} + \int_{t_0}^{t} \sum_{j\ge 1} \chi_{T_i^k(y_{\eta\xi,j}) \triangle T_i^k(y_{\eta\xi})}(s) N_{\eta\xi,k}(s)ds\right]dt \\
&\quad + \int_{t_0}^{T} \left(\left|\frac{d}{dt}\langle \eta, \Upsilon(y)(t)\xi\rangle - G_{k+1}(y_{\eta\xi})(t)\right|\right. \\
&\quad \left. + K_{\eta\xi}(t)|\langle \eta, \Upsilon(y)(t)\xi\rangle - \langle \eta, I_k(y)(t)\xi\rangle|\right)dt \\
&\le \alpha_{k+1} + \int_{t_0}^{T} \left(\left|\frac{d}{dt}\langle \eta, \Upsilon(y)(t)\xi\rangle - G_{k+1}(y_{\eta\xi})(t)\right|\right. \\
&\quad \left. + K_{\eta\xi}(t)|\langle \eta, \Upsilon(y)(t)\xi\rangle - \langle \eta, I_k(y)(t)\xi\rangle|\right)dt.
\end{aligned}$$

Taking the limit as $k \to \infty$, we have

$$d\left(\frac{d}{dt}\langle \eta, \Upsilon(y)(t)\xi\rangle, P(t, \Upsilon(y)(t))(\eta, \xi)\right) = 0.$$

Hence $\Upsilon(y) \in S^{(T)}(P)$ and therefore

$$\langle \eta, \Upsilon(y)(\cdot)\xi\rangle := R(y_{\eta\xi})(\cdot) \in S^{(T)}(P)(\eta, \xi). \qquad \square$$

The following corollary furnishes a generalization of our previous result [7] concerning the existence of a continuous selection from the multifunction $\langle \eta, x\xi\rangle \to S^{(T)}(x)(\eta, \xi)$ without any restriction on the domain of the selection map.

Corollary 3.2 *There exists a continuous map*:

$$\Phi : \tilde{\mathcal{A}}(\eta, \xi) \to wac(\tilde{\mathcal{A}})(\eta, \xi)$$

such that for each $x \in \tilde{\mathcal{A}}$, $\Phi(\langle \eta, x\xi \rangle) \in S^{(T)}(x)(\eta, \xi)$ *satisfying*

$$\Phi(\langle \eta, x_0\xi \rangle) = \langle \eta, X(\cdot)\xi \rangle$$

for each $X \in S^{(T)}(x_0)$.

Proof Let $x_0 \in \tilde{\mathcal{A}}$ and let $X \in S^{(T)}(x_0)$. As in the proof of the last theorem, we can write

$$X(t) = x_0 + \int_{t_0}^{t} V_0(s)ds, \quad \text{a.e. } t \in [t_0, T]$$

for some $V_0 \in Ad(\tilde{\mathcal{A}})_{wac} \cap L^1_{loc}(\tilde{\mathcal{A}})$ which depends on X. We consider the following non-empty set

$$M = \left\{ y \in wac(\tilde{\mathcal{A}}) : y(t) = y_0 + \int_{t_0}^{t} V_0(s)ds,\, y_0 \in \tilde{\mathcal{A}} \right\}.$$

Notice that the process $X \in M$. We apply Theorem 3.1 to the set

$$M(\eta, \xi) = \{\langle \eta, y(\cdot)\xi \rangle : y \in M\}$$

to obtain a continuous mapping $R : M(\eta, \xi) \to S^{(T)}(P)(\eta, \xi)$ satisfying

$$R(\langle \eta, X(\cdot)\xi \rangle) = \langle \eta, X(\cdot)\xi \rangle.$$

Define the map $\Phi(\cdot)$ by

$$\Phi(\langle \eta, y_0\xi \rangle) = R(\langle \eta, y(\cdot)\xi \rangle).$$

The map $\Phi(\cdot)$ is continuous on $\tilde{\mathcal{A}}(\eta, \xi)$ by the continuity of the map R and satisfies

$$\Phi(\langle \eta, x_0\xi \rangle) = \langle \eta, X(\cdot)\xi \rangle \in S^{(T)}(x_0)(\eta, \xi).$$

Since x_0 is arbitrary in $\tilde{\mathcal{A}}$, the conclusion of the corollary follows. □

The next result shows that if the map $(t, x) \to P(t, x)(\eta, \xi)$ is integrably bounded then the set of complex valued functions $S^{(T)}(x_0)(\eta, \xi)$ for each $x_0 \in \tilde{\mathcal{A}}$ is an absolute retract. This is understood in the sense that every continuous map $h : F \to S^{(T)}(x_0)(\eta, \xi)$ admits a continuous extension $\tilde{h} : H \to S^{(T)}(x_0)(\eta, \xi)$ for any subset $F = \bar{F} \subseteq H$ and for any separable metric space H where the restriction $\tilde{h}|_F = h$.

Corollary 3.3 *In addition to conditions of Theorem* 3.1, *assume that there exists a function* $b_{\eta\xi}(\cdot) \in L^1([t_0, T], \mathbb{R}_+)$ *such that for every element* $V_{\eta\xi}(\cdot) \in P(t, x)(\eta, \xi), t \in [t_0, T], |V_{\eta\xi}(t)| \le b_{\eta\xi}(t)$, *then the set of functions* $S^{(T)}(x_0)(\eta, \xi)$ *is an absolute retract.*

Proof We consider the set of complex valued functions defined as follows:

$$M(\eta, \xi) = \{y_{\eta\xi}(\cdot) \in wac(\tilde{\mathcal{A}})(\eta, \xi) : y_{\eta\xi}(t_0) = x_{\eta\xi,0}, |\dot{y}_{\eta\xi}(t)| \le b_{\eta\xi}(t), t \in [t_0, T]\}.$$

It is obvious that the set $S^{(T)}(x_0)(\eta, \xi) \subset M(\eta, \xi)$. By applying Theorem 3.1, there exists a continuous map $R : M(\eta, \xi) \to S^{(T)}(x_0)(\eta, \xi)$ such that $R(X_{\eta\xi}(\cdot)) = X_{\eta\xi}(\cdot), X_{\eta\xi}(\cdot) \in S^{(T)}(x_0)(\eta, \xi)$, showing that $S^{(T)}(x_0)(\eta, \xi)$ is the retract of the set $M(\eta, \xi)$. By convexity of the set $M(\eta, \xi)$ in the space $AC([t_0, T], \mathbb{C})$ and by Theorem 5 in [22, p. 341], the set $M(\eta, \xi)$ is an absolute retract. Hence, the set $S^{(T)}(x_0)(\eta, \xi)$ is an absolute retract by Theorem 6 in [22, p. 341]. □

Corollary 3.4 *Subject to all the conditions of Corollary* 3.3, *the set* $S^{(T)}(x_0)(\eta, \xi)$ *is contractible and locally contractible in itself. In addition, the set is locally and integrally connected in an arbitrary dimension.*

Proof Since the set $S^{(T)}(x_0)(\eta, \xi)$ is an absolute retract, then (by Theorem 3, p. 375, Theorem 5, p. 377 in [22]), the set is contractible and locally contractible in itself. Again (by Theorem 3, p. 376 in [22]) the set is locally and integrally connected in arbitrary dimension for each pair $\eta, \xi \in \mathbb{D} \underline{\otimes} \mathbb{E}$. □

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References

1. Aubin, J.P., Cellina, A.: Differential Inclusions. Springer, Berlin (1984)
2. Ayoola, E.O.: On convergence of one-step schemes for weak solutions of quantum stochastic differential equations. Acta Appl. Math. **67**(1), 19–58 (2001)
3. Ayoola, E.O.: Construction of approximate attainability sets for Lipschitzian quantum stochastic differential inclusions. Stoch. Anal. Appl. **19**(3), 461–471 (2001)
4. Ayoola, E.O.: Lagrangian quadrature schemes for computing weak solutions of quantum stochastic differential equations. SIAM J. Numer. Anal. **39**(6), 1835–1864 (2002)
5. Ayoola, E.O.: Error estimates for discretized quantum stochastic differential inclusions. Stoch. Anal. Appl. **21**(6), 1215–1230 (2003)
6. Ayoola, E.O.: Exponential formula for the reachable sets of quantum stochastic differential inclusions. Stoch. Anal. Appl. **21**(3), 515–543 (2003)
7. Ayoola, E.O.: Continuous selections of solution sets of Lipschitzian quantum stochastic differential inclusions. Int. J. Theor. Phys. **43**(10), 2041–2059 (2004)
8. Bressan, A., Cellina, A., Fryszkowski, A.: A class of absolute retracts in spaces of integrable functions. Proc. Am. Math. Soc. **112**(2), 413–418 (1991)
9. Broucke, M., Arapostathis, A.: Continuous interpolation of solutions of Lipschitz inclusions. J. Math. Anal. Appl. **258**, 565–572 (2001)
10. Cellina, A.: On the set of solutions to Lipschitzian differential inclusions. Diff. Integral Equ. **1**(4), 495–500 (1988)
11. Cellina, A., Ornelas, A.: Representation of the attainable set for Lipschitzian differential inclusions. Rocky Mt. J. Math. **22**(1), 117–124 (1992)
12. Colombo, R.M., Fryszkowski, A., Rzezuchowski, T., Staicu, V.: Continuous selections of solution sets of Lipschitzian differential inclusions. Funkc. Ekvacioj **34**, 321–330 (1991)
13. Ekhaguere, G.O.S.: Lipschitzian quantum stochastic differential inclusions. Int. J. Theor. Phys. **31**(11), 2003–2034 (1992)
14. Ekhaguere, G.O.S.: Quantum stochastic differential inclusions of hypermaximal monotone type. Int. J. Theor. Phys. **34**(3), 323–353 (1995)
15. Ekhaguere, G.O.S.: Quantum stochastic evolutions. Int. J. Theor. Phys. **35**(9), 1909–1946 (1996)
16. Fryszkowski, A.: Continuous selections for a class of non-convex multivalued maps. Stud. Math. **T. LXXVI**, 163–174 (1983)
17. Glockner, P.: Quantum stochastic differential equations on *-bialgebra. Math. Proc. Camb. Philos. Soc. **109**, 571–595 (1991)
18. Hudson, R.L., Parthasarathy, K.R.: Quantum Ito's formulae and stochastic evolutions. Commun. Math. Phys. **93**, 301–324 (1984)
19. Hudson, R.L.: Quantum stochastic calculus and some of its applications. In: Ekhaguere, G.O.S. (ed.) Contemporary Stochastic Analysis, pp. 31–70. World Scientific, Singapore (1991)

20. Hudson, R.L., Parthasarathy, K.R.: Construction of quantum diffusions. In: Accardi, L., Frigerio, A., Gorini, V. (eds.) Quantum Probability and Applications to the Quantum Theory of Irreversible Processes. Lecture Notes in Mathematics, vol. 1055, pp. 173–198. Springer, Berlin (1982)
21. Kuratowski, K.: Topology, vol. 1. Academic Press, New York (1966)
22. Kuratowski, K.: Topology, vol. 2. Academic Press, New York (1969)
23. Meyer, P.: Quantum Probability for Probabilists. Lecture Notes in Mathematics, vol. 1538. Springer, Berlin (1993)
24. Parthasarathy, K.R.: An Introduction to Quantum Stochastic Calculus. Monograph in Maths, vol. 85. Birkhauser, Basel (1992)
25. Repovs, D., Semenov, P.V.: Continuous Selections of Multivalued Mappings. Mathematics and Its Applications. Kluwer Academic, Dordrecht (1998)
26. Smirnov, G.V.: Topological properties of an integral funnel of a differential inclusion with a Lipschitz right side. Diff. Equ. **27**(2), 157–165 (1991)
27. Smirnov, G.V.: Introduction to the Theory of Differential Inclusions. Graduate Studies in Mathematics, vol. 41. AMS, Providence (2002)

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UPPER SEMICONTINUOUS QUANTUM STOCHASTIC DIFFERENTIAL INCLUSIONS VIA KAKUTANI-FAN FIXED POINT THEOREM

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ABSTRACT. The existence of solutions of Discontinuous Quantum Stochastic Differential Inclusions (QSDI) with upper semicontinuous coefficients is our concerned in this work. A non commutative generalization of Kakutani-Fan fixed point theorem is established in the work. By employing this result, the existence of solution of upper semicontinuous QSDI is established.

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1. INTRODUCTION

The problem of existence of solutions of Lipschitzian quantum stochastic differential inclusions of Hudson and Parthasarathy quantum stochastic calculus formulation was established in [7]. The properties of these solution sets were established in [3] and [4]. The quantum stochastic calculus is driven by quantum stochastic processes called annihilation, creation and gauge arising from quantum field operators. The multivalued generalization of this non commutative stochastic differential equation is essential in the applications of quantum control theory, quantum evolution inclusions[9] and differential equation with discontinuous coefficients.

For a classical differential equation with discontinuous coefficients the existence of solutions was established via a multivalued regularization procedure [2]. This multivalued regularization is upper semicontinuous. The existence of solutions of upper semicontinuous differential inclusions in the classical setting was established by using Kakutani fixed point approach [6] which is a multivalued generalization of Schauder fixed point theorem. The aim of this work is to establish this result in our non commutative setting. However, this result does not naturally transcends to our upper semicontinuous quantum stochastic differential inclusions. In this work we shall first establish a form of Kakutani-Fan fixed point theorem and then employ it to prove the existence of solution of our quantum stochastic differential inclusions. Hence we extend the existence of solution results in the literatures on quantum stochastic differential inclusions [7], [8] and [10] to discontinuous case.

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The work shall be arranged as follows; in section 2 we state the definitions and notations while section 3 shall be for results on the fixed point theorem and existence of solutions of upper semicontinuous quantum stochastic differential inclusions via this fixed point theorem.

2. PRELIMINARIES

2.1. **Notations and Definitions.** In what follows, if U is a topological space, we denote by $\text{clos}(U)$, the collection of all non-empty closed subsets of U.

To each pair (D, H) consisting of a pre-Hilbert space D and its completion H, we associate the set $L_w^+(D, H)$ of all linear maps x from D into H, with the property that the domain of the operator adjoint contains D. The members of $L_w^+(D, H)$ are densely-defined linear operators on H which do not necessarily leave D invariant and $L_w^+(D, H)$ is a linear space when equipped with the usual notions of addition and scalar multiplication.

To H corresponds a Hilbert space $\Gamma(H)$ called the boson Fock space determined by H. A natural dense subset of $\Gamma(H)$ consists of linear space generated by the set of exponential vectors(Guichardet, [11]) in $\Gamma(H)$ of the form

$$e(f) = \bigoplus_{n=0}^{\infty} (n!)^{-\frac{1}{2}} \bigotimes^{n} f, \;\; f \in H$$

where $\bigotimes^0 f = 1$ and $\bigotimes^n f$ is the n-fold tensor product of f with itself for $n \geq 1$.

In what follows, $\mathbb{D}$ is some pre-Hilbert space whose completion is $\mathcal{R}$ and γ is a fixed Hilbert. $L^2_\gamma(\mathbb{R}_+)$ (resp. $L^2_\gamma([0,t))$, resp. $L^2_\gamma([t,\infty))$ $t \in \mathbb{R}_+$) is the space of square integrable γ-valued maps on $\mathbb{R}_+$ (resp. $[0, t)$, resp. $[t, \infty)$).

The inner product of the Hilbert space $\mathcal{R} \otimes \Gamma(L^2_\gamma(\mathbb{R}_+))$ will be denoted by $\langle \cdot, \cdot \rangle$ and $\| \cdot \|$ the norm induced by $\langle \cdot, \cdot \rangle$. Let $\mathbb{E}, \mathbb{E}_t$ and $\mathbb{E}^t$, $t > 0$ be linear spaces generated by the exponential vectors in Fock spaces $\Gamma(L^2_\gamma(\mathbb{R}_+)), \Gamma(L^2_\gamma([0,t)))$ and $\Gamma(L^2_\gamma([t,\infty)))$ respectively;

$$\mathcal{A} \equiv L_w^+(\mathbb{D}\underline{\otimes}\mathbb{E}, \mathcal{R} \otimes \Gamma(L^2_\gamma(\mathbb{R}_+)))$$
$$\mathcal{A}_t \equiv L_w^+(\mathbb{D}\underline{\otimes}\mathbb{E}_t, \mathcal{R} \otimes \Gamma(L^2_\gamma([0,t)))) \otimes \mathbb{I}^t$$
$$\mathcal{A}^t \equiv \mathbb{I}_t \otimes L_w^+(\mathbb{E}^t, \Gamma(L^2_\gamma([t,\infty)))), \;\; t > 0$$

where $\underline{\otimes}$ denotes algebraic tensor product and $\mathbb{I}_t$ (resp. $\mathbb{I}^t$) denotes the identity map on $\mathcal{R} \otimes \Gamma(L^2_\gamma([0,t)))$ (resp. $\Gamma(L^2_\gamma([t,\infty)))$), $t > 0$. For every $\eta, \xi \in \mathbb{D}\underline{\otimes}\mathbb{E}$ define

$$\| x \|_{\eta,\xi} = | \langle \eta, x\xi \rangle |, \;\; x \in \mathcal{A}$$

then the family of seminorms

$$\{\| \cdot \|_{\eta\xi} \colon \eta, \xi \in \mathbb{D}\underline{\otimes}\mathbb{E}\}$$

generates a topology τ_w, weak topology. The completion of the locally convex spaces $(\mathcal{A}, \tau_w), (\mathcal{A}_t, \tau_w)$ and $(\mathcal{A}^t, \tau_w)$ are respectively denoted by $\widetilde{\mathcal{A}}, \widetilde{\mathcal{A}}_t$ and $\widetilde{\mathcal{A}}^t$.

We define the Hausdorff topology on $\mathrm{clos}(\widetilde{\mathcal{A}})$ as follows: For $x \in \widetilde{\mathcal{A}}$, $\mathcal{M}, \mathcal{N} \in \mathrm{clos}(\widetilde{\mathcal{A}})$ and $\eta, \xi \in \mathbb{D}\underline{\otimes}\mathbb{E}$, define

$$\rho_{\eta\xi}(\mathcal{M}, \mathcal{N}) \equiv \max(\delta_{\eta\xi}(\mathcal{M}, \mathcal{N}), \delta_{\eta\xi}(\mathcal{N}, \mathcal{M}))$$

where

$$\delta_{\eta\xi}(\mathcal{M}, \mathcal{N}) \equiv \sup_{x \in \mathcal{M}} \mathbf{d}_{\eta\xi}(x, \mathcal{N}) \text{ and}$$
$$\mathbf{d}_{\eta\xi}(x, \mathcal{N}) \equiv \inf_{y \in \mathcal{N}} \| x - y \|_{\eta\xi} .$$

The Hausdorff topology which shall be employed in what follows, denoted by, τ_H, is generated by the family of pseudometrics $\{\rho_{\eta\xi}(\cdot) : \eta, \xi \in \mathbb{D}\underline{\otimes}\mathbb{E}\}$. Moreover, if $\mathcal{M} \in clos(\widetilde{\mathcal{A}})$, then $\| \mathcal{M} \|_{\eta\xi}$ is defined by

$$\| \mathcal{M} \|_{\eta\xi} \equiv \rho_{\eta\xi}(\mathcal{M}, \{0\});$$

for arbitrary $\eta, \xi \in \mathbb{D}\underline{\otimes}\mathbb{E}$. For $A, B \in \mathrm{clos}(\mathbb{C})$ and $x \in \mathbb{C}$, a complex number, define

$$d(x, B) \equiv \inf_{y \in B} | x - y |$$
$$\delta(A, B) \equiv \sup_{x \in A} d(x, B)$$
$$\text{and } \rho(A, B) \equiv \max(\delta(A, B), \delta(B, A)).$$

Then ρ is a metric on $\mathrm{clos}(\mathbb{C})$ and induces a metric topology on the space. Let $I \subseteq \mathbb{R}_+$. A *stochastic process* indexed by I is an $\widetilde{\mathcal{A}}$-valued measurable map on I. A stochastic process X is called *adapted* if $X(t) \in \widetilde{\mathcal{A}}_t$ for each $t \in I$. We write $\mathrm{Ad}(\widetilde{\mathcal{A}})$ for the set of all adapted stochastic processes indexed by I.

Definition 2.1. A member X of $\mathrm{Ad}(\widetilde{\mathcal{A}})$ is called

(i) weakly absolutely continuous if the map $t \mapsto \langle \eta, X(t)\xi \rangle$, $t \in I$ is absolutely continuous for arbitrary $\eta, \xi \in \mathbb{D}\underline{\otimes}\mathbb{E}$

(ii) locally absolutely p-integrable if $\| X(\cdot) \|_{\eta\xi}^p$ is Lebesgue -measurable and integrable on $[0, t) \subseteq I$ for each $t \in I$ and arbitrary $\eta, \xi \in \mathbb{D}\underline{\otimes}\mathbb{E}$.

We denote by $\mathrm{Ad}(\widetilde{\mathcal{A}})_{wac}$ (resp. $L^p_{loc}(\widetilde{\mathcal{A}})$) the set of all weakly, absolutely continuous (resp. locally absolutely p-integrable) members of $\mathrm{Ad}(\widetilde{\mathcal{A}})$.

Stochastic integrators: Let $L^\infty_{\gamma,loc}(\mathbb{R}_+)$ [resp. $L^\infty_{B(\gamma),loc}(\mathbb{R}_+)$] be the linear space of all measurable, locally bounded functions from $\mathbb{R}_+$ to γ [resp. to B(γ), the Banach space of bounded endomorphisms of γ]. If $f \in L^\infty_{\gamma,loc}(\mathbb{R}_+)$ and $\pi \in L^\infty_{B(\gamma),loc}(\mathbb{R}_+)$, then πf is the member of $L^\infty_{\gamma,loc}(\mathbb{R}_+)$ given by $(\pi f)(t) = \pi(t)f(t)$, $t \in \mathbb{R}_+$.

For $f \in L^2_\gamma(\mathbb{R})_+$ and $\pi \in L^\infty_{B(\gamma),loc}(\mathbb{R}_+)$; the annihilation, creation and gauge operators, $a(f), a^+(f)$ and $\lambda(\pi)$ in $L^+_w(\mathbb{D}, \Gamma(L^2_\gamma(\mathbb{R})_+))$ respectively, are defined as:

$$
\begin{aligned}
a(f)\mathbf{e}(g) &= \langle f, g\rangle_{L^2_\gamma(\mathbb{R}_+)}\mathbf{e}(g)\\
a^+(f)\mathbf{e}(g) &= \frac{d}{d\sigma}\mathbf{e}(g+\sigma f)\mid_{\sigma=0}\\
\lambda(\pi)\mathbf{e}(g) &= \frac{d}{d\sigma}\mathbf{e}(e^{\sigma\pi} f)\mid_{\sigma=0}
\end{aligned}
$$

$g \in L^2_\gamma(\mathbb{R})_+$

For arbitrary $f \in L^\infty_{\gamma,loc}(\mathbb{R}_+)$ and $\pi \in L^\infty_{B(\gamma),loc}(\mathbb{R}_+)$, they give rise to the operator-valued maps A_f, A_f^+ and Λ_π defined by:

$$
\begin{aligned}
A_f(t) &\equiv a(f\chi_{[0,t)})\\
A_f^+(t) &\equiv a^+(f\chi_{[0,t)})\\
\Lambda_\pi(t) &\equiv \lambda(\pi\chi_{[0,t)})
\end{aligned}
$$

$t \in \mathbb{R}_+$, where χ_I denotes the indicator function of the Borel set $I \subseteq \mathbb{R}_+$. The maps A_f, A_f^+ and Λ_π are stochastic processes, called annihilation, creation and gauge processes, respectively, when their values are identified with their amplifications on $\mathcal{R} \otimes \Gamma(L^2_\gamma(\mathbb{R}_+))$. These are the stochastic integrators in Hudson and Parthasarathy [12] formulation of boson quantum stochastic integration.

For processes $p, q, u, v \in L^2_{loc}(\widetilde{\mathcal{A}})$, the quantum stochastic integral:

$$
\int_{t_0}^t \big(p(s)d\Lambda_\pi(s) + q(s)dA_f(s) + u(s)dA_g^+(s) + v(s)ds\big),\ t_0, t \in \mathbb{R}_+
$$

is interpreted in the sense of Hudson-Parthasarathy[12]. The definition of Quantum stochastic differential Inclusions follows as in [7]. A relation of the form

$$
\begin{aligned}
dX(t) &\in E(t, X(t))d\Lambda_\pi(t) + F(t, X(t))dA_f(t)\\
&\quad + G(t, X(t))dA_g^+(t) + H(t, X(t))dt \text{ almost all } t \in I \qquad (2.1)\\
X(t_0) &= x_0
\end{aligned}
$$

is called Quantum stochastic differential inclusions(QSDI) with coefficients E, F, G, H and initial data (t_0, x_0). Equation(2.1) is understood in the integral form:

$$
\begin{aligned}
X(t) \in x_0 + \int_{t_0}^t \big(&E(s, X(s))d\Lambda_\pi(s) + F(s, X(s))dA_f(s)\\
&+G(s, X(s))dA_g^+(s) + H(s, X(s))ds\big),\ t \in I
\end{aligned}
$$

called a stochastic integral inclusion with coefficients E, F, G, H and initial data (t_0, x_0). An equivalent form of (2.1) has been established in [7], Theorem 6.2 as:

$$
\begin{aligned}
(\mu E)(t,x)(\eta,\xi) &= \{\langle \eta, \mu_{\alpha\beta}(t)p(t,x)\xi\rangle : p(t,x) \in E(t,x)\}\\
(\nu F)(t,x)(\eta,\xi) &= \{\langle \eta, \nu_{\beta}(t)q(t,x)\xi\rangle : q(t,x) \in F(t,x)\}\\
(\sigma G)(t,x)(\eta,\xi) &= \{\langle \eta, \sigma_{\alpha}(t)u(t,x)\xi\rangle : u(t,x) \in G(t,x)\}\\
\mathbb{P}(t,x)(\eta,\xi) &= (\mu E)(t,x)(\eta,\xi) + (\nu F)(t,x)(\eta,\xi)\\
&\quad + (\sigma G)(t,x)(\eta,\xi) + H(t,x)(\eta,\xi)\\
H(t,x)(\eta,\xi) &= \{v(t,x)(\eta,\xi) : v(\cdot, X(\cdot))\\
&\quad \text{is a selection of } H(\cdot, X(\cdot)) \forall\ X \in L^2_{loc}(\widetilde{\mathcal{A}})\}
\end{aligned} \tag{2.2}
$$

Then Problem (2.1) is equivalent to

$$
\begin{aligned}
&\frac{d}{dt}\langle \eta, X(t)\xi\rangle \in \mathbb{P}(t, X(t))(\eta,\xi)\\
&X(t_0) = x_0
\end{aligned} \tag{2.3}
$$

for arbitrary $\eta, \xi \in \mathbb{D}\underline{\otimes}\mathbb{E}$, almost all $t \in I$. Hence the existence of solution of (2.1) implies the existence of solution of (2.3) and vice-versa. As explained in [7], for the map $\mathbb{P}$,

$$\mathbb{P}(t,x)(\eta,\xi) \neq \widetilde{\mathbb{P}}(t, \langle \eta, x\xi\rangle)$$

for some complex-valued multifunction $\widetilde{\mathbb{P}}$ defined on $I \times \mathbb{C}$ for $t \in I$, $x \in \widetilde{\mathcal{A}}$, $\eta, \xi \in \mathbb{D}\underline{\otimes}\mathbb{E}$.

Definition 2.2. Let $D \subset \widetilde{\mathcal{A}}$ be a non-empty bounded subset of $\widetilde{\mathcal{A}}$. For each $\eta, \xi \in \mathbb{D}\underline{\otimes}\mathbb{E}$, $\sup_{x\in D} \| x \|_{\eta\xi} < \infty$. We define the diameter of D with respect to $\eta, \xi \in \mathbb{D}\underline{\otimes}\mathbb{E}$ by,

$$diam.(D) = \sup_{x,y\in D} \| x - y \|_{\eta\xi} .$$

Definition 2.3. For arbitrary $\eta, \xi \in \mathbb{D}\underline{\otimes}\mathbb{E}$, let

$$\mathcal{B}_{\eta\xi} = \{D \subset \widetilde{\mathcal{A}} : \sup_{x,y\in D} \| x - y \|_{\eta\xi} < \infty\}$$

Then the map: $\alpha_{\eta\xi} : \mathcal{B}_{\eta\xi} \to \mathbb{R}_+$, defined by

$$\alpha_{\eta\xi}(D) = \inf\{d > 0 : D \text{ admits a finite cover by sets of diameter} \leq d\}, D \in \mathcal{B}_{\eta\xi}$$

is called (Kuratowski-)measure of non compactness.

The following are properties of $\alpha_{\eta\xi}$, established in [5]

Proposition 2.4. *Suppose $\alpha_{\eta\xi} : \mathcal{B}_{\eta\xi} \to \mathbb{R}_+$, then*

(a) $\alpha_{\eta\xi}(D) = 0$ *if and only if D is compact*

(b) $\alpha_{\eta\xi}$ *is a seminorm, that is; for* $\lambda > 0$,

$$\alpha_{\eta\xi}(\lambda D) = \mid \lambda \mid \alpha_{\eta\xi}(D) \text{ and } \alpha_{\eta\xi}(D_1 + D_2) \leq \alpha_{\eta\xi}(D_1) + \alpha(D_2)$$

(c) $D_1 \subset D_2$ *implies*

$$\alpha_{\eta\xi}(D_1) \leq \alpha_{\eta\xi}(D_2),\ \alpha_{\eta\xi}(D_1 \cup D_2) = \max\{\alpha_{\eta\xi}(D_1), \alpha_{\eta\xi}(D_2)\}$$

(d) $\alpha_{\eta\xi}(coD) = \alpha_{\eta\xi}(D)$.

(e) $\alpha_{\eta\xi}$ *is continuous with respect to the Hausdorff distance; that is*

$$\mid \alpha_{\eta\xi}(D_1) - \alpha_{\eta\xi}(D_2) \mid \leq \rho_{\eta\xi}(D_1, D_2)$$

for arbitrary $\eta, \xi \in \mathbb{D}\underline{\otimes}\mathbb{E}$ *where*

$$\rho_{\eta\xi}(D_1, D_2) = \max\big\{\sup_{x\in D_1} d_{\eta\xi}(x, D_2), \sup_{x\in D_2} d_{\eta\xi}(x, D_1)\big\},\ D_1, D_2 \subset \mathcal{B}_{\eta\xi}$$

Definition 2.5. (a) Let $v_0, v_1, \ldots, v_n$ be an affinely independent set of $n+1$ points in a vector space E. The convex hull

$$\{x \in E : x = \sum_{i=0}^{n} \lambda_i v_i, 0 \leq \lambda_i \leq 1, \sum_{i=0}^{n} \lambda_i = 1\}$$

is called (closed)n-simplex and is denoted by $v_0 v_1 \ldots v_n$. The points $v_0, v_1, \ldots, v_n$ are called the vertices of the simplex. For $0 \leq k \leq n$ and $0 \leq i_0 < i_1 < \cdots < i_k \leq n$, the k-simplex $v_{i_0} v_{i_1} \ldots v_{i_k}$ is a subset of the n-simplex $v_0 v_1 \ldots v_n$; it is called a k-dimensional face(or simply k-face) of $v_0 v_1 \ldots v_n$. In addition , if $y = \sum_{i=0}^{n} \lambda_i v_i$ we let $\chi(y) = \{i : \lambda_i > 0\}$

(b) A real-valued function ϕ on $\widetilde{\mathcal{A}}$ is lower (resp. upper) semicontinuous if the set $\{x \in \widetilde{\mathcal{A}} : \phi(x) \leq \lambda\}$ (resp. $\{x \in \widetilde{\mathcal{A}} : \phi(x) \geq \lambda\}$) is closed in $\widetilde{\mathcal{A}}$ for each $\lambda \in \mathbb{R}$. If Q is a convex set in a vector space then a real-valued function ϕ on Q is said to be quasiconcave (resp. quasiconvex) if $\{x \in Q : \phi(x) > \lambda\}$ (resp. $\{x \in Q : \phi(x) < \lambda\}$) is convex for each $\lambda \in \mathbb{R}$

(c) Let K be a non empty set, and $\Phi : K \to 2^K$ a multifunction, an element $x \in K$ is said to be a fixed point of Φ if $x \in \Phi(x)$.

(d) Let Q be a convex set in a vector space X, A a non-empty subset of Q and $F : A \to 2^Q$, a multivalued map. The family $\{F(x) : x \in A\}$ is said to be a KKM covering for Q if

$$co\{x : x \in N\} \subseteq \bigcup_{x\in N} F(x)$$

for any finite set $N \subseteq A$

2.2. **Preliminary results.** In the Locally convex spaces, Schauder-Tychonoff fixed point theorem is the generalization of Schauder fixed point theorem on Banach spaces[13]. For the case of multifunctions, Kakutani fixed point theorem is the multi-valued analogue of Schauder fixed point theorem and Kakutani-Fan fixed point theorem is the generalization of Schauder-Tychonoff theorem[1]. The following theorems due to Knaster, Kuratowski and Mazurkiewicz (KKM) shall be employed.

Theorem 2.6. [1] *Let $\{F_0, \ldots, F_n\}$ be a family of $n+1$ closed subsets of an n-simplex $v_0v_1\ldots v_n$. Suppose that for each $0 \le k \le n$ and $0 \le i_0 < i_1 < \cdots < i_k \le n$ we have*

$$v_{i_0}v_{i_1}\ldots v_{i_k} \subseteq F_{i_0} \cup F_{i_1} \cup \cdots \cup F_{i_k}$$

Then

$$\bigcap_{i=0}^{n} F_i \neq \emptyset$$

The infinite dimensional version of the KKM theorem, Theorem 2.1, above is:

Theorem 2.7. [1] *Let Q be a convex set in $\widetilde{\mathcal{A}}$, $\mathcal{N}$ a non-empty subset of Q, $F : \mathcal{N} \to 2^Q$ a multivalued map and $\{F(x) : x \in \mathcal{N}\}$ a KKM covering for Q. If there exists an $a \in \mathcal{N}$ with $\overline{F(a)}$ compact, then*

$$\bigcap_{x\in\mathcal{N}} \overline{F(x)} \neq \emptyset$$

The following is a non commutative analogue of the Ky Fan's minimax theorem, as established in[1]

Theorem 2.8. *Let $K \neq \emptyset$, convex and compact subset in $\widetilde{\mathcal{A}}$ and ϕ a real-valued function on the product space $K \times K$ satisfying the following conditions;*

$$\text{for each fixed } x \in K, \phi(x,\cdot) \text{ is lower semicontinuous on } K \text{ and} \tag{2.4}$$

$$\text{for each fixed } y \in K, \phi(\cdot,y) \text{ is quasiconcave on } K \tag{2.5}$$

Then there exists $y^* \in K$ with

$$\phi(x,y^*) \le \sup_{z\in K} \phi(z,z) \text{ for all } x \in K$$

(and therefore $\min_{y\in K}\sup_{x\in K}\phi(x,y) \le \sup_{x\in K}\phi(x,x)$)

Proof. Let $\lambda = \sup_{x\in K}\phi(x,x)$. We may assume that $\lambda \neq \infty$. For each $x \in K$ let

$$F(x) = \{y \in K : \phi(x,y) \le \lambda\}$$

condition 2.4 guarantees that each $F(x)$ is closed and hence compact in K (note that K is compact). We claim that $\{F(x) : x \in K\}$ is a KKM covering for K. If the claim is true then Theorem 2.2 guarantees that $\bigcap_{x\in K} F(x) \neq \emptyset$. Take $y^* \in \bigcap_{x\in K} F(x)$ and the proof is concluded.

To prove the claim . Suppose it is not true. Then there exists $\{x_1, \ldots, x_n\} \subset K$ and $\alpha_i > 0$ $(i = 0, 1, \ldots, n)$ with $\sum_{i=0}^{n} \alpha_i = 1$ such that

$$w = \sum_{i=0}^{n} \alpha_i x_i \in \big(\bigcup_{i=0}^{n} F(x_i)\big)'$$

This together with the definition of $F(x)$ yields

$$\phi(x_i, \sum_{i=0}^{n} \alpha_i x_i) = \phi(x_i, w) > \lambda, \text{ for } i = 0, 1, \ldots, n \tag{2.6}$$

Finally (2.4) together with the quasiconcavity of $\phi(\cdot, w)$ guarantees that $\phi(w, w) > \lambda$, a contradiction. □

In the following result, we shall employ the notation: $\langle x, g\rangle$ to denote the duality pairing for each $g \in \widetilde{\mathcal{A}}'$ and $x \in \widetilde{\mathcal{A}}$

Theorem 2.9. *Let $X : I \to \widetilde{\mathcal{A}}$, Q a non-empty subset of $\widetilde{\mathcal{A}}$ and $\Phi : Q \to 2^Q$ be upper semicontinuous with $\Phi(X(t))$ non-empty and bounded for each $X(t) \in Q$. Then for any $g \in \widetilde{\mathcal{A}}'$(dual), the map $\phi_g : Q \to \mathbb{R}$, defined by $\phi_g(Y(t)) = \sup_{X(t)\in\Phi(Y(t))} Re\langle X(t), g\rangle$ is upper semicontinuous in the sense of real-valued function.*

Proof. Fix $y_0 \in Q$. Let $\epsilon > 0$ be given and let

$$U_\epsilon = \{X(t) \in Q :\mid \langle X(t), g\rangle \mid < \frac{\epsilon}{2}\}$$

Notice that U_ϵ is an open neighbourhood of 0. Since $\Phi(y_0) + U_\epsilon$ is an open set containing $\Phi(y_0)$, it follows from the upper semicontinuity of Φ at y_0 that there exists a neighbourhood $N(y_0)$ of y_0 in Q with

$$\Phi(Y(t)) \subseteq \Phi(y_0) + U_\epsilon \text{ for all } Y(t) \in N(y_0)$$

Thus for each $Y(t) \in N(y_0)$ we have that

$$\begin{aligned}\phi_g(Y(t)) = \sup_{X(t)\in\Phi(Y(t))} Re\langle X(t), g\rangle &\le \sup_{X(t)\in\Phi(y_0)+U_\epsilon} Re\langle X(t), g\rangle \\ &\le \sup_{X(t)\in\Phi(y_0)} Re\langle X(t), g\rangle + \sup_{X(t)\in U_\epsilon} Re\langle X(t), g\rangle \\ &< \phi_g(y_0) + \epsilon\end{aligned}$$

therefore ϕ_g is upper semicontinuous. □

The following separation theorem shall be employed in what follows:

Theorem 2.10. [1] *Suppose that A and B are disjoint, non-empty, convex sets in $\widetilde{\mathcal{A}}$. If in addition A is compact and B is closed, then there exist $f :\in \widetilde{\mathcal{A}}'$ and $\gamma \in \mathbb{R}$ with*

$$\max Ref(A) < \gamma \le \inf Ref(B)$$

3. MAIN RESULTS

Theorem 3.1. *Suppose $K \neq \emptyset$, $K \subset \widetilde{\mathcal{A}}$ is a convex and compact subset of $\widetilde{\mathcal{A}}$, such that the following conditions hold:*

(i) *$X(t)$ is a stochastic process; $X : I \to \widetilde{\mathcal{A}}$ such that $X(t) \in K, \forall t \in I$*

(ii) *The map $\Phi : K \to 2^K$ is upper semicontinuous with respect to a pair $\eta, \xi \in \mathbb{D}\underline{\otimes}\mathbb{E}$, with $\Phi(X(t))$ a non-empty closed and convex subset of K for each $X(t) \in K$. Then there exists a $y(t) \in K$ with $y(t) \in \Phi(y(t))$.*

Proof. Suppose that the result is not true, that is suppose $y(t) \notin \Phi(y(t))$ for such $y(t) \in K$. Now for each $y(t) \in K$, Theorem 2.4 guarantees that there exists $f_{y(t)} \in \widetilde{\mathcal{A}}'$ with

$$Re\langle y(t), f_{y(t)}\rangle - \sup_{X(t)\in\Phi(y(t))} Re\langle X(t), f_{y(t)}\rangle > 0. \tag{3.1}$$

For each $g \in \widetilde{\mathcal{A}}'$, let

$$V(g) = \{y(t) \in K : Re\langle y(t), g\rangle - \sup_{X(t)\in\Phi(y(t))} \langle X(t), g\rangle > 0\}$$

We observe that (3.1) ensures that $K = \bigcup_{g\in\widetilde{\mathcal{A}}'} V(g)$. In addition Theorem 2.2 implies that $V(g)$ is open in K. The compactness of K guarantees the existence of $g_1, g_2, \ldots, g_n \in \widetilde{\mathcal{A}}'$ with $K = \bigcup_{i=1}^n V(g_i)$. Let $\{\lambda_1, \ldots, \lambda_n\}$ be a partition of unity on K subordinate to the covering $\{V(g_1), \ldots, V(g_n)\}$ (let $V_i = V(g_i)$ for $i - 1, \ldots, n$), that is $\lambda_1, \ldots, \lambda_n$ are continuous non negative real valued functions on K with λ_i vanishing on $K \setminus V_i$ for each $i = 1, \ldots, n$ and $\sum_{i=1}^n \lambda_i(X(t)) = 1$ for all $X(t) \in K$. Therefore K is a non-empty,convex and compact subset of $\widetilde{\mathcal{A}}$. Let $\phi : K \times K \to \mathbb{R}$ be given by

$$\phi(X(t), y(t)) = \sum_{i=1}^n \lambda_i(y(t))Re\langle y(t) - X(t), g_i\rangle$$

For each $X(t) \in K$ $\phi(X(t), \cdot)$ is lower semicontinuous on K and for each $y(t) \in K, \lambda \in \mathbb{R}$ the set, $\{X(t) \in K : \phi(X(t), y(t)) > \lambda\}$ is convex, then by Ky Fan's minimax theorem (Theorem 2.5), there exists $y_0 \in K$ with

$$\phi(X(t), y_0) \leq 0, \text{ for all } X(t) \in K$$

that is,

$$\sum_{i=1}^n \lambda_i(y_0)Re\langle y_0 - X(t), g_i\rangle \leq 0 \text{ for all } X(t) \in K \tag{3.2}$$

Suppose that $i \in \{1, 2, \ldots, n\}$ is such that $\lambda_i(y_0) > 0$. Then $y_i \in V(g_i)$ (since λ_i vanishes on $K \setminus V_i$) and consequently,

$$Re\langle y_0, g_i\rangle > \sup_{X(t)\in\Phi(y_0)} Re\langle X(t), g_i\rangle \geq Re\langle x_0, g_i\rangle$$

for all $x_0 \in \Phi(y_0)$ (that is, $Re\langle y_0 - x_0, g_i\rangle > 0$ for all $x_0 \in \Phi(y_0)$). Thus $\lambda_i(y_0)Re\langle y_0 - x_0, g_i\rangle > 0$ whenever $\lambda_i(y_0) > 0$ (for $i = 1, \dots, n$) for all $x_0 \in \Phi(y_0)$. Since $\lambda_i(y_0) > 0$ for at least one $i \in \{1, 2, \dots, n\}$, it follows that

$$\sum_{i=1}^{n} \lambda_i(y_0) Re\langle y_0 - x_0, g_i\rangle > 0$$

for all $x_0 \in \Phi(y_0)$. This contradicts (3.2). Therefore the conclusion of the theorem is true. □

Theorem 3.2. *Assume that the maps $E, F, G, H \in L^2_{loc}(I \times \widetilde{\mathcal{A}})_{mvs}$ and $\mathbb{P} : I \times \widetilde{\mathcal{A}} \to 2^{sesq(\mathbb{D}\underline{\otimes}\mathbb{E})^2}$, a sesquilinear form valued map with closed and convex values such that*

(a) $t \mapsto \mathbb{P}(t, X(t))(\eta, \xi)$ *has a measurable selection,*

(b) $X \mapsto \mathbb{P}(t, X(t))(\eta, \xi)$ *is upper semicontinuous,*

(c) $\rho(\mathbb{P}(t, X(t))(\eta, \xi), \{0\}) \leq c(t)(1+ \parallel X \parallel_{\eta\xi})$ *on* $I \times \widetilde{\mathcal{A}}$ *with* $c \in L^1_{loc}(I)$,

(d) $\lim_{\tau\to 0+} \alpha_{\eta\xi}\Big(\mathbb{P}(I_{t,\tau} \times B)(\eta, \xi)\Big) \leq k(t)\alpha_{\eta\xi}(B)$ *on* I, *where* $\mathbb{P}(I_{t,\tau} \times B)(\eta, \xi) = \{\mathbb{P}(t, X(t))(\eta, \xi) : (t, X) \in I_{t,\tau} \times B\}$, $I_{t,\tau} = [t - \tau, t + \tau] \cap I$ *for* $B \in \mathcal{B}_{\eta\xi}$, $\eta, \xi \in \mathbb{D}\underline{\otimes}\mathbb{E}$ *and* $k \in L^1_{loc}(I)$. *Then the quantum stochastic differential inclusion*

$$\frac{d}{dt}\langle \eta, X(t)\xi\rangle \in \mathbb{P}(t, X(t))(\eta, \xi) \ X(t_0) = x_0 \ a.e. \ on \ I$$

has a solution on I.

Proof. If $v \in Ad(\widetilde{\mathcal{A}})_{wac} \cap L^2_{loc}(\widetilde{\mathcal{A}})$, by (a), for an arbitrary pair of $\eta, \xi \in \mathbb{D}\underline{\otimes}\mathbb{E}$, $\mathbb{P}(\cdot, v(\cdot))(\eta, \xi)$ has a measurable selection. That is there exists $\omega_{\eta\xi}(\cdot) \in \mathbb{P}(\cdot, v(\cdot))(\eta, \xi)$, such that $t \to \omega_{\eta\xi}(t)$ is measurable. By (c) we find that there exists $\psi_1(t)$ and $\psi_2(t) = c(t)(1 + \psi_1(t))$. Now we define K as:

$$K = \{v \in Ad(\widetilde{\mathcal{A}})_{wac} \cap L^2_{loc}(\widetilde{\mathcal{A}}) : v(t_0) = x_0, \parallel v(t) \parallel_{\eta\xi} \leq \psi_1(t) \text{ and}$$
$$\parallel v(t) - v(s) \parallel_{\eta\xi} \leq \Big| \int_s^t \psi_2(\tau)d\tau \ \Big| \ \forall \eta, \xi \in \mathbb{D}\underline{\otimes}\mathbb{E}\}$$

Also, since $\omega_{\eta\xi}(\cdot) \in \mathbb{P}(\cdot, v(\cdot))(\eta, \xi)$, there exists $\omega : I \to \widetilde{\mathcal{A}}$ such that $\omega_{\eta\xi}(\cdot) = \langle \eta, \omega(\cdot)\xi\rangle$, for arbitrary $\eta, \xi \in \mathbb{D}\underline{\otimes}\mathbb{E}$. Let $\widehat{K} \subset K$ be defined as

$$\widehat{K} = \{u \in K : \text{ there exist } v(\cdot) \in Ad(\widetilde{\mathcal{A}})_{wac} \cap L^2_{loc}(\widetilde{\mathcal{A}}), \omega_{\eta\xi}(\cdot)$$
$$= \langle \eta, \omega(\cdot)\xi\rangle \in \mathbb{P}(\cdot, v(\cdot))(\eta, \xi),$$
$$\text{with } \langle \eta, u(t)\xi\rangle = \langle \eta, x_0\xi\rangle + \int_0^t \omega_{\eta\xi}(s)ds\} \tag{3.3}$$

and a multivalued map $G : \widehat{K} \to \widehat{K}$ defined by

$$G(v) = \{u \in Ad(\widetilde{\mathcal{A}})_{wac} \cap L^2_{loc}(\widetilde{\mathcal{A}}) : \langle \eta, u(t)\xi\rangle = \langle \eta, x_0\xi\rangle + \int_0^t \omega_{\eta\xi}(s)ds,$$
$$\omega_{\eta\xi}(\cdot) = \langle \eta, \omega(\cdot)\xi\rangle \in \mathbb{P}(\cdot, v(\cdot))(\eta, \xi) \forall \eta\xi \in \mathbb{D}\underline{\otimes}\mathbb{E}\}.$$

G maps $\widehat{K}$ into itself, since for any $u \in G(v)$; $\langle \eta, u(t)\xi\rangle = \langle \eta, x_0\xi\rangle + \int_0^t \omega_{\eta\xi}(s)ds$, $u(0) = x_0$, $\| u(t) \|_{\eta\xi} \leq \psi_1(t)$ and

$$\begin{aligned} | \langle \eta, (u(t) - u(s))\xi\rangle | &= | \int_0^t \omega_{\eta\xi}(s)ds - \int_0^s \omega_{\eta\xi}(s)ds | \\ &= | \int_s^t \omega_{\eta\xi}(s)ds | \\ &\leq | \int_s^t c(\tau)(1 + | \psi_1(\tau) |)d\tau | \\ &\leq | \int_s^t \psi_2(\tau)d\tau | \end{aligned}$$

$\widehat{K}$ is bounded and weakly-equicontinuous, since for any $v \in \widehat{K}$; $v \in Ad(\widetilde{\mathcal{A}})_{wac} \cap L^2_{loc}(\widetilde{\mathcal{A}}$, $t, s \in I$, given $\epsilon > 0$, there exists $\delta > 0$ such that $\| v(t) - v(s) \|_{\eta\xi} < \epsilon$ whenever $| t - s | < \delta$. The weak equicontinuity follows by setting $\delta = \frac{\epsilon}{\lambda}$ where $\lambda = \max_{\tau\in[s,t]} | \psi_2(\tau) |$.

Moreover let $\alpha_{\eta\xi,0}(\cdot) = \alpha_{\eta\xi}(\cdot)$ for $Ad(\widetilde{\mathcal{A}})_{wac} \cap L^2_{loc}(\widetilde{\mathcal{A}}$ and $B(t) = \{v(t) : v \in B\}$, then $\alpha_{\eta\xi,0}(B) = \max_I \alpha_{\eta\xi}(B(t))$ for $B \subseteq\subseteq \widehat{K}$. Let $K_0 = \widehat{K}$, $\widehat{K}_{n+1} = convG(\widehat{K}_n)$ for $n \geq 0$ and $\widehat{K}_\infty = \bigcap_{n\geq 0} \widehat{K}_n$. Then $\left(\widehat{K}_n\right)$ is a decreasing sequence of closed convex sets. To show that $\widehat{K}_\infty$ is compact. Let $\rho_{\eta\xi,n}(t) = \alpha_{\eta\xi}(\widehat{K}_n(t))$ and $\gamma_{\eta\xi,n}(t) = \alpha_{\eta\xi}(G(\widehat{K}_n)(t))$. $\gamma_{\eta\xi,n}$ is absolutely continuous with $\gamma_{\eta\xi,n}(0) = 0$ and for $0 < t - \tau < t \leq T$, we have

$$\gamma_{\eta\xi,n}(t) - \gamma_{\eta\xi,n}(t-\tau) \leq \alpha_{\eta\xi}\left(\{\int_{t-\tau}^t \omega_{\eta\xi}(s)ds; \langle \eta, \omega(\cdot)\xi\rangle \in \mathbb{P}(\cdot, v(\cdot))(\eta,\xi), v \in \widehat{K}_n\right).$$

Using

$$\int_{t-\tau}^t \omega_{\eta\xi}(s)ds \in \tau\overline{conv}\mathbb{P}(I_{t,\tau} \times \cup_{I_{t,\tau_0}} K_n(s))(\eta,\xi) \text{ for } \tau \leq \tau_0,$$

we obtain

$$\frac{d}{dt}\gamma_{\eta\xi,n}(t) \leq K(t)\alpha_{\eta\xi}(\bigcup_{I_{t,\tau_0}} \widehat{K}_n(s))$$

almost everywhere, from condition (c) and therefore

$$\frac{d}{dt}\gamma_{\eta\xi}(t) \leq K(t)\rho_n \ a.e.$$

by letting $\tau_0 \to 0+$, since $\widehat{K}_n$ is equicontinuous. But $(\overline{conv}A)(t) = \overline{conv}A(t)$, then

$$\rho_{n+1}(t) \leq \int_0^t K(s)\rho_n(s)ds$$

hence $\rho_n(t) \to 0$ uniformly, since (ρ_n) is decreasing. Consequently, $\alpha_{\eta\xi,0}(K_\infty) = \max_I \alpha_{\eta\xi}(K_\infty(t)) = 0$ that is $\widehat{K}_\infty$ is compact with respect to τ^{wac} and convex. We also have $\widehat{K}_\infty \neq \emptyset$, since we may pick $v_n \in \widehat{K}_n$ and proceed in the same way to get $v_m \to v_0$ for some subsequence, hence $v_0 \in \widehat{K}_\infty$. Now, $G : \widehat{K}_\infty \to 2^{\widehat{K}_\infty} \setminus \emptyset$ and has convex values. If $(u_n) \subset G(v)$ then the corresponding (ω_n) has a weakly convergent subsequence. Hence $G(v)$ is also compact, moreover $G |_{K_\infty}$ has closed graph, hence $G|_{K_\infty}$ is Upper semicontinuous and therefore G has a fixed point in K_∞ by Kakutani-Fan fixed point theorem (Theorem 3.1).

Let $\varphi \in \widehat{K}_\infty$ be a fixed point of G. Then $\varphi \in Ad(\widetilde{\mathcal{A}})_{wac} \cap L^2_{loc}(\widetilde{\mathcal{A}})$ and

$$\langle \eta, \varphi(t)\xi \rangle = \langle \eta, x_0 \xi \rangle + \int_0^t \omega_{\eta\xi}(s) ds$$

But, $\omega_{\eta\xi}(\cdot) \in \mathbb{P}(\cdot, \varphi(\cdot))(\eta, \xi)$. Therefore,

$$\frac{d}{dt}\langle \eta, \varphi(t)\xi \rangle = \langle \eta, \omega(t)\xi \rangle \in \mathbb{P}(t, \varphi(t))(\eta, \xi)$$

and $\varphi(t_0) = x_0$, a.e. $t \in I$. Hence the fixed point of G is a solution of the problem $\frac{d}{dt}\langle \eta, X(t)\xi \rangle \in \mathbb{P}(t, X(t))(\eta, \xi)\ X(t_0) = x_0$ a.e. on I. □

REFERENCES

[1] R. P. Agarwal, M. Meehan and D. O'Regan, *Fixed point theory and applications*, Cambridge Tracts in Mathematics, 141, Cambridge University Press, 2004.

[2] J. P. Aubin and A. Cellina, *Differential Inclusions*, Springer-Verlag, Berlin, 1984.

[3] E. O. Ayoola, Continuous selections of solution sets of Lipschitzian quantum stochastic differential inclusions.*Int. J. Theor.Phys.* 43, 10, 2041–2059, 2004.

[4] E. O. Ayoola, Topological properties of solution sets of Lipschitzian quantum stochastic differential inclusions.*Acta Appl.Math*, 100, 15–37, 2008.

[5] K. Deimling, *Non linear Functional Analysis*, Springer-Verlag, Berlin, 1985.

[6] K. Deimling, *Multivalued differential equations*, Walter de Gruyter, 1992.

[7] G. O. S. Ekhaguere, Lipschitzian quantum stochastic differential inclusions *Int. J. Theor. Phys.*, 31, 11, 2003–2034, 1992.

[8] G. O. S. Ekhaguere, Quantum stochastic differential inclusions of hypermaximal monotone type *Int. J. Theor. Phys.* 34, 3, 323–353, 1995.

[9] G. O. S. Ekhaguere, Quantum stochastic evolutions *Int. J. Theor. Phys.*, 35, 9, 1909–1946, 1996.

[10] G. O. S. Ekhaguere, Topological solutions of Non commutative stochastic differential equations *Stoch. Anal. and Appl.* 25, 9, 961–993, 2007.

[11] A. Guichardet, *Symmetric Hilbert spaces and related topics*Lecture Notes in Mathematics, 261, Springer-Verlag, Berlin, 1972.

[12] R. L. Hudson and K. R. Parthasarathy, Quantum Ito's formula and stochastic evolutions*Comm. Math. Phys.*, 93, 3, 301–323, 1984.

[13] M. Reed and B. Simon, *Methods of Modern Mathematical Physics I. Functional Analysis*, 2nd edn., Academic press, New York-London, 1980.

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In honour of Prof. Ekhaguere at 70

Existence of solution of impulsive quantum stochastic differential inclusion

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Abstract. By employing the non-commutative analoque of Leray-Schauder fixed point theorem, Arsela-Ascoli theorem and Michael selection theorem, we establish the existence of solution of impulsive quantum stochastic differential inclusions(IQSDI) in the framework of Hudson and Parthasarathy formulation of quantum stochastic calculus. The result hold in an infinite dimensional locally convex space. Important properties of these solutions are studied.

Keywords: non-commutative analogue, impulsive quantum stochastic differential inclusion, selections, fixed point, infinite dimensional locally convex space, Upper and lower semicontinuous operators.

1. Introduction

For well over a century, differential equations have been used in modeling the dynamics of changing processes. A great deal of the modeling development has been accompanied by a rich theory for differential equations. The dynamics of many evolving processes are subject to abrupt changes, such as shocks, harvesting and natural disasters. These phenomena involve short-term perturbations from continuous and smooth dynamics, whose duration is negligible in comparison with the duration of an entire evolution. In models involving such perturbations, it is natural to assume these perturbations act instantaneously or in the form of ”impulses”. As a consequence, classical impulsive differential equations have found application in modeling impulsive problems in physics, population dynamics, ecology, biological systems, biotechnology, industrial robotics, pharmacokinetics, optimal control, and so forth. Again, associated with this development, a classical theory of impulsive differential equations has been given extensive attention. Much attention has also been devoted to modeling natural phenomena with differential equations, both ordinary and functional, for which the part governing the derivative(s) is not known as a single-valued function. Our consideration in this paper concerns the establishment of a solution of impulsive quantum stochastic differential inclusions in the framework of Hudson-Phathasarathy formulation of quantum stochastic calculus.

The plan for the rest of the paper is as follows: section 2 contains fundamental structures and definitions that we use in the sequel. In section 3 we assemble some auxiliary results that are use in establishing the main result. The main result concerning the existence of solution to impulsive quantum stochastic differential inclusion is established in section 4.

2. Fundamental structures and definitions

In this section we state some fundamental structures and definition that will be use in the sequel. Given a multifunction $F : \mathbb{R}^m \to 2^{\mathbb{R}^m}$, a single valued map $f : \mathbb{R}^m \to \mathbb{R}^n$ is a selection if $f(x) \in F(x) \quad \forall x \in \mathbb{R}$.

(i) *Upper and Lower Semi continuous Multivalued Maps*: Let $\mathcal{N} \subseteq \tilde{\mathcal{A}}$ and $I \subseteq \mathbb{R}_+$. For arbitrary $\eta, \xi \in \mathbb{D}\underline{\otimes}\mathbb{E}, (t, x), (t_0, x_0) \in I \times \mathcal{N}$ and real numbers $\epsilon, \delta_{\eta,\xi} > 0$, we define the map $d_{\eta,\xi}$:

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$[I \times \mathcal{N}] \to \mathbb{R}_+$ by

$$d_{\eta,\xi}((t,x),(t_0,x_0)) = \max\{|t-t_0|, \|x-x_0\|_{\eta,\xi}\}.$$

The following shall be employed in what follows. For arbitrary $\eta,\xi \in \mathbb{D}\underline{\otimes}\mathbb{E}$

$$\tilde{\mathcal{A}}(\eta,\xi) = \{x_{\eta,\xi} = \langle \eta, x\xi\rangle, x \in \tilde{\mathcal{A}}\}$$

$$B_{\eta,\xi,\epsilon}(0) = \{x_{\eta,\xi} \in \tilde{\mathcal{A}}(\eta,\xi) : |x_{\eta,\xi}| < \epsilon\}$$

$$B_{\delta_{\eta,\xi}}(t_0,x_0) = \{(t,x) \in I \times \mathcal{N} : d_{\eta,\xi}((t,x),(t_0,x_0)) < \delta_{\eta,\xi}\}.$$

(ii) A map $\phi : I \times \mathcal{N} \to 2^{sesq(\mathbb{D}\otimes\mathbb{E})^2}$ will be said to be upper semi continuous at a point $(t_0,x_0) \in I \times \mathcal{N}$, if for each pair $\eta,\xi \in \mathbb{D}\underline{\otimes}\mathbb{E}$ and $\epsilon > 0$ there exists $\delta_{\eta,\xi} = \delta_{\eta,\xi}((t_0,x_0),\epsilon) > 0$ such that

$$\phi(t,x)(\eta,\xi) \subset \phi(t_0,x_0)(\eta,\xi) + B_{\eta,\xi,\epsilon}(0)$$

on $B_{\delta_{\eta,\xi}}(t_0,x_0)$. The map ϕ is said to be upper semi continuous if it is upper semi continuous at every point $(t,x) \in I \times \mathcal{N}$. Furthermore, for a sesquilinear formed valued map we define

$$B_{\mathbb{P},\epsilon}(0) = \{\varphi(t,x)(\eta,\xi) \in \mathbb{P}(t,x)(\eta,\xi) : |\varphi(t,x)(\eta,\xi)| < \epsilon\}$$

(iii) A sesquilinear form valued multifunction $\mathbb{P} : I \times \mathcal{N} \to 2^{sesq(\mathbb{D}\otimes\mathbb{E})^2}$ will be said to be upper semi continuous at a point $(t_0,x_0) \in I \times \mathcal{N}$ if for every $\eta,\xi \in \mathbb{D}\underline{\otimes}\mathbb{E}$ and $\epsilon > 0$ there exist $\delta_{\eta,\xi} = \delta_{\eta,\xi}((t_0,x_0),\epsilon) > 0$ such that

$$\mathbb{P}(t,x)(\eta,\xi) \subset \mathbb{P}(t_0,x_0)(\eta,\xi) + B_{\mathbb{P},\epsilon}(0)$$

on $B_{\delta_{\eta,\xi}}(t_0,x_0)$. The map $\mathbb{P}$ is said to be upper semi continuous if it is upper semi continuous at every point $(t,x) \in I \times \mathcal{N}$. For arbitrary $\eta,\xi \in \mathbb{D}\underline{\otimes}\mathbb{E}$, let $\Phi : I \times \tilde{\mathcal{A}} \to 2^{\tilde{\mathcal{A}}}$ be a closed multivalued map. For each pair $(t,x),(t',x') \in I \times \tilde{\mathcal{A}}$ we define

$$d_{\eta,\xi}((t,x),(t',x')) = \max\{|t-t'|, ||x-x'||_{\eta\xi}\}$$

$$B_{\eta,\xi,\delta}(t_0,x_o) = \{(t,x) \in I \times \tilde{\mathcal{A}} : d_{\eta,\xi}((t_0,x_0),(t,x))(\eta,\xi) < \delta\} \quad \text{and}$$

$$B_{\eta\xi,\epsilon}(\Phi(t,x) = \{y \in \tilde{\mathcal{A}} : \inf_{k\in\Phi(t,x)} ||y-k||_{\eta,\xi} < \epsilon\}.$$

(iv) A map $\Phi : I \times \mathcal{N} \to 2^{sesq(\mathbb{D}\otimes\mathbb{E})^2}$ will be said to be lower semi continuous at a point $(t_0,x_0) \in I \times \mathcal{N}$, with respect to the seminorm $||.||_{\eta\xi}$ if for each pair $\eta,\xi \in \mathbb{D}\underline{\otimes}\mathbb{E}$ and $\epsilon > 0$ there exists $\delta_{\eta,\xi} = \delta_{\eta,\xi}((t_0,x_0),\epsilon) > 0$ such that for each $y_o \in \Phi(t_o,x_o)$, $\inf_{y\in\Phi(t,x)} ||y-y_o||_{\eta\xi} < \epsilon$, $\forall \; y \in \mathcal{N}$, almost all $t \in I$ and $d_{\eta,\xi}((t,x),(t',x')) < \delta_{\eta,\xi}$. If Φ is lower semi continuous at every point $(t_0,x_0) \in I \times \mathcal{N}$ with respect to the seminorm $||.||_{\eta\xi}$, then it will be said to be lower semi continuous on $I \times \mathcal{N}$.

(v) A sesquilinear form valued multifunction $\mathbb{P} : I \times \mathcal{N} \to 2^{sesq(\mathbb{D}\otimes\mathbb{E})^2}$ will be said to be lower semi continuous at a point $(t_0,x_0) \in I \times \mathcal{N}$, with respect to the seminorm $||.||_{\eta\xi}$ if for every $\eta,\xi \in \mathbb{D}\underline{\otimes}\mathbb{E}$ and $\epsilon > 0$ there exist $\delta_{\eta,\xi} = \delta_{\eta,\xi}((t_0,x_0),\epsilon) > 0$ such that for each

$y_{\eta\xi,0} \in \mathbb{P}(t_0, x_0)(\eta,\xi)$ $\inf_{y_{\eta\xi}\in\mathbb{P}(t,x)} |y_{\eta\xi,0} - y_{\eta\xi}| < \epsilon, \forall y \in \mathcal{N}$, almost all $t \in I$ and $d_{\eta,\xi}((t,x),(t_0,x_0)) < \delta_{\eta,\xi}$

(vi) The space $Pad(I,\tilde{\mathcal{A}})_{wac} = \{X : I \to \tilde{\mathcal{A}} : X$ is adapted and weakly absolutely continuous everywhere except for some t_k at which $X(t_k^-)$ and $X(t_k^+)$, $k = 1,2,...,m$ exists and $X(t_k^-) = X(t_k^+)\}$.

(vii) For each pair $\eta,\xi \in \mathbb{D}\underline{\otimes}\mathbb{E}$, we define the space of complex valued numbers associated with (i) as $Pad(I,\tilde{\mathcal{A}})_{wac,\eta\xi} = \{\langle\eta,\Phi(.)\xi\rangle : \Phi \in Pad(I,\tilde{\mathcal{A}})_{wac}\}$.

(viii) On $Pad(I,\tilde{\mathcal{A}})_{wac}$, we define a seminorm

$$\|\Phi\|_{p,\eta\xi} = \sup\{\|\Phi(t)\|_{\eta\xi}, t \in I\}, \tag{2.0}$$

and denote by $P_{wac}(\tilde{\mathcal{A}})$ the completion of the locally convex space whose topology is generated by the seminorm in (2.0).

(ix) Let $E,F,G,H \in L^2_{loc}(I\times\tilde{\mathcal{A}})_{mvs}$ and (t_0,x_0) be a fixed point of $I\times\tilde{\mathcal{A}}$. Then a relation of the form

$$dX(t) \in E(t,X(t))d\Lambda_\pi(t) + F(t,X(t))dA_f(t) + G(t,X(t))dA_g^+(t) + H(t,x(t))dt$$

for almost all $t \in I\backslash\{t_k\}_{k=1}^m$,

$$\Delta X_{t=t_k} = J_k(X(t_k)), \qquad t = t_k, k = 1,2,...,m \tag{2.1}$$

$$X(t_0) = \Phi(t), t \in I$$

or equivalently

$$\frac{d}{dt}[\langle\eta, X(t)\xi\rangle] \in \mathbb{P}(t,X(t))(\eta,\xi) \qquad \text{almost all t } \in I\backslash\{t_k\}_{k=1}^m,$$

$$\langle\eta,\Delta X_{t=t_k}\xi\rangle = \langle\eta, J_k X(t_k)\xi\rangle, \quad t = t_k, k = 1,2,...,m, \tag{2.2}$$

$$\langle\eta, X(t_0)\xi\rangle = \langle\eta,\phi(t)\xi\rangle, t \in I,$$

where $0 = t_0 < t_1 < t_2 < ..... < t_m < t_{m+1} = T, I = [0,T]$, is called impulsive quantum stochastic differential inclusions (IQSDI). *Note*: The map $\mathbb{P} : I\times\tilde{\mathcal{A}} \to 2^{sesq(\mathbb{D}\otimes\mathbb{E})^2}$ is a multivalued sesquilinear form having non empty, compact values. $X(t_0) \in \tilde{\mathcal{A}}$, $J_k \in C(\tilde{\mathcal{A}},\tilde{\mathcal{A}})$, $k = 1,2,...,m. \Delta X|_{t=t_k} = X(t_k^+) - X(t_k^-)$, $X(t_k^-), X(t_k^+)$ represent the left and the right limit of $X(t)$.

For any process $X : I \to \tilde{\mathcal{A}}$ and any $t \in I$, $X(t)$ represents the history of the state from previous time up to the present time t , the map J_k characterize the jump of the solutions at impulse points $t_k, k = 1,2,...,m$.

(x) By solution of Impulsive quantum stochastic differential inclusion (2.1) or equivalently (2.2) we mean a stochastic process $\Phi : \tilde{\mathcal{A}} \to \tilde{\mathcal{A}}$ lying in the space $P_{wac}(\tilde{\mathcal{A}}) \cap wac((t_k,t_{k+1}),\tilde{\mathcal{A}}), 0 \le k \le m$, satisfying

$$\frac{d}{dt}[\langle\eta,\Phi(t)\xi\rangle] \in \mathbb{P}(t,\Phi(t))(\eta,\xi) \qquad \text{almost all t } \in I\backslash[t_k]_{k=1}^m$$

and the condition

$$\Delta\Phi|_{t=t_k} = J_k(\Phi(t_k^-)) \quad \text{and} \quad \Phi(0) = X_0.$$

The following theorems shall be employ to prove our main result.

3. Theorems

THEOREM 3.1 *Let U and $\overline{U}$ denote respectively the open and closed subsets of a convex set K of $\tilde{\mathcal{A}}$ such that $0 \in U$ and let $N : \overline{U} \to K$ be a compact and semi continuous map. Then either*
(i) The equation $x = Nx$ has a solution in $\overline{U}$ or
(ii) There exists a point $u \in \delta U$ such that $u = \lambda N u$ for some $\lambda \in \mathbb{C}$ such that $Re\lambda \in (0, 1)$ and $Im\lambda \in (0, 1)$, where δU is a boundary of U.

THEOREM 3.2 *Let $X : I \to \tilde{\mathcal{A}}$ be a stochastic process that satisfy the following conditions :*
(i) For any arbitrary pair $\eta, \xi \in \mathbb{D}\underline{\otimes}\mathbb{E}$,
let $K \subset \tilde{\mathcal{A}}$ such that $F : K \to K$ is a compact map.
(ii) $\|f(x)\|_{\eta\xi} \leq m$ for each $x \in X, f \in F$ and $m < \infty$.
(iii) For every $\epsilon > 0$ (depending on η, ξ) there exist $\delta_{\eta\xi}$ such that for every $x, y \in X$,

$$d(x, y)(\eta, \xi) < \delta_{\eta\xi}.$$

Then,

$$\langle \eta, (f(x) - f(y))\xi \rangle < \epsilon \quad \forall \quad f \in F, \quad x, y \in X.$$

Next, we shall establish the a priori estimates on possible solutions of problem (2.1)-(2.2).

THEOREM 3.3 *Suppose that the following hold for arbitrary pair $\eta, \xi \in \mathbb{D}\underline{\otimes}\mathbb{E}$. (i) There exists a continuous non-decreasing function $\phi : \mathbb{R}_+ \to \mathbb{R}_+$ and*

$$p \in L^1(I, \mathbb{R}_+) \quad \textit{such that} \quad |\mathbb{P}(t, x)(\eta, \xi)| \leq p(t)\phi(||X||_{\eta,\xi})$$

$$\textit{for a.e} \quad t \in I \quad \textit{and} \quad x \in \tilde{\mathcal{A}} \tag{3.3.1}$$

with

$$\int_{t_{k-1}}^{t_k} p(s)ds < \int_{N_{k-1,\eta,\xi}}^{\infty} \frac{du}{\phi(u)}, \quad k = 1, ..., m+1, \tag{3.3.2}$$

where

$$N_{0,\eta,\xi} = ||x_0||_{\eta,\xi},$$

$$N_{k-1,\eta,\xi} = \sup_{||x||_{\eta,\xi} \in [-M_{k-2}, M_{k-2}]} ||J_{k-1}(x)||_{\eta,\xi} + M_{k-2},$$

$$M_{k-2} = \Gamma_{k-1}^{-1} \int_{t_{k-2}}^{t_{k-1}} p(s)ds, \quad \textit{for} \quad k = 1, ..., m+1, \tag{3.3.3}$$

and

$$\Gamma_l(z) = \int_{N_{l-1,\eta,\xi}}^{z} \frac{du}{\phi(u)}, \quad z \geq N_{l-1} \quad l \in [1, ..., m+1]. \tag{3.3.4}$$

Then, for each $k = 1, ..., m+1$ there exists a constant $M_{k-1,\eta,\xi}$ such that

$$\sup\{||X(t)||_{\eta,\xi} : t \in [t_k, t_{k+1}]\} \leq M_{k-1,\eta,\xi} \tag{3.3.5}$$

for each solution X of the problem (2.1 - 2.2).

Proof. Let X be a possible solution of (2.2). Then $X|_{[0,t_1]}$ is a solution to

$$\frac{d}{dt}\langle \eta, X(t)\xi\rangle \in \mathbb{P}(t, X(t))(\eta, \xi) \qquad \text{almost all} \quad t \in [0, t_1], X(0) = X_0. \tag{3.3.7}$$

Since

$$\frac{d}{dt}|\langle \eta, X(t)\xi\rangle| \leq |\frac{d}{dt}\langle \eta, X(t)\xi\rangle|, \tag{3.3.8}$$

we have

$$\frac{d}{dt}|\langle \eta, X(t)\xi\rangle| \leq p(t)\phi(\|X(t)\|_{\eta\xi}), \quad \text{for a.e} \quad t \in [0, t_1]. \tag{3.3.9}$$

Let $t^* \in [0, t_1]$ such that

$$\sup\{\|X(t)\|_{\eta\xi} : t \in [0, t_1]\} = \|X(t^*)\|_{\eta\xi}\}, \tag{3.3.10}$$

then

$$\frac{\frac{d}{dt}|\langle \eta, X(t)\xi\rangle|}{\phi(\|X(t)\|_{\eta\xi})} \leq p(t) \text{ for a.e} \quad t \in [0, t_1]). \tag{3.3.11}$$

From inequality (3.3.11), it follows that

$$\int_0^{t^*} \frac{\frac{d}{dt}|\langle \eta, X(s)\xi\rangle|}{\phi(\|X(s)\|_{\eta\xi})} \leq \int_0^{t^*} p(s)ds. \tag{3.3.12}$$

Using change of variable formula, we get

$$\Gamma_1(\|X(t^*)\|_{\eta\xi}) = \int_{||X_0||_{\eta,\xi}}^{||X(t^*||_{\eta,\xi}} \frac{du}{\phi(u)} \leq \int_0^{t^*} p(s)ds \leq \int_0^{t_1} p(s)ds. \tag{3.3.13}$$

Given that $\phi : \mathbb{R}_+ \to \mathbb{R}_+$ and $p \in L^1(I, \mathbb{R}_+)$ such that

$$|\mathbb{P}(t, X)(\eta, \xi)| \leq p(t)\phi(||X||_{\eta,\xi}),$$

we obtain that

$$\|X(t^*)\|_{\eta\xi} \leq \Gamma_1^{-1}\left(\int_0^{t_1} p(s)ds\right)$$

Hence,

$$\|X(t^*)\|_{\eta\xi} = sup\{[\|X(t)\|_{\eta\xi} : t \in [0, t_1]\} \leq \Gamma_1^{-1}\left(\int_0^{t_1} p(s)ds\right) := M_0.$$

Now $X|_{[t_1,t_2]}$ is a solution to

$$\frac{d}{dt}\langle \eta, X(t)\xi\rangle \in \mathbb{P}(t, X(t))(\eta,\xi) \qquad \text{almost all} \quad t \in [t_1,t_2]$$

$$\Delta X|_{t=t_1} = J_k(X(t_1)). \tag{3.3.14}$$

Then

$$\frac{d}{dt}|\langle \eta, X(t)\xi\rangle| \leq p(t)\phi(||X(t)||_{\eta,\xi}) \qquad \text{almost all} \quad t \in [t_1,t_2]. \tag{3.3.16}$$

Let $t^* \in [t_1,t_2]$ such that

$$\sup\{\|X(t)\|_{\eta\xi} : t \in [t_1,t_2]\} = \|X(t^*)\|_{\eta\xi}. \tag{3.3.17}$$

Then

$$\frac{\frac{d}{dt}|\langle \eta, X(t)\xi\rangle|}{\phi(\|X(t)\|_{\eta\xi}} \leq p(t). \tag{3.3.18}$$

From this inequality, it follows that

$$\int_{t_1}^{t^*} \frac{\frac{d}{dt}|\langle \eta, X(s)\xi\rangle|}{(|X(s)|} ds \leq \int_{t_1}^{t^*} p(s)ds. \tag{3.3.19}$$

Proceeding as above we obtain

$$\Gamma_2|X(t^*)| = \int_{N_1}^{|X(t^*)|} \frac{du}{\phi(u)} \leq \int_{t_1}^{t^*} p(s)ds \leq \int_{t_1}^{t_2} p(s)ds. \tag{3.3.20}$$

This yields

$$|X(t^*)| = \sup\{|X(t)| : t \in [t_1,t_2]\} \leq \Gamma_2^{-1}\left(\int_{t_1}^{t_2} p(s)ds\right) := M_1. \tag{3.3.21}$$

Continuing this process and taken into account that $X|_{[t_m,T]}$ is a solution to the problem,

$$\frac{d}{dt}\langle \eta, X(t)\xi\rangle \in \mathbb{P}(t, X(t))(\eta,\xi) \qquad \text{almost all} \quad t \in [t_m,T]$$

$$\Delta X|_{t=t_m} = J_k(X(t_m)), \tag{3.3.22}$$

then there exist a constant M_m such that

$$\sup\{\|X(t)\|_{\eta\xi} : t \in [t_m,T]\} \quad \leq \quad \Gamma_{m+1}^{-1}\left(\int_{t_m}^{T} p(s)ds\right) := M_m. \tag{3.3.23}$$

Consequently for each X to (2.2), we have

$$\|X\|_{\eta,\xi} \quad \leq \quad \max\{\|X_0\|_{\eta\xi}, M_{k-1} : k = 1, ..., m+1\}. \tag{3.3.24}$$

■

THEOREM 3.4 *Assume that the map $\mathbb{P} : (I \times \tilde{\mathcal{A}}) \to 2^{sesq(\mathbb{D}\underline{\otimes}\mathbb{E})^2}$ satisfies the following conditions :*
(i) for each pair $\eta, \xi \in \mathbb{D}\underline{\otimes}\mathbb{E}, \mathbb{P}(t, x)(\eta, \xi)$ is closed and convex in $\mathbb{C}$.
(ii) The map $(t, x) \to \mathbb{P}(t, x)(\eta, \xi)$ is lower semi continuous on $(I \times \tilde{\mathcal{A}})$,
then there exists a continuous map $f : (I \times \tilde{\mathcal{A}}) \to sesq(\mathbb{D} \otimes \mathbb{E})^2$
which is a selection of $\mathbb{P}(t, x)(\eta, \xi)$.

4. Main result

The following theorem furnish our main result.

THEOREM 4.1 *Suppose that the following hypothesis are satisfied*
(i) The map $\mathbb{P} : I \times \tilde{\mathcal{A}} \to 2^{sesq(\mathbb{D}\underline{\otimes}\mathbb{E})^2}$ is such that for each pair $\eta, \xi \in \mathbb{D}\underline{\otimes}\mathbb{E}, (t, x) \in I \times \tilde{\mathcal{A}}, \mathbb{P}(t, x)(\eta, \xi)$ is closed and convex in $\mathbb{C}$, the space of complex numbers.
(ii) The map $(t, x) \to \mathbb{P}(t, x)(\eta, \xi)$ is lower semi continuous and measurable on $(I \times \tilde{\mathcal{A}})$.
(iii) For every $r > 0$, there exists function $h_{\eta\xi,r} : I \to \mathbb{R}$ lying in $L^{'}(I, \mathbb{R}_+)$, such that $|\mathbb{P}(t, x)(\eta, \xi)| =$ $\sup\{|v_{\eta\xi}| : v_{\eta\xi} \in \mathbb{P}(t, x)(\eta, \xi)\} \leq h_{\eta\xi,r}$, for a.e $t \in I$ and $x \in \tilde{\mathcal{A}}$ with $||x||_{\eta\xi} \leq r$.
Then the impulsive problem (2.1) -(2.2) has a solution.

Proof. Let

$$f : P_{wac}(\tilde{\mathcal{A}}) \to L'_{loc}(\tilde{\mathcal{A}})$$

such that

$$f(x) \in \mathcal{F}(x) \quad \forall \quad y \in P_{wac}(\tilde{\mathcal{A}}).$$

Consider the single valued problem

$$\left.\begin{array}{l} \dfrac{d}{dt}\langle\eta, X(t)\xi\rangle = F(X(t))(\eta, \xi) \; t \in I, t \neq t_k, k = 1, 2, ...m \\[2ex] \Delta X|_{t=t_k} = J_k(X(t_k^-))\xi \qquad t = t_k, k = 1, 2, ..., m \\[2ex] X(0) = X_0. \end{array}\right\} \tag{4.1}$$

Let

$$N(X)(t)(\eta, \xi) = \|N(X)(t)\|_{\eta\xi} = |\langle\eta, N(X)(t)\xi\rangle| = \langle\eta, x_0\xi\rangle + \int_0^t |\langle\eta, (E(t, X(t))d\Lambda_\pi(t)+$$

$$F(t, X(t))dA_f(t) + G(t, X(t))dA_g^+(t) + H(t, X(t))dt)\xi\rangle| + \sum_{0<t_k<t} \langle\eta, J_k(X(t_k^-))\xi\rangle,$$

where

$$(E(t, X(t))d\Lambda_\pi(t) + F(t, X(t))dA_f(t) + G(t, X(t))dA_g^+(t) + H(t, X(t))dt \equiv \mathbb{P}(t, X(t))(\eta, \xi).$$

We now transform problem (4.1) into a fixed point problem by considering the operators

$$N_{\eta\xi}(X)(t) = x_0 + \int_0^t f(X(s))(\eta,\xi) + \sum_{0<t_k<t} \langle \eta, J_k(X(t_k^-))\xi\rangle. \tag{4.2}$$

We show that $N_{\eta\xi}$ is compact for each pair $\eta, \xi \in \mathbb{D}\underline{\otimes}\mathbb{E}$. That is

$$N(X)(t) = x_0 + \sum \int_t^{t_0} E(t, X(t))d\Lambda_\pi(t) + F(t, X(t))dA_f(t) +$$

$$G(t, X(t))dA_g^+(t) + H(t, X(t))dt + \sum_{0<t_k<t} J_k(X(t_k^-))$$

$$N : P_{wac}(\tilde{\mathcal{A}}) \to P_{wac}(\tilde{\mathcal{A}}).$$

Step 1 : N is continuous. Let $\{X_n\}$ be a sequence such that $X_n \to X \in P_{wac}(\tilde{\mathcal{A}})$.

$$\begin{aligned}\|N(X_n(t)) - N(X(t))\|_{\eta,\xi} &\leq \int_0^t |\mathbb{P}(s, X_n(s))(\eta,\xi) - \mathbb{P}(s, X(s))(\eta,\xi)|ds \\ &\quad + \sum_{0<t_k<t} ||J_k(X_n(t_k^-)) - J_k(X(t_k^-))||_{\eta,\xi} \\ &\leq \int_0^T |\mathbb{P}(s, X_n(s))(\eta,\xi) - \mathbb{P}(s, X(s))(\eta,\xi)|ds \\ &\quad + \textstyle\sum_{0<t_k<t} ||J_k(X_n(t_k^-)) - J_k(X(t_k^-))||_{\eta,\xi}. \end{aligned} \tag{4.3}$$

Since $\mathbb{P}$ and $J_K, k = 1, 2, ..., m$ are continuous, then

$$\begin{aligned}\|N(X_n) - N(X))||_{\eta,\xi} &\leq \|\mathbb{P}(t, X_n(s))(\eta,\xi) - \mathbb{P}(t, X(s))(\eta,\xi)\|_{\eta,\xi} \\ &\quad + \textstyle\sum_{0<t_k<t} |J_k(X_n(t_k^-)) - J_k(X(t_k^-))| \to 0 \end{aligned} \tag{4.5}$$

as $n \to \infty$ which implies that N is continuous.

Step 2 : N maps bounded set into bounded sets in $P_{wac}(\tilde{\mathcal{A}})$. Let $X \in B_q = [x \in P_{wac}(\tilde{\mathcal{A}}) : ||x||_{\eta,\xi} \leq q]$ for arbitrary $\eta, \xi \in \mathbb{D}\underline{\otimes}\mathbb{E}$ we have that

$$\|N(X)\|_{\eta,\xi} \leq q,$$

since $J_k, k = 1, ..., m$ are continuous from assumption (iii), we have

$$\|N(X(t))\|_{\eta,\xi} \leq \|X_0\|_{\eta\xi} + \int_0^t |\mathbb{P}(t, X(s))(\eta,\xi)|ds + \sum_{0<t_k<t} ||J_k(X(t_k^-))||_{\eta,\xi}$$

$$\leq \|X_0\|_{\eta\xi} + ||h_q||_{L'} + \textstyle\sum_{k=1}^m ||J_k(x(t_k^-)||_{\eta,\xi} := l. \tag{4.6}$$

Step 3 : N maps bounded set into equicontinuous sets of $P_{wac}(\tilde{\mathcal{A}})$. Let $r_1, r_2, \in I$ and let $B_q = [X \in P_{wac}(\tilde{\mathcal{A}}) : ||X||_{\eta,\xi} \leq q]$ be a bounded set of $P_{wac}(\tilde{\mathcal{A}})$. Then

$$\|N(X)(r_2) - N(X)(r_1)\|_{\eta,\xi} \leq \int_{r_1}^{r_2} |h_q(s))|ds + \sum_{0<t_k<r_2-r_1} ||J_k(x(t_k^-)||_{\eta,\xi}. \tag{4.7}$$

As $r_2 \to r_1$, the right hand side of the above inequality tends to zero. This established the equicontinuity of the case where $t \neq t_i, i = 1, 2, ..., m$. To examine equicontinuity at $t = t_i$ we have

$$\begin{aligned}
&\|N(X)(r_2) - N(X)(r_1)\|_{\eta,\xi} \leq |\mathbb{P}(r_2, X_n(s))(\eta, \xi) - \mathbb{P}(r_1, X(s))(\eta, \xi)|ds \\
&\sum_{0<t_k<t} \|J_k(X_n(t_k^-)) - J_k(X(t_k^-))\|_{\eta,\xi}\|X_0\|_{\eta,\xi} \\
&+ \int_{r_1}^{r_2} |\mathbb{P}(r_2 - s, X(s))(\eta, \xi) - \mathbb{P}(r_1 - s, X(s))(\eta, \xi)|ds \\
&+ \sum_{0<t_k<t} \|J_k(X_n(t_k^-)) - J_k(X(t_k^-))\|_{\eta,\xi}(B(X(s))|ds \\
&+ \int_{r_1}^{r_2} |\mathbb{P}(r_2 - s, X(s))(\eta, \xi)|(B(x(s))|ds \\
&+ \int_{0}^{r_1} |\mathbb{P}(r_2 - s, X(s))(\eta, \xi) - \mathbb{P}(r_1 - s, X(s))(\eta, \xi)|\phi_q(s)ds \\
&+ \int_{r_1}^{r_2} |\mathbb{P}(r_2 - s, X(s))(\eta, \xi)| \\
&+ \sum_{r_1<0<r_2} J_k|\mathbb{P}(r_2 - t_k, X(s)) - \mathbb{P}(r_1 - t_k, X(s))|. \qquad (4.8)
\end{aligned}$$

The right hand side of (4.8) tends to zero as $r_2 - r_1 \to 0$. To show equicontinuity at the left limit $t = t_k^-$ fix $\delta_1 > 0$ such that $[t_k : k \neq i] \cap [t_i - \delta_1, t_i + \delta_1] = \emptyset$. For $0 < h < \delta_1$, we have

$$\begin{aligned}
&|N(X)(t_i) - N(X)(t_i - h)| \leq |\mathbb{P}((t_i, X(s))(\eta, \xi) - \mathbb{P}((t_i - h, X(s))(\eta, \xi)|||X_0||_{\eta,\xi} \\
&+ \int_{0}^{t_i-h} |\mathbb{P}(t_i - s, X(s))(\eta, \xi) - \mathbb{P}(t_i - h - s, X(s))(\eta, \xi)|(B(X(s))|ds \\
&+ \int_{t_i-h}^{t_i} |\mathbb{P}(t_i - h, X(s))(\eta, \xi)|(B(x(s))|ds \\
&+ \int_{0}^{t_i-h} |\mathbb{P}(t_i - s, X(s)) - \mathbb{P}(t_i - h - s, X(s))|(B(X(s))|ds \\
&+ \int_{t_i-h}^{t_i} |\mathbb{P}(t - i - s, X(s))(\eta, \xi)|(B(X(s))|ds \\
&+ \int_{0}^{t_i-h} |[\mathbb{P}(t - i - h - s, X(s)) - \mathbb{P}(t_i - s, X(s))]\phi_q(s)|ds \\
&+ \int_{0}^{t_i-h} |[\mathbb{P}(t_i - h - s, X(s))]\phi_q(s)|ds \\
&+ \sum_{k=1}^{l-1} [\mathbb{P}(t_i - h - t_k, X(s)) - \mathbb{P}(t_i - t_k s, X(s))] J_k(X(t_k^-)).
\end{aligned}$$

To show equicontinuity at the right limit $t = t_k^+$, fix $\delta_2 > 0$ such that $[t_k : k \neq i] \cap [t_i - \delta_2, t_i + \delta_2] = \emptyset$. For $0 < h < \delta_2$, we have

$$\begin{aligned}
|N(x)(t_i + h) - N(x)(t_i)| &\leq |[\mathbb{P}(t_i + h, X(s)) - \mathbb{P}(t_i, X(s))||X_0||_{\eta,\xi} + \textstyle\int_0^{t_i} |\mathbb{P}(t_i + h - s, X(s)) - \\
&\mathbb{P}(t_i - s, X(s))|(B(X(s))|ds + \textstyle\int_{t_i}^{t_i+h} |\mathbb{P}(t_i - h, X(s))|(B(X(s))|ds + \textstyle\int_0^{t_i} |\mathbb{P}(t_i + h - s, X(s)) \\
&- \mathbb{P}(t_i - s, X(s))|\phi_q(s)ds| + \textstyle\int_{t_i}^{t_i+h} |\mathbb{P}(t_i - h, X(s))(\phi_q(s)|ds + \textstyle\sum_{0<t_k<t_i} [\mathbb{P}(t_i - h - t_k) - \\
&\mathbb{P}(t_i - t_k, X(s))] + \textstyle\sum_{t_i<t_k<t_{i+1}} |\mathbb{P}(t_i - h - t_k, X(s)) J_k(X(t_k^+))|. \qquad (4.9)
\end{aligned}$$

The right hand tends to zero as $h \to 0$. Set

$$U = [X \in P_{wac}(\tilde{\mathcal{A}}) : \|x\|_{P_{wac}} \leq \max[X_0, M_{k-1} : k = 1, ..., m+1].$$

As a consequence of steps 1,2 and 3, we can conclude that

$$N : \overline{U} \to P_{wac}(\tilde{\mathcal{A}})$$

is compact. From the choice of U there is no $y \in \delta U$ such that $x = \lambda N x$ for any $\lambda \in \mathbb{C}$ such that $Re\lambda \in (0,1)$ and $Im\lambda \in (0,1)$ As a result , we deduce that N has a fixed point $x \in U$ which is a solution to problem (2.1 - 1.2). ■

References

[1] Aubin, J.P. and Cellina, A. (1984). *Differential Inclusions.* Springer-Verlag, Berlin.

[2] Aubin, J.P. and Franskowska, H. (1990). *Set Valued Analysis.* Springer-Verlag, Berlin.

[3] Ayoola, E. O. (1998). On numerical procedures for solving Lipschitzian quantum stochastic differential equation. *Ph.D. Thesis*, University of Ibadan, Ibadan, Nigeria.

[4] Ayoola, E. O. (2000). Converging multistep methods for weak solution of quantum stochastic differential equations. *Stochastic Analysis Application*, 18, 525-554.

[5] Ayoola, E. O. (2002). Lagragian quadrature schemes for computing weak solution of quantum stochastic differential equations. *SIAN I Number ANAK*, 39(6), 1835-1864.

[6] Ayoola, E. O. (2003). Error estimates for discretized quantum stochastic differential inclusions. *Stochastic Analysis and Applications*, 21(6), 1215-1230.

[7] Ayoola, E. O. (2004). Continuous selections of solution sets of Lipschitzian quantum stochastic differential inclusions. *International Journal of Theoretical Physics*, 43(10), 2041-2059.

[8] Ayoola, E. O. and Adeyeye J. O. (2007). Continuous interpolation of solution sets of Lipschitzian quantum stochastic differential inclusions. *Journal of Applied Mathematics and Stochastic Analysis*, 1-13.

[9] Ayoola, E. O. (2008). Topological properties of solution sets of Lipschitzian quantum differential inclusions. *Acta Applicanche Mathematical*, 100, 15-37.

[10] Deimling, K. (1985). *Non-Linear Functional Analysis.* Springer-Verlag, Berlin.

[11] Deimling, K. (1992). *Multivalued Differential Equations.* Walter de Gruyter, Berlin.

[12] Ekhaguere, G. O. S. (1992). Lipschitzian quantum stochastic differential inclusions. *Internat. S. Preoret Phys.*, 31, 2003-2034.

[13] Ekhaguere, G. O. S. (1995). Quantum stochastic differential inclusions of hypermaximal monotone type. *Internt. J. Theoret. Phys*, 34, 322-353.

[14] Ekhaguere, G. O. S. (1996). Quantum stochastic evolutions. *Internat. J. Theoret. Phys.*, 35, 1909-1946.

[15] Ekhaguere, G. O. S. (2007). Topological solutions of non-commutative stochastic differential equations. *Stochastic analysis and Applications*, 25(9), 961-993.

[16] Lakshmikantham, V., Bainov, D. D. and Simeonov, P. S. (1989). Theory of Impulsive Dierential Equations, Series in Modern Applied Mathematics, vol. 6, World Scientic, New Jersy.

[17] Herderson, J. and Quahab, A. (2011). On the solution sets for first order impulsive neutral functional differential inclusions in Banach spaces. *Dynamics of Continuous, Discrete and Impulsive Systems. Series A: Mathematical Analysis*, 18, 53-114.

[18] Hudson, R. L. and Parthasarathy, K. R.. Construction of Quantum Diffusion, in Quantum Theory of Irreversible Processes, Lecture Notes in Math, 1055.

[19] Hudson, R. L. (1991). Quantum Stochastic Calculus and Some of Its Applications, in Contemporary Stochastic Analysis, World Scientific, River Edge, NJ., pp. 31-70.

[20] Hudson, R. L. and Parthasarathy (1984). Quantum Ito formulae and stochastic evolutions. *Comm. Math. & Phys.*, 93, 301-324.

[21] Ito, K. (1951). On stochastic differential equations. *Memoirs American Mathematical Society* 4.

[22] Micheal, E. (1956). Continuous selections I and II. *Annal of Mathematics*, 63/64, 361-382, 562-580.

[23] Ogundiran, M. O. (2006). Solutions of quantum stochastic differential inclusions. *Ph.D Thesis*, University of Ibadan.

[24] Samoilenko, A. M. and Perestyuk, N. A. (1995). Impulsive Differential Equations, World Scientic Series on Nonlinear Science. Series A: Monographs and Treatises, vol. 14, World Scientic, New Jersey.

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AN APPLICATION OF MICHAEL SELECTION THEOREMS

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Abstract

The existence of continuous selection for a lower semicontinuous multifunction reduces problems of existence of solutions in differential inclusion involving such multifunction into a corresponding differential equation. In this paper, an extension of Michael selection theorem to a non-commutative setting is established. The result is then applied in the establishment of the existence of solutions of quantum stochastic evolution inclusion. The quantum stochastic evolution inclusion has its coefficients to be operator-valued stochastic processes which are hypermaximal monotone and lower semicontinuous multifunctions.

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1. Introduction

This paper is concerned with the establishment of the existence of continuous selections for a non-commutative multivalued stochastic process in the problem of existence of solutions of differential inclusions. The problem concerned can be reduced to an ordinary differential equation such that its right hand side consists of a selection of the multifunction. The celebrated Michael selection theorem established the existence of a continuous selection. Each selection strategy depends on the topological property of the domain of definition of the multifunction and the regularity property of the multifunction itself [1, 9].

In [5], the existence of solutions of hypermaximal monotone quantum stochastic differential inclusions was established via the unique adapted limit of the Yosida approximations of the multifunction. The result was later generalized to non-linear quantum stochastic evolution inclusions

$$\frac{d}{dt}\langle \eta, X(t)\xi \rangle \in -\mathbb{P}(t, X(t))(\eta, \xi) + \langle \eta, p(t)\xi \rangle,$$

$$\langle \eta, X(0)\xi \rangle = \langle \eta, x_0 \xi \rangle \text{ for almost all } t \in [0, T]$$

in [6], where $\mathbb{P}$ is a hypermaximal monotone multifunction and p is continuous. In this work, we established the existence of a solution for a more general case

$$\frac{d}{dt}\langle \eta, X(t)\xi \rangle \in -\mathbb{P}(t, X(t))(\eta, \xi) + \Phi(t, X(t))(\eta, \xi),$$

$$\langle \eta, X(0)\xi \rangle = \langle \eta, x_0 \xi \rangle \text{ for almost all } t \in [0, T],$$

where Φ is a lower semicontinuous multivalued stochastic process. It is worthy of note that the stochastic differential inclusion is driven by operator-valued stochastic processes which are annihilation, creation and gauge processes arising from quantum field operator as in Hudson and Parthasarathy formulation of quantum stochastic calculus [8]. The existence of the continuous selection in this work is guaranteed by the pseudometric property of the domain which implies its paracompactness. The remaining sections are arranged as follows: in Section 2, notation and definitions are stated while Section 3 is meant for the main results.

2. Preliminaries

In this section, we shall introduce the notation and definitions on quantum stochastic differential inclusions as applicable in subsequent sections.

2.1. Notation and Definitions

Let $\mathbb{D}$ be some pre-Hilbert space whose completion is $\mathcal{R}$; γ be a fixed Hilbert and $L^2_\gamma(\mathbb{R}_+)$ be the space of square integrable γ-valued maps on $\mathbb{R}_+$.

The inner product of the Hilbert space $\mathcal{R} \otimes \Gamma(L^2_\gamma(\mathbb{R}_+))$ will be denoted by $\langle \cdot, \cdot \rangle$ and $\| \cdot \|$ the norm induced by $\langle \cdot, \cdot \rangle$.

Let $\mathbb{E}$ be a linear space generated by the exponential vectors in Fock space $\Gamma(L^2_\gamma(\mathbb{R}_+))$. We define the *locally convex space* $\mathcal{A}$ of non-commutative stochastic processes whose topology τ_w is generated by the family of seminorms $\{\| x \|_{\eta\xi} = | \langle \eta, x\xi \rangle |, \ x \in \mathcal{A}, \ \eta, \xi \in \mathbb{D} \underline{\otimes} \mathbb{E}\}$. The completion of $(\mathcal{A}, \tau_w)$ is denoted by $\tilde{\mathcal{A}}$. The underlying elements of $\tilde{\mathcal{A}}$ consist of linear maps from $\mathbb{D} \underline{\otimes} \mathbb{E}$ into $\mathcal{R} \otimes \Gamma(L^2_\gamma(\mathbb{R}_+))$ having domains of their adjoints containing $\mathbb{D} \underline{\otimes} \mathbb{E}$. For a fixed Hilbert space γ, the spaces $L^p_{loc}(\tilde{\mathcal{A}})$, $L^\infty_{\gamma, loc}(\mathbb{R}_+)$ and $L^p_{loc}(I \times \tilde{\mathcal{A}})$ are adopted as in [4].

For a topological space $\mathcal{N}$, let $clos(\mathcal{N})$ be the collection of all non-empty closed subsets of $\mathcal{N}$; we shall employ the Hausdorff topology on $clos(\tilde{\mathcal{A}})$ as defined in [4]. Moreover, for $A, B \in clos(\mathbb{C})$ and $x \in \mathbb{C}$, a complex number, we define the Hausdorff distance, $\rho(A, B)$ as

$$\mathbf{d}(x, B) \equiv \inf_{y \in B} | x - y |, \quad \delta(A, B) \equiv \sup_{x \in A} \mathbf{d}(x, B)$$

and

$$\rho(A, B) \equiv \max(\delta(A, B), \delta(B, A)).$$

Then ρ is a metric on $clos(\mathbb{C})$ and induces a metric topology on the space.

By a multivalued stochastic process indexed by $I = [0, T] \subseteq \mathbb{R}_+$, we mean a multifunction on I with values in $clos(\tilde{\mathcal{A}})$.

If Φ is a multivalued stochastic process indexed by $I \subseteq \mathbb{R}_+$, then a selection of Φ is a stochastic process $X : I \to \tilde{\mathcal{A}}$ with the property that $X(t) \in \Phi(t)$ for almost all $t \in I$.

A multivalued stochastic process Φ will be called

(i) *adapted* if $\Phi(t) \subseteq \tilde{\mathcal{A}}_t$ for each $t \in \mathbb{R}_+$; (ii) *measurable* if $t \mapsto d_{\eta\xi}(x, \Phi(t))$ is measurable for arbitrary $x \in \tilde{\mathcal{A}}$, $\eta, \xi \in \mathbb{D} \underline{\otimes} \mathbb{E}$; (iii) *locally absolutely p-integrable* if $t \mapsto \| \Phi(t) \|_{\eta\xi}$, $t \in \mathbb{R}_+$, lies in $L^p_{loc}(\tilde{\mathcal{A}})$ for arbitrary $\eta, \xi \in \mathbb{D} \underline{\otimes} \mathbb{E}$.

The set of all absolutely p-integrable multivalued stochastic processes will be denoted by $L^p_{loc}(\tilde{\mathcal{A}})_{mvs}$ and for $p \in (0, \infty)$, $L^p_{loc}(I \times \tilde{\mathcal{A}})_{mvs}$ is the set of maps $\Phi : I \times \tilde{\mathcal{A}} \to clos(\tilde{\mathcal{A}})$ such that $t \mapsto \Phi(t, X(t))$, $t \in I$ lies in $L^p_{loc}(\tilde{\mathcal{A}})_{mvs}$ for every $X \in L^p_{loc}(\tilde{\mathcal{A}})$.

Consider multivalued stochastic processes $E, F, G, H \in L^p_{loc}(I \times \tilde{\mathcal{A}})_{mvs}$ and $(0, x_0)$ be a fixed point in $[0, T] \times \tilde{\mathcal{A}}$. Then, a relation of the form

$$X(t) \in x_0 + \int_0^t (E(s, X(s))d\Lambda_\pi(s) + F(s, X(s))dA_f(s)$$

$$+ G(s, X(s))dA_g^+(s) + H(s, X(s))ds, \quad t \in [0, T])$$

will be called a *stochastic integral inclusion* with coefficients E, F, G and H.

The stochastic differential inclusion corresponding to the integral inclusion above is

$$dX(t) \in E(t,\ X(t))d\Lambda_{\pi}(t) + F(t,\ X(t))dA_{f}(t)$$

$$+ G(t,\ X(t))dA_{g}^{+}(t) + H(t,\ X(t))dt,$$

$$X(0) = x_0 \text{ for almost all } t \in [0,\ T]. \tag{2.1}$$

Let $\mathbb{P} : [0,\ T] \times \tilde{\mathcal{A}} \to 2^{sesq(\mathbb{D} \underline{\otimes} \mathbb{E})^2}$ be sesquilinear form valued stochastic process defined in [4] in terms of *E*, *F*, *G*, *H* by using the matrix elements in Hudson and Parthasarathy quantum stochastic calculus [8], it was established that problem (2.1) is equivalent to

$$\frac{d}{dt}\langle \eta,\ X(t)\xi \rangle \in \mathbb{P}(t,\ X(t))(\eta,\ \xi),$$

$$\langle \eta,\ X(0)\xi \rangle = \langle \eta,\ x_0\xi \rangle \text{ for almost all } t \in [0,\ T]. \tag{2.2}$$

In what follows, if *U* is a topological space, we denote by $clos(U)$, the collection of all non-empty closed subsets of *U*.

As explained in [4], the map $\mathbb{P}$ cannot in general be written in the form:

$$\mathbb{P}(t,\ x)(\eta,\ \xi) = \tilde{\mathbb{P}}(t,\ \langle \eta,\ x\xi \rangle)$$

for some complex-valued multifunction $\tilde{\mathbb{P}}$ defined on $I \times \mathbb{C}$ for $t \in I$, $x \in \tilde{\mathcal{A}},\ \ \eta,\ \xi \in \mathbb{D} \underline{\otimes} \mathbb{E}$.

2.2. Lower semicontinuous and hypermaximal monotone multifunctions

Let $S \subset \tilde{\mathcal{A}}$. Then a multivalued stochastic process $\Phi : S \to 2^{\tilde{\mathcal{A}}}$ is said to be *lower semicontinuous* (*l.s.c.*) if for every closed subset *C* of $\tilde{\mathcal{A}}$, the set $\{s \in S : \Phi(s) \subset C\}$ is closed in *S*.

For an arbitrary $\eta,\ \xi \in \mathbb{D} \underline{\otimes} \mathbb{E}$, a sesquilinear form valued map $\Psi : S \to 2^{sesq(\mathbb{D} \underline{\otimes} \mathbb{E})^2}$ will be said to be *lower semicontinuous* if for every closed subset *C* of $\mathbb{C}$ the set $\{s \in S : \Psi(s)(\eta,\ \xi) \subset C\}$ is closed in *S*.

The following proposition is an obvious implication of the definitions above.

Proposition 2.1. *Assume that the following holds*:

(i) *The coefficients E*, *F*, *G*, *H belong to the space* $L^2_{loc}(I \times \widetilde{\mathcal{A}})_{mvs}$.

(ii) *E*, *F*, *G*, *H are lower semicontinuous on* $I \times \widetilde{\mathcal{A}}$.

Then, *the map* $(t, x) \to \mathbb{P}(t, x)(\eta, \xi)$ *is lower semicontinuous on* $I \times \widetilde{\mathcal{A}}$.

Hence we shall be considering the sesquilinear form in the sequel.

Suppose $P : \widetilde{\mathcal{A}} \to 2^{sesq(\mathbb{D} \underline{\otimes} \mathbb{E})^2}$ is a multifunction; the domain of *P*; $D(P) = \{x \in \widetilde{\mathcal{A}} : P(x)(\eta, \xi) \neq \emptyset\}$; range of *P*; $range(P) = \bigcup_{x \in \widetilde{\mathcal{A}}} P(x)(\eta, \xi)$; graph of *P*; $graph(P) = \{(x, y) \in \widetilde{\mathcal{A}} \times \mathbb{C} : y \in P(x)(\eta, \xi)\}$.

We shall adopt the definition of hypermaximal monotone multifunction for regular multifunction $Reg(\widetilde{\mathcal{A}}_0)$ in [5].

(Monotone multifunctions)

A sesquilinear form valued map $\mathbb{P}$ is said to be

(i) *Monotone* if

$$\mathrm{Re}(\langle (a - b)(\eta \otimes \xi), \Phi_{\eta, \xi}(x, y)\rangle_{(2)}) \geq 0$$

and $a \in P_{\alpha, \beta}(x) \otimes 1$, $b \in P_{\alpha, \beta}(y) \otimes 1$, $x, y \in D(\mathbb{P})$, and $\eta, \xi \in \mathbb{D} \underline{\otimes} \mathbb{E}$, with $\eta = u \otimes e(\alpha)$, $\xi = v \otimes e(\beta)$, $\alpha, \beta \in L^\infty_{\gamma, loc}(\mathbb{R}_+)$, $u, v \in \mathbb{D}$.

(ii) *Maximal monotone* if the graph of $\mathbb{P}$ is not properly contained in the graph of any other monotone member of $Reg(\widetilde{\mathcal{A}})_0$.

(iii) *Hypermaximal monotone* if $\mathbb{P}$ is maximal monotone and

(a) the range of the map

$$x \mapsto id_{\widetilde{\mathcal{A}}}(x) \otimes 1 + P_{\alpha\beta}(x) \otimes 1, \quad x \in D(\mathbb{P}), \; \alpha, \beta \in L^\infty_{\gamma, loc}(\mathbb{R}_+)$$

is all of $\widetilde{\mathcal{A}} \otimes 1$ and

(b) $(id_{\widetilde{A}}(\cdot) + P_{\alpha\beta} \otimes 1)^{-1}$, $\alpha, \beta \in L^{\infty}_{\gamma, loc}(\mathbb{R}_+)$ is a continuous single-valued map from $\widetilde{\mathcal{A}} \otimes 1$ to $D(\mathbb{P})$. $id_{\widetilde{A}}(\cdot)$ is the identity map on $\widetilde{\mathcal{A}}$.

$\mathbb{P}$ generates a strongly continuous semigroup of contractions $\{S(t) : t \geq 0\}$ on $\overline{D(\mathbb{P})}$.

Then, for all $x, y \in \overline{D(\mathbb{P})}$ and $t, s \geq 0$, the following conditions are satisfied:

(i) $S(t+s)x = S(t)S(s)x$, (ii) $S(0)x = x$, (iii) $\tau \mapsto S(\tau)x$ is continuous,

(iv) $\| S(t)x - S(t)y \|_{\eta\xi} \leq \| x - y \|_{\eta\xi}$ for arbitrary $\eta, \xi \in \mathbb{D} \underline{\otimes} \mathbb{E}$. (2.3)

In [5], it was established that if E, F, G and H are hypermaximal monotone, then $\mathbb{P}$ is hypermaximal monotone. Let $s > 0$ be fixed. Consider the initial value stochastic differential inclusion

$$dX(t) \in -(E(t, X(t))d\Lambda_{\pi}(t) + F(t, X(t))dA_f(t)$$

$$+ G(t, X(t))dA_g^+(t) + H(t, X(t))dt) + p(t)dt$$

$$\text{for almost all } t \in (s, T],$$

$$X(s) = x_s \text{ for some } x_s \in \widetilde{\mathcal{A}} \tag{2.4}$$

which is equivalent to the differential inclusion

$$\frac{d}{dt}\langle \eta, X(t)\xi \rangle \in -\mathbb{P}(t, X(t))(\eta, \xi) + \langle \eta, p(t)\xi \rangle,$$

$$\langle \eta, X(s)\xi \rangle = \langle \eta, x_s\xi \rangle \text{ for almost all } t \in (s, T], \tag{2.5}$$

$\eta, \xi \in \mathbb{D} \underline{\otimes} \mathbb{E}$, with $\eta = u \otimes e(\alpha)$, $\xi = v \otimes e(\beta)$, $\alpha, \beta \in L^{\infty}_{\gamma, loc}(\mathbb{R}_+)$, $u, v \in \mathbb{D}$.

In Ekhaguere [5], the case corresponding to $p \equiv 0$ was considered and it was shown that problem (2.5) has a unique adapted solution. This same condition applies also to any non-zero $p \in C([s, T], \mathbb{D} \underline{\otimes} \mathbb{E})$ which was considered in Ekhaguere [6].

Suppose that problem (2.5) has a unique adapted solution φ. We may interpret (2.5) as describing a system whose state at time s is $\varphi(s) = x_s$, while $\varphi(t)$ is the state of the system at some later time $t \geq s$. We say that the system has evolved from the state $\varphi(s)$ to the state $\varphi(t)$, $t \geq s$. This transition may be described by means of transformation $U(t, s)$ which moves $\varphi(s)$ to $\varphi(t)$ thus:

$$U(t, s)\varphi(s) = \varphi(t), \quad t \geq s.$$

A map U from the set $\{(t, s) \in \mathbb{R}_+^2 : 0 \leq s \leq t \leq T\}$ to the set of all operators on $\widetilde{\mathcal{A}}$ is called an *evolution operator* if it satisfies

(i) $U(s, s)\varphi(s) = \varphi(s)$;

(ii) $U(t, r)U(r, s)\varphi(s) = U(t, s)\varphi(s)$ for $s \leq r \leq t$.

It was shown in Ekhaguere [6], that by uniqueness of the solution of problem (2.5), these conditions, called *evolution conditions*, are satisfied.

Moreover, the contraction condition

$$\| U(t, s)x_0 - U(t, s)y_0 \|_{\eta\xi} \leq \| x_0 - y_0 \|_{\eta\xi}, \quad \forall s \in (0, T],$$

$$x_0, y_0 \in \widetilde{\mathcal{A}}, \quad \forall s \leq t \leq T, \quad \eta, \xi \in \mathbb{D} \underline{\otimes} \mathbb{E}$$

was shown to be satisfied.

The family of multifunctions $\{\mathbb{P}(t, \cdot); t \in [0, T]\}$ is called the *generator* of the evolution operator, $U(\cdot, \cdot)$, $D(\mathbb{P}(t, \cdot)) = \{x \in \widetilde{A} : \mathbb{P}(t, x)(\eta, \xi) \neq \emptyset\}$.

Consider the initial value problem

$$\frac{d}{dt}\langle \eta,\ X(t)\xi\rangle \in -\mathbb{P}(t,\ X(t))(\eta,\ \xi) + \langle \eta,\ p(t)\xi\rangle,$$

$$\langle \eta,\ X(0)\xi\rangle = \langle \eta,\ x_0\xi\rangle \text{ for almost all } t \in [0,\ T]. \tag{2.6}$$

By a solution of (2.6), we mean a unique adapted solution φ such that

$$\frac{d}{dt}\langle \eta,\ \varphi(t)\xi\rangle \in -\mathbb{P}(t,\ \varphi(t))(\eta,\ \xi) + \langle \eta,\ p(t)\xi\rangle.$$

Given a multifunction $\Phi : [0,\ T] \times \overline{D(\mathbb{P}(t,\ \cdot))} \to 2^{sesq(\mathbb{D} \underline{\otimes} \mathbb{E})^2}$ and $x_0 \in \overline{D(\mathbb{P}(t,\ \cdot))}$, we consider the initial value problem

$$\frac{d}{dt}\langle \eta,\ X(t)\xi\rangle \in -\mathbb{P}(t,\ X(t))(\eta,\ \xi) + \Phi(t,\ X(t))(\eta,\ \xi),$$

$$\langle \eta,\ X(0)\xi\rangle = \langle \eta,\ x_0\xi\rangle \text{ for almost all } t \in [0,\ T]. \tag{2.7}$$

By a *solution* of the problem (2.7), we mean an adapted weakly absolutely continuous function $\varphi : [0,\ T] \to \overline{D(\mathbb{P}(t,\ \cdot))}$ with the property that there exists $p \in C([0,\ T],\ sesq(\mathbb{D} \underline{\otimes} \mathbb{E}))$ such that

$\langle \eta,\ p(t)\xi\rangle \in \Phi(t,\ \varphi(t))(\eta,\ \xi)$ for an arbitrary $\eta,\ \xi \in \mathbb{D} \underline{\otimes} \mathbb{E}$, a.e. $t \in [0,\ T]$

and φ is a solution of problem (2.6).

3. Main Results

Proposition 3.1. *Assume that the following holds*:

(i) *For arbitrary* $\eta, \xi \in \mathbb{D} \underline{\otimes} \mathbb{E}$, *the map* $(t,\ x) \to G(t,\ x)(\eta,\ \xi)$ *is lower semicontinuous with respect to a seminorm* $\|\cdot\|_{\eta\xi}$;

(ii) $g : I \times \tilde{\mathcal{A}} \to sesq(\mathbb{D} \underline{\otimes} \mathbb{E})$ *is continuous, and*

(iii) $\varepsilon : \tilde{\mathcal{A}} \to \mathbb{R}_+$ *is lower semicontinuous.*

Then the map $(t, x) \to \Phi(t, x)(\eta, \xi)$ *defined by*

$$\Phi(t, x)(\eta, \xi) = B_{\varepsilon(x)}(g(t, x)(\eta, \xi)) \bigcap G(t, x)(\eta, \xi)$$

is lower semicontinuous with respect to the seminorm $\|\cdot\|_{\eta\xi}$ *on its domain*.

Proof. Fix (t^*, x^*) in $\mathrm{Dom}\Phi$, $y^*_{\eta\xi} \in \Phi(t^*, x^*)(\eta, \xi)$ and $\omega > 0$. For some $\sigma > 0$, $|y^*_{\eta\xi} - g(t^*, x^*)(\eta, \xi)| = \varepsilon(x^*) - \sigma$.

There exists δ_1 such that to any $(t, x) \in I \times \tilde{\mathcal{A}}$ with $d_{\eta\xi}((t, x), (t^*, x^*)) < \delta_1$, we can associate $y(t, x)(\eta, \xi)$ in $G(t, x)(\eta, \xi)$ so that

$$|y_{\eta\xi, (t, x)} - y^*_{\eta\xi}| < \min\left\{\omega, \frac{\sigma}{3}\right\}$$

and δ_2 such that

$$d_{\eta\xi}((t, x), (t^*, x^*)) < \delta_2$$

implies

$$\varepsilon(x) > \varepsilon(x^*) - \frac{\sigma}{3}$$

and δ_3 such that

$$d_{\eta\xi}((t, x), (t^*, x^*)) < \delta_3$$

implies $|g(t^*, x^*)(\eta, \xi) - g(t, x)(\eta, \xi)| < \frac{\sigma}{3}$.

Then when $d_{\eta\xi}((t, x), (t^*, x^*)) < \min\{\delta_1, \delta_2, \delta_3\}$,

$$|y(t, x)(\eta, \xi) - g(t, x)(\eta, \xi)|$$

$$\leq |y(t, x)(\eta, \xi) - y^*_{\eta\xi}| + |y^*_{\eta\xi} - g(t^*, x^*)(\eta, \xi)|$$

$$+ |g(t^*, x^*)(\eta, \xi) - g(t, x)(\eta, \xi)|$$

$$< \frac{\sigma}{3} + \varepsilon(x^*) - \sigma + \frac{\sigma}{3}$$

$$= \varepsilon(x^*) - \frac{\sigma}{3} < \varepsilon(x),$$

that is, $y(t, x)(\eta, \xi) \in \Phi(t, x)(\eta, \xi)$ and

$$| y^*(t, x)(\eta, \xi) - y(t, x)(\eta, \xi) | < \omega.$$

Proposition 3.2. *For an arbitrary* $\eta, \xi \in \mathbb{D} \underline{\otimes} \mathbb{E}$, *suppose a map* $(t, x) \to \Phi(t, x)(\eta, \xi)$ *is convex and lower semicontinuous.*

Then, for every $\varepsilon > 0$, *there exists a* (*jointly*) *continuous map,*

$$\varphi : I \times \tilde{\mathcal{A}} \to sesq(\mathbb{D} \underline{\otimes} \mathbb{E})^2$$

such that for $(\tau, \zeta) \in I \times \tilde{\mathcal{A}}$, $d(\varphi(\tau, \zeta)(\eta, \xi), \Phi(\tau, \zeta)(\eta, \xi)) \leq \varepsilon$.

Proof. For every $(t, x) \in I \times \tilde{\mathcal{A}}$, let $y_{(t, x)} \in \Phi(t, x)(\eta, \xi)$ and let $\delta_{(t, x)} > 0$ be such that $(y_{(t, x)} + B_\varepsilon) \cap \Phi(t', x')(\eta, \xi) \neq \emptyset$ for $(t', x') \in B_{\eta\xi, \delta_{(t, x)}}(t, x)$.

Since $I \times \tilde{\mathcal{A}}$ is paracompact, there exists a locally finite refinement $\{\mathcal{U}_{(t, x)}\}_{(t, x)} \in I \times \tilde{\mathcal{A}}$ of $\{B_{\eta\xi, \delta_{(t, x)}}(t, x)\}_{(t, x)}$.

Let $\{\pi_{(t, x)}(\cdot, \cdot)(\eta, \xi)\}_{(t, x)}$ be a partition of unity subordinate to it.

The mapping $\varphi : I \times \tilde{\mathcal{A}} \to sesq(\mathbb{D} \underline{\otimes} \mathbb{E})$ given by

$$\varphi(\tau, \zeta)(\eta, \xi) = \sum \pi_{(t, x)}(\tau, \zeta) y_{(t, x)}(\eta, \xi)$$

is continuous since it is a locally finite sum of continuous functions.

Fix (τ, ζ). Whenever

$$| \pi_{(t, x)}(\tau, \zeta)(\eta, \xi) | > 0, \ (\tau, \zeta) \in \mathcal{U}_{(t, x)} \subset B_{\eta\xi, \delta_{(t, x)}}(t, x),$$

hence $y_{(t, x)} \in \Phi(\tau, \zeta)(\eta, \xi) + B_\varepsilon$.

Since this latter set is convex, any convex combination of such $y_{(t,x)}$'s in particular, $\varphi(\tau, \zeta)$ belongs to it.

$\varphi(\tau, \zeta)(\eta, \xi) \in \Phi(\tau, \zeta)(\eta, \xi) + B_\varepsilon$ implies for any $\eta, \xi \in \mathbb{D} \underline{\otimes} \mathbb{E}$, we have $d(\varphi(\tau, \zeta)(\eta, \xi), \Phi(\tau, \zeta)(\eta, \xi)) \leq \varepsilon$.

The following theorem is a non-commutative generalization of Michael selection theorems [9], Theorem 1.11.1 [1].

Theorem 3.1. *Suppose that* $\Psi : I \times \widetilde{\mathcal{A}} \to 2^{sesq(\mathbb{D} \underline{\otimes} \mathbb{E})^2}$ *is a multivalued stochastic process such that for arbitrary elements* $\eta, \xi \in \mathbb{D} \underline{\otimes} \mathbb{E}$,

(i) $(t, x) \to \Psi(t, x)(\eta, \xi)$ *is lower semicontinuous with respect to a seminorm* $\|\cdot\|_{\eta\xi}$;

(ii) Ψ *is closed and convex valued.*

Then there exists $\psi : I \times \widetilde{\mathcal{A}} \to sesq(\mathbb{D} \underline{\otimes} \mathbb{E})^2$, *a continuous selection from* Ψ.

Proof. The proof shall be in steps:

We claim that we can define a sequence $\{\psi_n\}$ of continuous mappings from $I \times \widetilde{\mathcal{A}}$ into $sesq(\mathbb{D} \underline{\otimes} \mathbb{E})$ with the following properties:

(i) for each $(\tau, \gamma) \in I \times \widetilde{\mathcal{A}}$, $\eta, \xi \in \mathbb{D} \underline{\otimes} \mathbb{E}$,

$$d(\psi_n(\tau, \gamma)(\eta, \xi), \Psi(\tau, \gamma)(\eta, \xi)) \leq \frac{1}{2^n}, \quad n = 1, 2, \ldots;$$

(ii) for each $(\tau, \gamma) \in I \times \widetilde{\mathcal{A}}$, $\eta, \xi \in \mathbb{D} \underline{\otimes} \mathbb{E}$,

$$|\psi_n(\tau, \gamma)(\eta, \xi) - \psi_{n-1}(\tau, \gamma)(\eta, \xi)| \leq \frac{1}{2^{n-2}}, \quad n = 2, \ldots.$$

For $n = 1$, it is enough to take in Proposition 3.1 above $\Phi = \Psi$ and $\varepsilon = \frac{1}{2}$.

Assume we have defined mappings ψ_n satisfying (i) up to $n = \nu$. We shall define $\psi_{\nu+1}$ satisfying (i) and (ii) as follows.

Consider the set

$$\Phi(\tau, \gamma)(\eta, \xi) = (\psi_\nu(\tau, \gamma)(\eta, \xi) + B_{\frac{1}{2^\nu}}) \bigcap \Psi(\tau, \gamma)(\eta, \xi).$$

By (i), it is not empty, and it is a convex set. The map $(\tau, \gamma) \to \Phi(\tau, \gamma)(\eta, \xi)$ is lower semicontinuous, by Propositions 3.1 and 3.2, there exists a continuous φ such that $d(\varphi(t, x)(\eta, \xi), \Phi(t, x)(\eta, \xi)) \leq \frac{1}{2^{\nu+1}}$.

Set

$$\psi_{\nu+1}(\tau, \gamma)(\eta, \xi) = \varphi(\tau, \gamma)(\eta, \xi).$$

Hence $d(\psi_{\nu+1}(\tau, \gamma)(\eta, \xi), \Psi(\tau, \gamma)(\eta, \xi)) \leq \frac{1}{2^{\nu+1}}$ proving (i). Also,

$$\psi_{\nu+1}(\tau, \gamma)(\eta, \xi) \in \Phi(\tau, \gamma)(\eta, \xi) + B_{\frac{1}{2^{\nu+1}}}$$

$$\subset \psi_\nu(\tau, \gamma)(\eta, \xi) + \left(\frac{1}{2^\nu} + \frac{1}{2^{\nu+1}}\right)B,$$

that is;

$$|\psi_{\nu+1}(\tau, \gamma)(\eta, \xi) - \psi_\nu(\tau, \gamma)(\eta, \xi)| \leq \frac{1}{2^{\nu-1}}$$

proving (ii).

Since the series $\sum \frac{1}{2^n}$ converges, $\{\psi_n(\cdot)(\eta, \xi)\}$ is a Cauchy sequence, uniformly converging to a continuous map $\psi(\cdot)(\eta, \xi)$.

Since the values of Ψ are closed, by (i) above, ψ is a selection of Ψ.

Corollary 3.1. *Suppose the following holds*:

(i) *The map* $\mathbb{P} : [0, T] \times \widetilde{\mathcal{A}} \to 2^{sesq(\mathbb{D} \underline{\otimes} \mathbb{E})^2}$ *is a hypermaximal monotone multifunction*.

(ii) $\Phi : [0, T] \times \overline{D(\mathbb{P}(t, \cdot))} \to 2^{sesq(\mathbb{D} \underline{\otimes} \mathbb{E})^2}$ *is a lower semicontinuous multifunction with closed non-empty convex values*.

Then there exists a solution for a non-linear evolution inclusion

$$\frac{d}{dt}\langle \eta, X(t)\xi \rangle \in -\mathbb{P}(t, X(t))(\eta, \xi) + \Phi(t, X(t))(\eta, \xi),$$

$$\langle \eta, X(0)\xi \rangle = \langle \eta, x_0\xi \rangle. \tag{3.1}$$

Proof. Since Φ satisfies the hypotheses of Theorem 3.1, there exists a continuous selection $p \in C([0, T], sesq(\mathbb{D} \underline{\otimes} \mathbb{E}))$ such that

$\langle \eta, p(t)\xi \rangle \in \Phi(t, \varphi(t))(\eta, \xi)$ for an arbitrary $\eta, \xi \in \mathbb{D} \underline{\otimes} \mathbb{E}$, a.e. $t \in [0, T]$.

By [6], there exists a solution for (2.6). Suppose φ is such solution. Then φ is a solution of (3.1).

Remark 3.1. The result is applicable in Quantum Physics. If a quantum system evolves from a state space to another over a period of time such that there is continuous creation, annihilation and gauge. The inclusions describing the stochastic evolutions of the quantum system can be solved via the result in this work.

References

[1] J. P. Aubin and A. Cellina, Differential Inclusions, Springer-Verlag, Berlin, 1984.

[2] E. O. Ayoola, Topological properties of solution sets of Lipschitzian quantum stochastic differential inclusions, Acta Appl. Math. 100 (2008), 15-37.

[3] A. Cernea and V. Staicu, Directionally continuous selections and nonconvex evolution inclusions, Set-Valued Anal. 11 (2003), 9-20.

[4] G. O. S. Ekhaguere, Lipschitzian quantum stochastic differential inclusions, Inter. J. Theo. Phys. 31(11) (1992), 2003-2034.

[5] G. O. S. Ekhaguere, Quantum stochastic differential inclusions of hypermaximal monotone type, Inter. J. Theo. Phys. 34(3) (1995), 323-353.

[6] G. O. S. Ekhaguere, Quantum stochastic evolutions, Inter. J. Theo. Phys. 35(9) (1996), 1909-1946.

[7] G. O. S. Ekhaguere, Topological solutions of non commutative stochastic differential equations, Stoch. Anal. Appl. 25(9) (2007), 961-993.

[8] R. L. Hudson and K. R. Parthasarathy, Quantum Ito's formula and stochastic evolutions, Comm. Math. Phys. 93(3) (1984), 301-323.

[9] E. Michael, Continuous selections, I and II, Ann. Math. 63; 64 (1956), 361-382; 562-580.

[10] M. Reed and B. Simon, Methods of Modern Mathematical Physics I. Functional Analysis, 2nd ed., Academic Press, New York, London, 1980.

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Arcwise Connectedness of Solution Sets of Lipschitzian Quantum Stochastic Differential Inclusions

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Abstract. In the framework of the Hudson - Parthasarathy quantum stochastic calculus, we employ some recent selection results to prove that the function space of the matrix elements of solutions to quantum stochastic differential inclusion (QSDI) is arcwise connected both locally and globally.

1. Introduction

This paper considers Quantum Stochastic Differential Inclusion(QSDI) in the framework of the Hudson and Parthasarathy formulation [11] of Quantum Stochastic Calculus. It has found applications in the study of quantum stochastic control theory [13] and often occurs as regularization of quantum stochastic differential equations with discontinuous coefficients.

In [3,4,8,9] some topological properties of solution sets of QSDI have been achieved. These were subject to some conditions on the coefficients of their inclusions.

There are some of the interesting motivations [1,2,13,14,15] for studying connectedness, path connectedness and arcwise connectedness of solution sets in the classical differential inclusions with their applications.This provides the possibility of moving from one solution to another. However as established in [1, 14, 15] for the case of differential inclusions on finite dimensional Euclidian spaces, this work concerns the establishment of arcwise connectedness of solution sets of quantum stochastic differential inclusion in the integral form:

$$\begin{aligned} X(t) \in\ & a + \int_0^t \Big(E(s,X(s))d\wedge_\pi(s) + F(s,X(s))dA_f(s) \\ & + G(s,X(s))dA_g^+(s) + H(s,X(s))ds\Big), \quad \text{almost all} \quad t \in [0,T]. \quad (1.1) \end{aligned}$$

In equation (1.1), the coefficients $\{E, F, G, H\}$ lie in a certain class of stochastic processes for which quantum stochastic integrals against the guage, creation and annihilation processes

$\bigwedge_\pi, A_f^+, A_g$ and the Lebesgue measure are defined. Equation(1.1) involves unbounded linear operators on a Hilbert space and it is a noncommutative generalization of the classical stochastic integral equations of the form

$$X(t,w) \;=\; x_o + \int H(t,X)dt + \int F(t,X)dQ(t), \quad X(t_0) = x_0, \;\; t \in [t_0, T] \quad (1.2)$$

where the driving process $Q(t)$ is a martingale and H, F are sufficiently smooth ordinary functions.

We shall employ the various spaces of quantum stochastic processes introduced in [3, 4, 8].The remaining part of the work shall be arranged as follows; In section 2, some notations and fundamental structures shall be stated which shall be employed in the sequel. In section 3 some results and assumptions shall be stated and in section 4 the main result of this paper shall be established.

2. Notations and Fundamental Structures

In what follows, if N is a topological space, we denote by clos(N), the collection of all non-empty closed subsets of N. To each pair (D, H) which consists of a pre - Hilbert space D with completion H , we associate $\mathcal{L}_w^+(D, H)$ the set of all linear maps x from a pre-Hilbert space D to its completion H. With the property that the domain of the operator adjoint x^* of x contains D.

The members of $\mathcal{L}_w^+(D, H)$ are densely-defined linear operators on H which do not necessarily leave D invariant and $\mathcal{L}_w^+(D, H)$ is a linear space when equipped with the usual notions of addition and scalar multiplication.

To H also corresponds a Hilbert space $\Gamma(H)$, called the Boson Fock space determined by H. A natural dense subset of $\Gamma(H)$ consists of the linear space generated by the set of exponential vectors in $\Gamma(H)$ of the form

$$e(f) = \bigoplus_{n=0}^{\infty} (n!)^{-\frac{1}{2}} \bigotimes^{n} f, \quad f \in H$$

where $\otimes^0 f = 1$ and $\otimes^n f$ is the n-fold tensor product of f with itself for $n \geq 1$.

In what follows, $I\!\!D$ is some pre-Hilbert space whose completion is $\mathcal{R}$ and γ is a fixed Hilbert space. We shall write $L^2_\gamma(I\!\!R_+)$ (resp. $L^2_\gamma([0,t]))$' resp. $L^2_\gamma([t,\infty)), t \in I\!\!R_+ \equiv [0,\infty)$ for the Hilbert space of square integrable, γ-valued maps on $I\!\!R_+ = [0,\infty)$ (resp. on $[0,t)$; resp. on $[t,\infty); t \in I\!\!R_+$)

The noncommutative stochastic processes discussed in the sequel are densely-defined linear operators on $\mathcal{R} \otimes \Gamma(L^2_\gamma(I\!\!R_+))$; the inner product of this Hilbert space will be denoted by $\langle \cdot, \cdot \rangle$. For each $t > 0$, the direct sum decomposition

$$L^2_\gamma(I\!\!R_+) = L^2_\gamma([0,t)) \oplus L^2_\gamma([t,\infty))$$

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induces a factorization

$$\Gamma(L^2_\gamma(I\!\!R_+)) = \Gamma(L^2_\gamma([0,t))) \otimes \Gamma(L^2_\gamma([t,\infty)))$$

of Fock space.

Let $I\!\!E$, $I\!\!E_t$ and $I\!\!E^t$, $t > 0$ be the linear spaces generated by the exponential vectors in $\Gamma(L^2_\gamma(I\!\!R_+))$, $\Gamma(L^2_\gamma([0,t)))$ and $\Gamma(L^2_\gamma([t,\infty)))$, $t > 0$ respectively.

Then we define

$$\begin{aligned} \mathcal{A} &\equiv L^+_w(I\!\!D\underline{\otimes}I\!\!E,\ \mathcal{R}\otimes\Gamma(L^2_\gamma(I\!\!R_+))) \\ \mathcal{A}_t &\equiv L^+_w(I\!\!D\underline{\otimes}I\!\!E_t,\ \mathcal{R}\otimes\Gamma(L^2_\gamma([0,t)))) \otimes 1^t \\ \mathcal{A}_t &\equiv 1_t \otimes L^+_w(I\!\!E^t, \Gamma(L^2_\gamma([t,\infty)))),\quad t > 0 \end{aligned}$$

where $\underline{\otimes}$ denotes algebraic tensor product and 1_t (resp. 1^t) denotes the identify map on $\mathcal{R}\otimes\Gamma(L^2_\gamma([0,t)))$ (resp. $\Gamma(L^2_\gamma([t,\infty)))$ $t > 0$. We note that the spaces $\mathcal{A}_t$ and $\mathcal{A}^t$, $t > 0$, may be naturally identified with subspaces of $\mathcal{A}$.

For $\eta,\xi \in I\!\!D\underline{\otimes}I\!\!E$, define $\|\cdot\|_{\eta\xi}$ by

$$\|x\|_{\eta\xi} = |\langle\eta, x\xi\rangle|,\quad x \in \mathcal{A}$$

Then, $\{\|\cdot\|_{\eta\xi},\ \eta,\xi \in I\!\!D\underline{\otimes}I\!\!E\}$ is a family of locally convex seminorms on $\mathcal{A}$; we write τ_w for the locally convex topology on $\mathcal{A}$ determined by this family.

In the foregoing $\bar{\mathcal{A}}, \bar{\mathcal{A}}_t$ and $\bar{\mathcal{A}}^t$ denote the completions of the locally convex spaces $(\mathcal{A},\tau_w)$, $(\mathcal{A}_t,\tau_w)$, $\mathcal{A}^t,\tau_w)$, $t > 0$ respectively we then note that $\{\tilde{\mathcal{A}}_t,\ t \in I\!\!R_+\}$ is a filtration of $\tilde{\mathcal{A}}$.

Hausdorff topology: If A is a topological space, then $Clos(A)$ [resp. $Comp(A)$] denotes the collection of all nonvoid closed (resp. Compact) Subsets of A. We shall employ the Hausdorff topology on $Clos(\bar{A})$ which is defined as follows.

For $x \in \tilde{\mathcal{A}}$, $\mathcal{M},\mathcal{N} \in Clos(\tilde{\mathcal{A}})$, and $\eta,\xi \in I\!\!D \otimes I\!\!E$, set

$$d_{\eta\xi}(x,\mathcal{N}) = \inf_{y\in\mathcal{N}} \|x-y\|_{\eta\xi}$$

$$\delta_{\eta\xi}(\mathcal{M},\mathcal{N}) = \sup_{x\in\mathcal{M}} d_{\eta\xi}(x,\mathcal{N})$$

and

$$\rho_{\eta\xi}(\mathcal{M},\mathcal{N}) = \max(\delta_{\eta\xi}(\mathcal{M},\mathcal{N}), \delta_{\eta\xi}(\mathcal{N},\mathcal{M}))$$

Then $\{\rho_{\eta\xi}(\cdot) : \eta,\xi \in I\!\!D\underline{\otimes}I\!\!E\}$ is a family of pseudometrics which determines a Hausdorff topology on $Clos(\bar{A})$ denoted in the sequel by τ_H. If $\mathcal{M} \in Clos(\bar{A})$, then $\|\mathcal{M}\|_{\eta\xi}$ is defined by

$$\|\mathcal{M}\|_{\eta\xi} \equiv \rho_{\eta\xi}(\mathcal{M},\{0\})$$

for arbitrary $\eta,\xi \in I\!\!D\underline{\otimes}I\!\!E$.

For $A, B \in clos(\mathbb{C})$ and $x \in \mathbb{C}$, a complex number, define

$$d(x, B) \equiv \inf_{y \in B} |x - y|$$

,

$$\delta(A, B) \equiv \sup_{x \in A} d(x, B)$$

and

$$\rho(A, B) \equiv max(\delta(A, B), \delta(B, A)).$$

Then ρ is a metric on $clos(\mathbb{C})$ and induces a metric topology on the space.

Let $I \subseteq \mathbb{R}_+$. A stochastic process indexed by I is an $\tilde{\mathcal{A}}$-valued map on I. A stochastic process X is called adapted if $X(t) \in \tilde{\mathcal{A}}_t$ for each $t \in I$. We write $Ad(\tilde{\mathcal{A}})$ for the set of all adapted stochastic processes indexed by I.

Definition: A member X of $Ad(\tilde{\mathcal{A}})$ is called

(i) Weakly absolutely continuous if the map $t \longmapsto \langle \eta, X(t)\xi \rangle, t \in I$, is absolutely continuous for arbitrary $\eta, \xi \in \mathbb{D}\underline{\otimes}\mathbb{E}$.

(ii) Locally absolutely p-integrable if $\|X(\cdot)\|^p_{\eta\xi}$ is Lebesgue-measurable and integrable on $[0, t) \subseteq I$ for each $t \in I$ and arbitrary $\eta, \xi \in \mathbb{D}\underline{\otimes}\mathbb{E}$.

Notation.

We write $Ad(\tilde{\mathcal{A}})_{wac}$ [resp. $L^p_{loc}(\tilde{\mathcal{A}})$] for the set of all weakly absolutely continuous (resp. locally absolutely p-integrable) members of $Ad(\tilde{\mathcal{A}})$.

Stochastic Integrators

Let $L^\infty_{\gamma,loc}(\mathbb{R}_+)$ [resp. $L^\infty_{B(Y),loc}(\mathbb{R}_+)$] be the linear space of all measurable, locally bounded functions from $\mathbb{R}_+$ to γ [resp. to $B(\gamma)$, the Banach space of Bounded endomorphisms of γ]. If $f \in L^\infty_{\gamma,loc}(\mathbb{R}_+)$ and $\pi \in L^\infty_{B(\gamma),loc}(\mathbb{R}_+)$, then πf is the member of $L^\infty_{\gamma,loc}(\mathbb{R}_+)$ given by $(\pi f)(t) = \pi(t)f(t), t \in \mathbb{R}_+$.

For $f \in L^2_\gamma(\mathbb{R}_+)$ and $\pi \in L^\infty_{B(\gamma),loc}(\mathbb{R}_+)$, define the operators $a(f)$, $a^+(f)$ and $\lambda(\pi)$ in $L^+_w(\mathbb{D}, \Gamma(L^2_\gamma(\mathbb{R}_+)))$ as follows;

$$\begin{aligned} a(f)e(g) &= \langle f, g \rangle_{L^2_\gamma(\mathbb{R}_+)} e(g) \\ a^+(f)e(g) &= \frac{d}{d\sigma} e(g + \sigma f)\Big|_{\sigma=0} \\ \lambda(\pi)e(g) &= \frac{d}{d\sigma} e(e^{\sigma\pi} f)\Big|_{\sigma=0} \end{aligned}$$

for $g \in L^2_\gamma(\mathbb{R}_+)$.

These are the annihilation, creation and gauge operators of quantum field theory.

For arbitrary $f \in L^{\infty}_{\gamma,loc}(I\!R_+)$ and $\pi \in L^{\infty}_{B(\gamma),loc}(I\!R_+)$, they give rise to the operator-valued maps A_f, A_f^+, and A_π defined by

$$\begin{aligned} A_f(t) &\equiv a(f\chi_{[0,t)}) \\ A_f^+(t) &\equiv a^+(f\chi_{[0,t)}) \\ \wedge_\pi(t) &\equiv \lambda(\pi\chi_{[0,t)}) \end{aligned}$$

$t \in I\!R_+$, where χ_I denotes the indicator function of the Borel set $I \subseteq I\!R_+$.

The maps A_f, A_f^+ and $\wedge_\pi$ are stochastic processes, called the annihilation, creation and gauge processes, respectively, when their values are identified with their ampliations on $\mathcal{R}\otimes\Gamma(L^2_\gamma(I\!R_+))$. These are the stochastic integrators in the Hudson and Parthasarathy [11]formulation of Boson quantum stochastic integration, which we shall adopt in the sequel.

Accordingly, if $p, q, u, v \in L^2_{loc}(\tilde{\mathcal{A}})$, $f, g \in L^{\infty}_{\gamma,loc}(I\!R_+)$ and $\pi \in L^{\infty}_{B(\gamma),loc}(I\!R_+)$ then we interpret the integral.

$$\int_{t_0}^{t} p(s)d\wedge_\pi(s) + q(s)dA_f(s) + u(s)dA_g^+(s) + v(s)ds;\ t_0, t \in I\!R_+$$

as it is in the Hudson and Parthasarathy [11] formulation.

Stochastic Differential Inclusions

Definition:

(i) By a multivalued stochastic process indexed by $I \subseteq I\!R_+$, we mean a multifunction on I with values in $Clos(\tilde{\mathcal{A}})$.

(ii) If Φ is a multivalued stochastic process indexed by $I \subseteq I\!R_+$, then a selection of Φ is a stochastic process $X : I \to \tilde{\mathcal{A}}$ with the property that $X(t) \in \Phi(t)$ for almost all $t \in I$.

(iii) A multivalued stochastic process Φ will be called

 (a) adapted if $\Phi(t) \subseteq \tilde{\mathcal{A}}$, for each $t \in I\!R^+$;

 (b) measurable if $t \longmapsto d_{\eta\xi}(x, \Phi(t))$ is measurable for arbitrary $x \in \tilde{\mathcal{A}}$, $\eta, \xi \in I\!D\underline{\otimes}I\!E$

(c) locally absolutely p-integrable if $t \mapsto \|\phi(t)\|_{\eta\xi}$, $t \in I\!R_+$, lies in $L^p_{loc}(I)$ for arbitrary $\eta, \xi \in I\!D\underline{\otimes}I\!E$.

We note that

(1) the set of all locally absolutely p-integrable multivalued stochastic processes will be denoted by $L^p_{loc}(\tilde{\mathcal{A}})_{mvs}$

(2) For $p \in (0, \infty)$ and $I \subseteq I\!R_+$, $L^p_{loc}(I \times \tilde{\mathcal{A}})_{mvs}$ is the set of maps

$$\Phi : I \times \tilde{\mathcal{A}} \longrightarrow Clos(\tilde{\mathcal{A}})$$

such that

$$t \longmapsto \Phi(t, X(t)),$$

$t \in I$, lies in $L^p_{loc}(\tilde{\mathcal{A}})_{mvs}$ for every $X \in L^p_{loc}(\tilde{\mathcal{A}})$,

(3) If $\Phi \in L^p_{loc}(I \times \tilde{\mathcal{A}})_{mvs}$, then
$L_p(\Phi) \equiv \{\varphi \in L^p_{loc}(\tilde{\mathcal{A}}) : \varphi \text{ is a selection of } \Phi\}$.

(4) For $f, g \in L^\infty_{\gamma,loc}(I\!R_+)$, $\pi \in L^\infty_{B(Y),loc}(I\!R_+)$, 1 is the identity map on $\mathcal{R} \otimes \Gamma(L^2_\gamma(I\!R_+))$, and M is any of the stochastic processes A_f, A_g^+, A_π and $s \longmapsto s1$, $s \in I\!R_+$.

Thus, we introduce stochastic integral (resp. differential) expressions as follows.
If $\Phi \in L^2_{loc}(I \times \tilde{\mathcal{A}})_{mvs}$ and $(t, X) \in I \times L^2_{loc}(\tilde{\mathcal{A}})$, then we make the definition

$$\int_{t_0}^t \Phi(s, X(s))dM(s) \equiv \left\{\int_{t_0}^t \varphi(s)dM(s) : \varphi \in L_2(\Phi)\right\}$$

This leads to the following notion.

Definition: Let $E, F.G, H \in L^2_{loc}(I \times \tilde{\mathcal{A}})_{mvs}$ and (t_0, x_0) be a fixed point of $I \times \tilde{\mathcal{A}}$, then a relation of the form

$$X(t) \in x_0 + \int_{t_0}^t (E(s, X(s))d\wedge_\pi + F(s, X(s))dA_f(s) \\ + G(s, X(s))dA_g^+(s) + H(s, X(s))ds); t \in I$$

be called a stochastic integral inclusion with coefficient E, F, G and H initial data (t_0, x_0). We shall sometimes write the foregoing inclusion as follows;

$$dX(t) \in E(t, X(t))d\wedge_\pi (t) + F(t, X(t))dA_f(t) \\ + G(t, X(t))dA_g^+(t) + H(t, X(t))dt \tag{2.1}$$

for almost all $t \in I$, $X(t_0) = x_0$.
This we refer to as stochastic differential inclusions with coefficients E, F, G and H and initial data (t_0, X_0).

Definition: By a solution of (2.1) we mean a weakly absolutely continuous stochastic process $\varphi \in L^2_{loc}(\tilde{\mathcal{A}})$ such that

$$d\varphi(t) \in E(t, \varphi(t))d\wedge_\pi (t) + F(t, \varphi(t))dA_f(t) \\ + G(t, \varphi(t))dA_g^+(t) + H(t, \varphi(t))dt$$

almost all $t \in I$, $\varphi(t_0) = x_0$.
Remarks

(i) The existence of solution to a stochastic differential inclusion with Lipschitzian coefficients has been proved in [8].

(ii) If $\mathcal{M}$ is a subset of $\tilde{\mathcal{A}}$, we write $co\mathcal{M}$ for the closed convex hull of $\mathcal{M}$ and if $\Phi : I \times \tilde{\mathcal{A}} \longrightarrow Clos(\tilde{\mathcal{A}})$, we define $co\ \Phi : I \times \tilde{\mathcal{A}} \longrightarrow Clos(\tilde{\mathcal{A}})$ by

$$(co\ \Phi)(t,x) = co\ \Phi(t,x), \quad t,x) \in I \times \tilde{\mathcal{A}}$$

(iii) Related to (2.1) is the following stochastic differential inclusion:

$$dX(t) \in co\ E(t,X(t))dA_\pi(t) + co\ F(t,X(t))dA_f(t)$$

$$+co\ G(t,X(t))dA_g^+(t) + co\ H(t,X(t))dt$$

almost all $t \in I$

$$X(t_0) = X_0 \tag{2.2}$$

(iv) In [8], Ekhaguere established equivalent form of (1.1) and (2.1) as follows; $E, F, G, H \in L^2_{loc}(I \times \tilde{\mathcal{A}})_{mus}$ and (t_0, x_0) is some fixed point of $I \times \tilde{\mathcal{A}}$. Taking theorems 4.1 and Theorem 4.4 of Hudson and Parthasarathy which describes the matrix elements of the quantum stochastic integral.
For $\eta, \xi \in I\!\!D \otimes I\!\!E$, with $\eta = c \otimes e(\alpha)$ and $\xi = d \otimes c(\beta)$, define $\mu_{\alpha\beta}$, V_β, $\sigma_\alpha : I \to \mathbb{C}$, $I \subset I\!\!R_+$, by

$$\begin{aligned} \mu_{\alpha\beta}(t) &= \langle \alpha(t), \pi(t)\beta(t)\rangle_\gamma \\ V_\beta(t) &= \langle f(t), \beta(t)\rangle_\gamma \\ \sigma_\alpha(t) &= \langle \alpha(t), g(t)\rangle_\gamma \\ t \in I. \end{aligned}$$

To these functions, are associated the maps $\mu I\!\!E, vF, \sigma G, P$ and coP from $I \times \tilde{\mathcal{A}}$ into the set of multivalued sesquilinear forms on $I\!\!D \underline{\otimes} I\!\!E$ defined by

$$\begin{aligned} (\mu E)(t,x)(\eta,\xi) &= \{\langle \eta, \mu_{\alpha\beta}(t)p(t,x)\xi\rangle : p(t,x) \in E(t,x)\} \\ (vF)(t,x)(\eta,\xi) &= \{\langle \eta, v_\beta(t)q(t,x)\xi\rangle : q(t,x) \in F(t,x)\} \\ (\sigma G)(t,x)(\eta,\xi) &= \{\langle \eta, \sigma_\alpha(t)u(t,x)\xi\rangle : u(t,x) \in G(t,x)\} \\ P(t,x)(\eta,\xi) &= (\mu E)(t,x)(\eta,\xi) + (vF)(t,x)(\eta,\xi) \\ &\quad +(\sigma G)(t,x)(\eta,\xi) + H(t,x)(\eta,\xi) \\ (coP)(t,x)(\eta,\xi) &= \text{closed convex/hull of } P(t,x)(\eta,\xi) \end{aligned}$$

$$\eta, \xi \in I\!\!D \underline{\otimes} I\!\!E, \ (t,x) \in I \times \tilde{\mathcal{A}}$$

where

$$H(t,x)(\eta,\xi) = \{v(t,x)(\eta,\xi) : v(\cdot, X(\cdot))$$

is a selection of $H(\cdot, X(\cdot))\ \forall\, X \in L^2_{loc}(\tilde{\mathcal{A}})$
$\eta,\xi \in I\!\!D\underline{\otimes} I\!\!E,\ (t,x) \in I \times \tilde{\mathcal{A}}$.

As in [3, 4, 5, 6] we shall consider the equivalent form of (1.1) given by

$$\begin{aligned} \frac{d}{dt}\langle \eta, X(t)\xi\rangle &\in P(t, X(t))(\eta,\xi) \\ X(0) &= a, \quad t \in [0,T]. \end{aligned} \tag{2.3}$$

Inclusion (2.3) is a nonclassical ordinary differential inclusion and the map $(\eta,\xi) \to P(t,x)(\eta,\xi)$ is a multivalued sesquilinear form on $(I\!\!D\underline{\otimes} I\!\!E)^2$ for $(t,x) \in [0,T] \times \tilde{\mathcal{A}}$. We refer the reader to [8, 9, 10] for the explicit forms of the map and the existence results for solutions of QSDI (1.1) of Lipschitz, hypermaximal monotone and of evolution types.

3. Preliminary Results and Assumptions

As in [3, 4, 8], we let clos($\mathcal{N}$) denote the family of all nonempty closed subsets of a topological space $\mathcal{N}$. For $\mathcal{N} \in \{\tilde{\mathcal{A}},\ I\!\!\!C\}$, we adopt the Hausdorff topology on clos($\mathcal{N}$) as explained in the references above. We denote by $d(x, A)$, the distance from a point $x \in I\!\!\!C$ to a set $A \subseteq I\!\!\!C$. For $A, B \in$ clos($I\!\!\!C$), $\rho(A, B)$ denote the Hausdorff distance between the sets.

As in the references above,we shall employ the space $\mathbf{wac}(\tilde{\mathcal{A}})$ which is the completion of the locally convex topological space $(Ad(\tilde{\mathcal{A}})_{wac}, \tau)$ of adapted weakly absolutely continuous stochastic processes $\Phi : [0,T] \to \tilde{\mathcal{A}}$ whose topology τ is generated by the family of seminorms given by :

$$|\Phi|_{\eta\xi} := \|\Phi(0)\|_{\eta\xi} + \int_0^T |\frac{d}{dt}\langle \eta, \Phi(t)\xi\rangle| dt, \ \text{ for } \ \eta,\ \xi \in I\!\!D\underline{\otimes} I\!\!E. \tag{3.1}$$

Associated with the space $\mathbf{wac}(\tilde{\mathcal{A}})$, we shall employ the space $\mathbf{wac}(\tilde{\mathcal{A}})(\eta,\xi)$ consisting of absolutely continuous complex valued functions $\langle \eta, \Phi(\cdot)\xi\rangle$, where $\Phi \in \mathbf{wac}(\tilde{\mathcal{A}})$ for arbitrary pair of points $\eta,\xi \in I\!\!D\underline{\otimes} I\!\!E$. By a solution of QSDI (1.1) or its equivalent form (2.3), we mean a stochastic process $\Phi : [0,T] \to \tilde{\mathcal{A}}$ lying in the space $Ad(\tilde{\mathcal{A}})_{wac} \bigcap L^2_{loc}(\tilde{\mathcal{A}})$ satisfying QSDI (1.1) or its equivalent form (1.2).

We assume the following conditions in what follows:
$\mathcal{S}_{(1)}$ The coefficients E, F, G, H appearing in QSDI (1.1) are continuous.
$\mathcal{S}_{(2)}$ The multivalued map $(t,x) \to P(t,x)(\eta,\xi)$ has nonempty and closed values as subsets of the field $I\!\!\!C$ of complex numbers.

$\mathcal{S}_{(3)}$ For each $x \in \tilde{\mathcal{A}}$, the map $t \to P(t,x)(\eta,\xi)$ is measurable.

$\mathcal{S}_{(4)}$ There exists a map $K^P_{\eta\xi} : [0,T] \to I\!R_+$ lying in $L^1_{loc}([0,T])$ such that

$$\rho(P(t,x)(\eta,\xi), P(t,y)(\eta,\xi)) \leq K^P_{\eta\xi}(t)\|x-y\|_{\eta\xi} \tag{3.2}$$

for $t \in [0,T]$, and for each pair $x,\ y \in \tilde{\mathcal{A}}$.

$\mathcal{S}_{(5)}$ There exists a stochastic process $Y : [0,T] \to \tilde{\mathcal{A}}$ lying in $Ad(\tilde{\mathcal{A}})_{wac}$ such that for each pair $\eta,\xi \in I\!D\underline{\otimes} I\!E$,

$$d\left(\frac{d}{dt}\langle\eta, Y(t)\xi\rangle\ , P(t,Y(t))(\eta,\xi)\right) \leq \rho_{\eta\xi}(t), \tag{3.3}$$

for almost all $t \in [0,T]$ and for some locally integrable map $\rho_{\eta\xi} : [0,T] \to R_+$.

Associated with the space $\tilde{\mathcal{A}}$, we define the space of complex numbers $\tilde{\mathcal{A}}(\eta,\xi) := \{\langle\eta, a\xi\rangle : a \in \tilde{\mathcal{A}}\}$. We shall denote by $S^{(T)}(a)$, the subset of $\mathbf{wac}(\tilde{\mathcal{A}})$ consisting of the set of solutions of QSDI (1.1) corresponding to the initial value $a \in \tilde{\mathcal{A}}$ and write $S^{(T)}(a)(\eta,\xi) = \{\langle\eta, \Phi(\cdot)\xi\rangle : \Phi \in S^{(T)}(a)\}$. Moreover, $S^{(T)}(P)(\eta,\xi) := \bigcup_{a\in\tilde{\mathcal{A}}} S^{(T)}(a)(\eta,\xi)$. In what follows, $a \to S^{(T)}(a)$ is the multivalued solution map of QSDI (1.1) corresponding to the initial value $x = a$. Under the conditions $\mathcal{S}_{(1)} - \mathcal{S}_{(5)}$ above, it is well known that the set $S^{(T)}(a)$ is not empty for arbitrary $a \in \tilde{\mathcal{A}}$ (see [8, 9, 10]).

Next, we employ Corollary 3.2 in [3] to establish an auxiliary result needed for the proof of the arcwise connectedness of the entire space $S^{(T)}(P)(\eta,\xi)$. To this end, for any family of linear maps $\{a_\alpha,\ \alpha \in [0,1]\}$ in $\tilde{\mathcal{A}}$, we define $a_{\eta\xi,\alpha} = \langle\eta, a_\alpha\xi\rangle,\ \alpha \in [0,1]$ for arbitrary elements $\eta,\xi \in I\!D\underline{\otimes} I\!E$.

Proposition 3.1: Let $a_0,\ a_1 \in \tilde{\mathcal{A}}$ such that $a_0 \neq a_1$. Let $X_0 \in S^{(T)}(a_0),\ \ X_1 \in S^{(T)}(a_1)$. Then there exists a continuous map $h : [0,1] \to \mathbf{wac}(\tilde{\mathcal{A}})(\eta,\xi)$ such that $h(0) = X_{\eta\xi,0},\ \ h(1) = X_{\eta\xi,1}$ and for $\alpha \in [0,1]$, $h(\alpha) \in S^{(T)}(a_\alpha)(\eta,\xi)$ where $a_\alpha = (1-\alpha)a_0 + \alpha a_1$ and $a_{\eta\xi,\alpha} = (1-\alpha)a_{\eta\xi,0} + \alpha a_{\eta\xi,1}$.

Proof: By Corollary (3.2) in [3], there exists a continuous map $\Phi : \tilde{\mathcal{A}}(\eta,\xi) \to wac(\tilde{\mathcal{A}})(\eta,\xi)$ such that for each $a \in \tilde{\mathcal{A}}$, $\Phi(a_{\eta\xi}) \in S^{(T)}(a)(\eta,\xi)$, and $\Phi(a_{\eta\xi,0}) = X_{\eta\xi,0},\ \ \Phi(a_{\eta\xi,1}) = X_{\eta\xi,1}$. Then, the map $h : [0,1] \to wac(\tilde{\mathcal{A}})(\eta,\xi)$ defined by $h(\alpha) = \Phi(a_{\eta\xi,\alpha})$ is the required map.

Definition : A space X is said to be arcwise connected if any two distinct points can be joined by an arc, that is a path f which is a homeomorphism between the unit interval and its image $f([0,1])$

4. Main Result

In order to establish the arcwise connectedness of the space $S^{(T)}(P)(\eta,\xi)$, some idea from [14, 15] were employed in what follows.

Theorem 4.1: Assume that the conditions $\mathcal{S}_{(1)} \; - \; \mathcal{S}_{(5)}$ above are satisfied.
Then, for every $a \in \tilde{\mathcal{A}}$ and arbitrary pair $\eta, \xi \in I\!\!D \underline{\otimes} I\!\!E$, the set $S^{(T)}(a)(\eta,\xi)$ is arcwise connected in $C([0,T]; \mathbb{C})$.

Proof: Fix a_0 in $\tilde{\mathcal{A}}$ and let $X, Y \in S^{(T)}(a_0)$. Then the functions $X_{\eta\xi}(\cdot),\ Y_{\eta\xi}(\cdot) \in S^{(T)}(a_0)(\eta,\xi)$. By Corollary (3.2) in [3], there exists a continuous map

$$\Phi : \tilde{\mathcal{A}}(\eta,\xi) \to wac(\tilde{\mathcal{A}})(\eta,\xi)$$

such that $\Phi(a_{\eta\xi,0}) = X_{\eta\xi}(\cdot)$ and

$$\Phi(a_{\eta\xi}) \in S^{(T)}(a)(\eta,\xi), \forall a_{\eta\xi} \in \tilde{\mathcal{A}}(\eta,\xi). \tag{4.1}$$

Since $Y_{\eta\xi}(\cdot)$ is continuous on $[0,T]$, the map $\lambda \to \Phi(Y_{\eta\xi}(\lambda T))$ is continuous from $[0,1]$ to $wac(\tilde{\mathcal{A}})(\eta,\xi)$, being the composition of continuous maps

$$h : [0,1] \to [0,T]; Y_{\eta\xi} : [0,T] \to \tilde{\mathcal{A}}(\eta,\xi); \Phi : \tilde{\mathcal{A}}(\eta,\xi) \to wac(\tilde{\mathcal{A}})(\eta,\xi)$$

where $h(\lambda) = \lambda T$. Moreover,

$$\Phi(Y_{\eta\xi}(\lambda T)) \;\; \in S^{(T)}(Y(\lambda T))(\eta,\xi), \tag{4.2}$$

for each $\lambda \in [0,T]$. Thus there exists a stochastic process
$\phi(Y(\lambda T)) \in S^{(T)}(Y(\lambda T))$ such that

$$\Phi(Y_{\eta\xi}(\lambda T)(t) = \langle \eta, \phi(Y(\lambda T))(t)\xi\rangle,\ t \in [0,T]. \tag{4.3}$$

Equation (4.3) implies that

$$\begin{array}{rcl} \dfrac{d}{dt}\langle \eta, \phi(Y(\lambda T))(t)\xi\rangle & \in & P(t, \phi(Y(\lambda T))(t))(\eta,\xi) \\ \phi(Y(\lambda T))(0) & = & Y(\lambda T),\ t \in [0,T]. \end{array} \tag{4.4}$$

Next, we define the following pair of maps. For each $\lambda \in [0,1]$,

$$X_\lambda(t) = \begin{cases} Y(t), & \text{if } \ 0 \le t \le \lambda T \\ \phi(Y(\lambda T))(t-\lambda T), & \text{if } \ \lambda T \le t \le T. \end{cases} \tag{4.5}$$

Setting $X_{\eta\xi,\lambda}(\cdot) = \langle \eta, X_\lambda(\cdot)\xi\rangle$, we obtain the inner product form of equation (4.1) given by:

$$X_{\eta\xi,\lambda}(t) = \begin{cases} Y_{\eta\xi}(t), & \text{if } \ 0 \le t \le \lambda T \\ \Phi(Y_{\eta\xi}(\lambda T))(t-\lambda T), & \text{if } \ \lambda T \le t \le T. \end{cases} \tag{4.6}$$

Notice that $X_0(\cdot) = X(\cdot)$, and $X_1(\cdot) = Y(\cdot)$, and

$$\begin{aligned} \frac{d}{dt}\langle \eta, X_\lambda(t)\xi\rangle &\in P(t, X_\lambda(t))(\eta,\xi) \\ X_\lambda(0) &= a_0, \text{ almost all } t \in [0,T]. \end{aligned} \tag{4.7}$$

By definition, for each $\lambda \in [0,1]$,

$$X_\lambda \in Ad(\tilde{\mathcal{A}})_{wac} \bigcap L^2_{loc}(\tilde{\mathcal{A}}). \tag{4.8}$$

Hence, $X_\lambda \in S^{(T)}(a_0)$. Therefore, $X_{\eta\xi,\lambda} \in S^{(T)}(a_0)(\eta,\xi)$. To complete the proof, it remains to be proved that the map $\lambda \to X_{\eta\xi,\lambda}$ is continuous from $[0,T]$ to the space $\mathbf{wac}(\tilde{\mathcal{A}})(\eta,\xi)$ in the topology of the space $C([0,T];\mathbb{C})$. To this end, we employ a similar idea from [15]. Let $\epsilon > 0$ be given and let $\lambda_0 \in [0,1]$ be fixed. We show that there exists $\delta > 0$ such that for any $\lambda \in [0,1]$ with $|\lambda - \lambda_0| < \delta$, we have

$$\sup_{[0,T]} |X_{\eta\xi,\lambda}(t) - X_{\eta\xi,\lambda_0}(t)| < \epsilon. \tag{4.9}$$

For $t \in [0,T]$, we distinguish three cases as follows:

$$\text{(i) } 0 \le t \le \lambda_0 T \le \lambda T, \text{ (ii) } \lambda_0 T \le t \le \lambda T, \text{ (iii) } \lambda_0 T \le \lambda T \le t \le T. \tag{4.10}$$

In the case of (i) we have

$$|X_{\eta\xi,\lambda}(t) - X_{\eta\xi,\lambda_0}(t)| = |Y_{\eta\xi}(t) - Y_{\eta\xi}(t)| = 0. \tag{4.11}$$

For case (ii) where $\lambda_0 T \le t \le \lambda T$, then

$$|X_{\eta\xi,\lambda_0}(t) - X_{\eta\xi,\lambda}(t)| = |\Phi(Y_{\eta\xi}(\lambda_0 T))(t - \lambda_0 T) - Y_{\eta\xi}(t)| \tag{4.12}$$

Since the map $t \to \Phi(Y_{\eta\xi}(\lambda_0 T))(t)$ and $t \to Y_{\eta\xi}(t)$ are uniformly continuous on the interval $I = [0,T]$, there exists $\delta_1 > 0$ such that for any t' and t'' in $[0,T]$ with $|t' - t''| < \delta_1$, we have

$$|\Phi(Y_{\eta\xi}(\lambda_0 T))(t') - \Phi(Y_{\eta\xi}(\lambda_0 T))(t'')| < \frac{\epsilon}{2} \tag{4.13}$$

and

$$|Y_{\eta\xi}(t') - Y_{\eta\xi}(t'')| < \frac{\epsilon}{2}. \tag{4.14}$$

Let $|\lambda_0 - \lambda| < \frac{\delta_1}{T}$. Then $|t - \lambda_0 T| \le |\lambda - \lambda_0| T \le \delta_1$ and since
$|\Phi(Y_{\eta\xi}(\lambda_0 T)(0) = Y_{\eta\xi}(\lambda_0 T)$, we have

$$\begin{aligned} & |X_{\eta\xi,\lambda_0}(t) - X_{\eta\xi,\lambda}(t)| \\ \le\ & |\Phi(Y_{\eta\xi}(\lambda_0 T)(t - \lambda_0 T) - \Phi(Y_{\eta\xi}(\lambda_0 T))(0)| + |\Phi(Y_{\eta\xi}(\lambda_0 T))(0) - Y_{\eta\xi}(t)| \\ \le\ & \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon. \end{aligned} \tag{4.15}$$

For case (iii), then

$$\begin{aligned}|X_{\eta\xi,\lambda_0}(t) - X_{\eta\xi,\lambda}(t)| &= |\Phi(Y_{\eta\xi}(\lambda_0 T))(t-\lambda_0 T) - \Phi(Y_{\eta\xi}(\lambda T))(t-\lambda T)| \\ &\leq |\Phi(Y_{\eta\xi}(\lambda_0 T))(t-\lambda_0 T) - \Phi(Y_{\eta\xi}(\lambda_0 T))(t-\lambda T)| \\ &+ |\Phi(Y_{\eta\xi}(\lambda_0 T))(t-\lambda T) - \Phi(Y_{\eta\xi}(\lambda T))(t-\lambda T)|. \quad (4.16)\end{aligned}$$

Since the map $\lambda \to \Phi(Y_{\eta\xi}(\lambda T))$ is continuous from $[0,1]$ to $\mathbf{wac}(\tilde{\mathcal{A}})(\eta,\xi) \subseteq C([0,T];\mathbb{C})$, there exists $\delta_2 > 0$ such that

$$|\lambda - \lambda_0| < \delta_2 \tag{4.17}$$

implies that

$$\sup_{t\in[0,T]} |\Phi(Y_{\eta\xi}(\lambda T))(t) - \Phi(Y_{\eta\xi}(\lambda_0 T))(t)| < \frac{\epsilon}{2}, \tag{4.18}$$

so that for $|\lambda - \lambda_0| < \delta_2$, we have

$$|\Phi(Y_{\eta\xi}(\lambda T))(t-\lambda T) - \Phi(Y_{\eta\xi}(\lambda_0 T))(t-\lambda T)| < \frac{\epsilon}{2}. \tag{4.19}$$

Furthermore, since the map $t \to \Phi(Y_{\eta\xi}(\lambda_0 T))(t)$ is uniformly continuous on $[0,T]$, there exists $\delta_3 > 0$ such that for any pair of points t', t'' in $[0,T]$, $|t' - t''| < \delta_3$ implies that

$$|\Phi(Y_{\eta\xi}(\lambda_0 T))(t') - \Phi(Y_{\eta\xi}(\lambda_0 T))(t'')| < \frac{\epsilon}{2}. \tag{4.20}$$

Then if $|\lambda - \lambda_0| < \frac{\delta_3}{T}$, we have

$$|t - \lambda T - t + \lambda_0 T| \leq |\lambda - \lambda_0| T \leq \delta_3, \tag{4.21}$$

and

$$|\Phi(Y_{\eta\xi}(\lambda_0 T))(t-\lambda T) - \Phi(Y_{\eta\xi}(\lambda_0 T))(t-\lambda_0 T)| < \frac{\epsilon}{2}. \tag{4.22}$$

By Equations (4.16), (4.19) and (4.22), if $|\lambda - \lambda_0| < \min\{\delta_2, \frac{\delta_3}{T}\}$, then

$$|X_{\eta\xi,\lambda}(t) - X_{\eta\xi,0}(t)| < \epsilon. \tag{4.23}$$

Let

$$\delta = \min\{\frac{\delta_1}{T},\ \delta_2,\ \frac{\delta_3}{T}\}, \tag{4.24}$$

then we have proved that if $\lambda_0 \leq \lambda$ and $|\lambda - \lambda_0| < \delta$, then for any $t \in [0,T]$,

$$|X_{\eta\xi,\lambda_0}(t) - X_{\eta\xi,\lambda}(t)| < \epsilon. \tag{4.25}$$

This implies that

$$\sup_{t\in[0,T]} |X_{\eta\xi,\lambda}(t) - X_{\eta\xi,\lambda_0}(t)| < \epsilon. \tag{4.26}$$

For $\lambda \leq \lambda_0$, the proof is similar.

Theorem 4.2: Corresponding to an arbitrary pair of points $\eta, \xi \in \mathbb{D}\underline{\otimes}\mathbb{E}$, the function space $S^{(T)}(P)(\eta, \xi)$ is arcwise connected in $C([0, T];\ \mathbb{C})$.

Proof: Let $X,\ Y\ \ \in S^{(T)}(P) := \bigcup_{a\in\tilde{\mathcal{A}}} S^{(T)}(a)$ such that for any pair of distinct elements $a,\ a_0 \in \tilde{\mathcal{A}}$, $X \in S^{(T)}(a_0)$ and $Y \in S^{(T)}(a)$. Then by Proposition 3.1, there exists a continuous map $h : [0, 1] \to wac(\tilde{\mathcal{A}})(\eta, \xi)$ such that $h(0) = X_{\eta\xi}$, $h(1) = Y_{\eta\xi}$ and for each $\alpha \in [0, 1]$, $h(\alpha) \in S^{(T)}(a_\alpha)(\eta, \xi)$, where $a_\alpha = (1 - \alpha)a_0 + \alpha a$. If $a = a_0$, then $X,\ Y \in S^{(T)}(a_0)$ and the existence of a continuous map $h : [0, 1] \to wac(\tilde{\mathcal{A}})(\eta, \xi)$ such that $h(0) = X_{\eta\xi}$ and $h(1) = Y_{\eta\xi}$, $h(\alpha) \in S^{(T)}(a_0)(\eta, \xi)$ follows from Theorem 4.1 above.

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References

[1] Aubin J P and Cellina A 1984 *Differential Inclusions* (Berlin:Springer - Verlag)
[2] Aurelian Cernea 2008 *Miskolc Math. Notes* **9** No.1 pp. 33-39
[3] Ayoola E O 2008 *Acta Appl. Math.* **100** 15 - 37
[4] Ayoola E O 2004 *Internat. Journal Theoret. Physics* **43** 2041 - 2059
[5] Ayoola E O 2003 *Stochastic Anal.Applic.* **21** 1215 - 1230
[6] Ayoola E O 2003 *Stochastic Anal. Applic.***21** 515 - 543
[7] Ayoola E O 2001 *Stochastic Anal. Applic.* **19** 461 - 471
[8] Ekhaguere G O S 1992 *Inter. J. Theor. Phys.* **31** 2003-2034
[9] Ekhaguere G O S 1995 *Inter. J. Theor. Phys.* **34** 323-353
[10] Ekhaguere G O S 1996 *Inter. J. Theor. Phys.* **35** 1909-1946
[11] Hudson R L and Parthasarathy K R 1984 *Comm. Math. Phys.* **93** 301-324
[12] Ogundiran M O and Ayoola E O 2010 *J. Math. Phys.* **51** 023521
[13] Ren-you Zhong, Nan-jing Huang and Mu-ming Wong 2009 *Taiwanese J. Math.* **13** 821–836
[14] Staicu V 1991 *Proc. AMS* **113** 403 - 413
[15] Staicu V and Wu H 1991 *Bollettino U. M. I.* **7** 253-256

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Existence and Uniqueness of Solutions of a Class of Quantum Stochastic Partial Differential Equations

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ABSTRACT: By employing the theory of iterated stochastic integration with respect to quantum martingale measures taking values in a linear space $\mathcal{A}$ of unbounded linear operators on a Hilbert space, we present a rigorous formulation of quantum stochastic partial differential equations (QSPDE). The solutions of certain classes of these equations are closable operators and they are known to provide examples of irreversible quantum dynamics which have found applications as models of open quantum systems and models of electric currents in neutrons among many other applications. Existence and uniqueness of a class of semi-linear quantum stochastic partial differential equations are studied.
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1 INTRODUCTION

It is well known that when modelling classical physical systems that are susceptible to noise, non-linear stochastic partial differential equations (SPDEs) often arise. A convenient approach to analyzing nonlinear SPDE's is to first reduce them to stochastic integral equations. Availability of a theory of iterated stochastic integration, over time and space, with respect to a cylindrical Wiener process (G. Da Prato 1992)or a martingale measure [Walsh 1986] subsequently becomes central.

In analogy to the classical context, open quantum systems and other physical systems (Davies, 1976; Lindblad 1976] are invariably subject to quantum noise. The temporal evolution of such systems maybe modelled by means of a noncommutative stochastic calculus [Hudson-Parthasarathy 1984; Parthasarathy 1992]that generalises the classical Ito stochastic calculus. To be able to take account of both the temporal and spatial variation of the observables of the quantum systems, a framework involving iterated stochastic integration has been established (Applebaum 1995, 1998). In this paper, we exploit the notion of a quantum martingale measure. Iterated stochastic integration with respect to quantum martingales is finally employed to establish the existence and uniqueness of a class of quantum stochastic partial differential equations in this work.

The organization of the paper is as follows. Section 2 highlights some of the fundamental notions and notation which we use throughout the discussion. Our notion of martingale measures is discussed in section 3. The main results of the paper are assembled in Sections 4 ad 5. The main results concern the existence and uniqueness of the solutions of a semilinear quantum stochastic heat equation, which generalizes the classical semilinear heat equation.

2 PRELIMINARIES

This section is devoted to the explanation of some of the basic structures which are employed in what follows. We shall begin by describing some relevant spaces of vector valued functions.
In the sequel, $\mathcal{R}$ and Υ are two fixed Hilbert spaces and $I\!\!D$ is a dense subspace of $\mathcal{R}$. The inner product and norm of Υ will be written as $\langle\cdot,\cdot\rangle_\Upsilon$ and $\|\cdot\|_\Upsilon$ respectively. We denote by

$$\mathcal{H} = L^2_\Upsilon(I\!\!R_+);\ \ \mathcal{H}_t = L^2_\Upsilon([0,t));\ \ \mathcal{H}^t = L^2_\Upsilon([t,\infty)),\ t \in I\!\!R_+$$

the Hilbert spaces of Lebesgue square integrable, Υ-valued maps respectively on
$I\!\!R_+ := [0,\infty)$; $[0,t)$; and $[t,\infty),\ t \in I\!\!R_+$.

Similarly, S_Υ (resp. $S_{B(\Upsilon)}$) denotes the linear space $L^\infty_{\Upsilon,loc}(I\!\!R_+)$ (resp.$L^\infty_{B(\Upsilon),loc}(I\!\!R_+)$) of all measurable, locally bounded functions from $I\!\!R_+$ to Υ (resp. to $B(\Upsilon)$), the Banach space of all bounded endormorphisms of Υ; the norm of $B(\Upsilon)$ will be denoted by $\|\cdot\|_{B(\Upsilon)}$.

The spaces S_Υ and $S_{B(\Upsilon)}$ will be equipped with the weak topologies which we define as follows: For $\alpha,\ \beta \in S_\Upsilon$ and $t \in I\!\!R_+$, define the linear forms $T_{\alpha t}$ and $T_{\alpha\beta t}$ respectively on S_Υ and $S_{B(\Upsilon)}$, as

$$T_{\alpha t}(h) = \int_0^t \langle \alpha(s), h(s)\rangle_\Upsilon ds,\ \ h \in S_\Upsilon,\ \ T_{\alpha\beta t}(U) = \int_0^t \langle \alpha(s), U(s)\beta(s)\rangle_\Upsilon ds,\ U \in S_{B(\Upsilon)}.$$

We introduce the sets:

$$S^{'}_\Upsilon = \{T_{\alpha t} : \alpha \in S_\Upsilon, t \in I\!\!R_+\},\ \text{ and }\ S^{'}_{B(\Upsilon)} = \{T_{\alpha\beta t} : \alpha,\beta \in S_\Upsilon,\ t \in I\!\!R_+\}.$$

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Then, we endow S_Υ with the $\sigma(S_\Upsilon, S_\Upsilon^{'})$-topology and $S_{B(\Upsilon)}$ with the $\sigma(S_{B(\Upsilon)}, S_{B(\Upsilon)}^{'})$-topology. With these topologies, $S_\Upsilon^{'}$ is the dual of S_Υ and $S_{B(\Upsilon)}^{'}$ is the dual of $S_{B(\Upsilon)}$.

Boson Fock Space

If $\mathbf{D}$ is a pre-Hilbert space with completion $\mathbf{H}$, the symbol $L_w^+(\mathbf{D}, \mathbf{H})$ denotes the set of all linear maps X from $\mathbf{D}$ to $\mathbf{H}$ such that domain of the operator adjoint X^* contains $\mathbf{D}$. We shall employ the following linear spaces of operators as in Ayoola 2001, 2008, Ekhaguere 1992.

(i) $\mathcal{A} \equiv L_w^+(I\!\!D \underline{\otimes} I\!\!E, \mathcal{R} \otimes \Gamma(\mathcal{H}))$, (ii) $\mathcal{A}_t \equiv L_w^+(I\!\!D \underline{\otimes} I\!\!E_t, \mathcal{R} \otimes \Gamma(\mathcal{H}_t)) \otimes 1^t$, (iii) $\mathcal{A}^t \equiv 1_t \otimes L_w^+(I\!\!E^t, \Gamma(\mathcal{H}^t)), t > 0$

where $\underline{\otimes}$ denotes algebraic tensor product and $\mathbf{1}_t$ (resp. $\mathbf{1}^t$) denotes the identity map on

$$\mathcal{R} \otimes \Gamma(\mathcal{H}_t)) \text{ (resp.}\Gamma(\mathcal{H}^t)), t > 0.$$

As usual, the net $\mathcal{A}(I\!\!R_+) \equiv \{\mathcal{A}_t : \ t \in I\!\!R_+\}$ is a filtration of $\mathcal{A}$. That is $\mathcal{A}_s \subseteq \mathcal{A}_t$, if $t \geq s \geq 0$ and $\cup_{t \in I\!\!R_+} \mathcal{A}_t$ generates $\mathcal{A}$.

Definition 2.1. (i) If $t \in I\!\!R_+$, and $a \in \mathcal{A}$, the member $E(a|\mathcal{A}_t)$ of $\mathcal{A}_t$ satisfying

$$\langle \eta, E(a|\mathcal{A}_t)\xi \rangle = \langle \eta, a\xi \rangle, \ \forall \eta \in \mathcal{R} \otimes \Gamma(\mathcal{H}_t), \ \ \xi \in I\!\!D \underline{\otimes} I\!\!E_t,$$

is called the conditional expectation of a given $\mathcal{A}_t$. In terms of the projection P_t of $\mathcal{R} \otimes \Gamma(\mathcal{H})$ onto $\mathcal{R} \otimes \Gamma(\mathcal{H}_t)$, the condition expectation satisfies: $E(a|\mathcal{A}_t) = P_t a P_t$.

(ii). We shall employ the topology τ_s, called the strong topology, on $\mathcal{A}$ whose family of seminorms $\{\| \cdot \|_\xi, \ \xi \in I\!\!D \underline{\otimes} I\!\!E\}$ defined by $\|a\|_\xi = \|a\xi\|, \ a \in \mathcal{A}, \ \xi \in I\!\!D \underline{\otimes} I\!\!E$.

We denote the completion of the locally convex space $(\mathcal{A}, \tau_s)$ by $\tilde{\mathcal{A}}$. The following definition and structures concerns our framework for stochastic processes and integration.

Definition 2.2. An $\tilde{\mathcal{A}}$-valued map X on an interval $j \subset I\!\!R_+$ is called a τ_s-stochastic process indexed by J if $t \to X(t)\xi, \ t \in J$ is measurable for arbitrary $\xi \in I\!\!D \underline{\otimes} I\!\!E$, where $\mathcal{R} \otimes \Gamma(\mathcal{H})$ is endowed with its natural Borel structure.

A stochastic process X indexed by J is called adapted if $X(t) \in \mathcal{A}_t$ for each $t \in J$.

The set of all adapted processes X on $I\!\!R_+$ such that

$$\|X\|_\xi^2 \equiv \int_0^t \|X(s)\|_\xi^2 ds < \infty$$

for each $t \in I\!\!R_+, \ \xi \in I\!\!D \underline{\otimes} I\!\!E$ will be denoted by $L^2_{loc}(\mathcal{A})$.

To each $f \in S_\Upsilon$ and $\pi \in S_{B(\Upsilon)}$, we adopt the basic field operators $a(f), \ a^\dagger(f)$, and $\lambda(\pi)$ in $L_w^+(I\!\!E, \Gamma(\mathcal{H}))$ and in terms of these operators, we introduce the following stochastic processes as in (Hudson-Parthasarathy 1984).

$$A_f(t) = a(f\chi_{[0,t)}), \ A_f^\dagger(t) = a^\dagger(f\chi_{[0,t)}), \ \ \wedge_\pi(t) = \lambda(\pi\chi_{[0,t)})$$

$t \in I\!\!R_+$ where χ_I denotes the indicator function of the Borel set $I \subseteq I\!\!R_+$. Employing the notation:

$$dA_f(t) \equiv A_f(dt), \ \ dA_f^\dagger(t) \equiv A_f^\dagger(dt), \ \ d\wedge_\pi(t) \equiv \wedge_\pi(dt), \ \ t \in I\!\!R_+,$$

then we interpret the stochastic integral

$$\int_0^t \Big(p(s) \wedge_\pi (ds) + q(s) A_f(ds) + k(s) A_g^\dagger(ds) + h(s) ds \Big)$$

as in (Hudson-Parthasarathy 1984, Ekhaguere 1992) for certain admissible integrands $p, q, k, h \in L^2_{loc}(\mathcal{A})$.

3 MARTINGALE MEASURES

Let (X, τ) be a complex, Hausdorff, locally convex space with topology τ and dual X', and $(\Omega, \mathcal{B})$ a measurable space, with $\mathcal{B}$ a σ-algebra of subsets of Ω. In the sequel, $M(\mathcal{B}, X)$ is the set of all X-valued additive set functions μ on $\mathcal{B}$ such that $B \longmapsto x'(\mu(B))$ is σ-additive and regular on $\mathcal{B}$, for all $x' \in X'$. Then, for each $\mu \in M(\mathcal{B}, X)$ and arbitrary $x' \in X'$, $x' \circ \mu$ is a complex, and hence automatically finite, measure on $(\Omega, \mathcal{B})$.

Definition 3.1 Let $\tilde{\mathcal{A}}$ and $(\Omega, \mathcal{B})$ be as above. A map $W : I \times \mathcal{B} \to \tilde{\mathcal{A}}$. $I \subseteq I\!\!R_+$, will be called a τ_s-martingale measure relative to the filtration $\mathcal{A}(I)$ provided that

(i) $W(0, B) = 0, \ \ \forall \ B \in \mathcal{B}$;

(ii) for each $B \in \mathcal{B}$, the map $t \longmapsto W(t, B), t \in I$, is adapted to $\mathcal{A}(I)$;

(iii) for each $B \in \mathcal{B}$, the map $t \longmapsto W(t, B), \ t \in I$, is a martingale, i.e.

$$E(W(t, B)|\mathcal{A}_s) = W(s, B) \ \text{ whenever } s, t \in I \text{ satisfy } t \geq s;$$

(iv) for each $t \in I$, the map $B \longmapsto W(t, B), B \in \mathcal{B}$, is in $M(\mathcal{B}, \tilde{\mathcal{A}})$;

(v) for each $\xi \in I\!\!D\underline{\otimes}I\!\!E$, there is a regular, σ-finite measure ω_ξ on $(\Omega, \mathcal{B})$ and a locally bounded function c_ξ on I such that

$$\|W(t,B)\|_\xi \leq c_\xi(t)\omega_\xi(B),\ t \in I,\ B \in \mathcal{B}.$$

Notation 3.2 The symbol $MM(I, \mathcal{B}, \tilde{\mathcal{A}})$ will denote the set of all $\tilde{\mathcal{A}}$-valued, $\mathcal{A}(I)$-adapted martingale measures on $(\Omega, \mathcal{B})$. The set is not empty as some examples of quantum martingale measures can be found in Applebaum (1998).
Let $\mu, \sigma \in M(\mathcal{B}(I\!\!R^d), \mathcal{S}_\Upsilon)$ and $\pi \in M(\mathcal{B}(I\!\!R^d), \mathcal{S}_{B(\Upsilon)})$. In what follows, we shall assume the validity of the following inequality.

$$\max\{\|\mu(s,B)\|_\Upsilon, \|\sigma(s,B)\|_\Upsilon, \|\pi(s,B)\|_{B(\Upsilon)}\} \leq c_{\mu\sigma\pi}(s)\omega_{\mu\sigma\pi}(B). \tag{3.1}$$

for some nonnegative, locally bounded functions $c_{\mu\sigma\pi}$ on $I\!\!R_+$, σ-finite measure $\omega_{\mu\sigma\pi}$ on $(I\!\!R^d, \mathcal{B}(I\!\!R^d)$, and arbitrary $\alpha, \beta \in \mathcal{S}_\Upsilon,\ (s,B) \in I\!\!R_+ \times \mathcal{B}(I\!\!R^d)$. Next, we introduce the maps

$$A_\mu(t,B) = a(\chi_{[0,t)}\mu(B));\ A^\dagger_\sigma(t,B) = a^\dagger(\chi_{[0,t)}\sigma(B)),\ \Lambda_\pi(t,B) = \lambda(\chi_{[0,t)}\pi(B)) \tag{3.2}$$

for arbitrary $(s,B) \in I\!\!R_+ \times \mathcal{B}(I\!\!R^d)$, where χ_C denotes the indicator function of C. Then the following holds.
Proposition 3.3 The maps A_μ, $A^\dagger_\sigma$, and Λ_π are τ_s-martingale measures, i.e. $\{A_\mu, A^\dagger_\sigma, \Lambda_\pi\}$ is a subset of $MM(I\!\!R_+, \mathcal{B}(I\!\!R^d), \tilde{\mathcal{A}})$.
Definition 3.4 The triplet $\{A_\mu\ \ A^\dagger_\sigma,\ \wedge_\pi\}$ with $\mu, \sigma \in M(\mathcal{B}(I\!\!R^d), \mathcal{S}_\Upsilon)$, and $\pi \in M(\mathcal{B}(I\!\!R^d), \mathcal{S}_{B(\Upsilon)})$ will be called quantum τ_s-martingale measures.

4 ITERATED STOCHASTIC INTEGRATION WITH RESPECT TO QUANTUM MARTINGALE MEASURES

Throughout the rest of the paper, we employ the quantum τ_s-martingale measures as defined above. The space $L^2_{ucloc}(I\!\!R_+ \times I\!\!R^d, \tilde{\mathcal{A}})$ is the set of maps $h : I\!\!R_+ \times I\!\!R^d \to \tilde{\mathcal{A}}$ satisfying the following properties:
(i) for each $x \in I\!\!R^d$, the map $t \to h(t,x),\ t \in I\!\!R_+$, is in $L^2_{loc}(\tilde{\mathcal{A}})$,
(ii) the map h is locally uniformly continuous in the sense that given any $\epsilon > 0$, there is a $\delta > 0$ such that

$$\sup_{0 \leq s \leq t} \|h(s,x) - h(s,y)\|_\xi < \epsilon,$$

whenever $t \in I\!\!R_+$, $\xi \in I\!\!D\underline{\otimes}I\!\!E$, $x, y \in B$, a bounded member of $\mathcal{B}(I\!\!R^d)$, and $\|x - y\| < \delta$, where we write $\|z\|$ for the norm of $z \in I\!\!R^d$. We endow $L^2_{ucloc}(I\!\!R_+ \times I\!\!R^d, \tilde{\mathcal{A}})$ with the locally convex topology τ_{ucloc} whose family $\{\|\cdot\|_{\xi t},\ \xi \in I\!\!D\underline{\otimes}I\!\!E,\ t \in I\!\!R_+\}$ of seminorms is defined by

$$\|h\|_{\xi t} = \left(\int_0^t \sup_{x \in I} \|h(s,x)\|^2_\xi\right)^{\frac{1}{2}},\ h \in L^2_{ucloc}(I\!\!R_+ \times I\!\!R^d, \tilde{\mathcal{A}}),\ \xi \in I\!\!D\underline{\otimes}I\!\!E,\ t \in I\!\!R_+.$$

Remarks.(i) Let $E, F, G \in L^2_{ucloc}(I\!\!R_+ \times I\!\!R^d, \tilde{\mathcal{A}})$, we define an iterated stochastic integral of the form:

$$M(t,B) = \int_B \int_{[0,t]} \Big(E(s,x) \wedge_\pi (ds,dx) + F(s,x)A_\mu(ds,dx) + G(s,x)A^\dagger_\sigma(ds,dx)\Big), \tag{4.1}$$

$(t,B) \in [0,T] \times \mathcal{B}(I\!\!R^d),\ T > 0$, B bounded.

(ii) The integral $M(t,B)$ is developed as follows:
For $x = (x_1 x_2, \cdots, x_d)$, $y = (y_1, y_2, \cdots, y_d) \in I\!\!R^d$, we write $x \leq y$ (resp. $x < y$) if and only if $x_j \leq y_j$, $j = 1, 2 \cdots d$ (resp. $x \leq y$ and $x \neq y$). Let $\mathcal{B}_{rec}(I\!\!R^d)$ be the subring of $\mathcal{B}(I\!\!R^d)$ consisting of all rectangles, i.e sets of the form: $[a,b) \equiv \Pi^d_{j=1}[a_j, b_j)$ for some $a, b \in I\!\!R^d$. We first develop the integral for the case where $B \in \mathcal{B}_{rec}(I\!\!R^d)$. It is well known that the extension of the formulation to the case where B is an arbitrary subset of $I\!\!R^d$ follows a standard procedure. We have the following results on iterated quantum stochastic integration with respect to the processes $\wedge_\pi,\ A_\mu,\ A^\dagger_\sigma$. The establishment of the result is similar to that of Theorem 3.1 in Applebaum (1998).

Theorem 4.1: (i). For $t \in [0,T],\ I \in \mathcal{B}_{rec}(I\!\!R^d),\ \eta = v \otimes e(\alpha),\ \xi = u \otimes e(\beta) \in I\!\!D\underline{\otimes}I\!\!E$, we have:

$$\begin{aligned}\langle \eta, M(t,I)\xi\rangle &= \int_I \int_{[0,t]} \{\langle \alpha(s), \pi(s,dx)\beta(s)\rangle_\Upsilon \langle \eta, E(s,x)\xi\rangle \\ &+ \langle \mu(s,dx), \beta(s)\rangle_\Upsilon \langle \eta, F(s,x)\xi\rangle + \langle \alpha(s), \sigma(s,dx)\beta(s)\rangle_\Upsilon \langle \eta, G(s,x)\xi\rangle\}.\end{aligned}$$

(ii) The order of integration may be changed. Defining the integral:

$$\tilde{M}(t,I) = \int_{[0,t]} \int_I \Big(E(s,x) \wedge_\pi (ds,dx) + F(s,x)A_\mu(ds,dx) + G(s,x)A^\dagger_\sigma(ds,dx)\Big), \tag{4.2}$$

then $\tilde{M}(t,I) = M(t,I)$ on $I\!\!D\underline{\otimes}I\!\!E$, for arbitrary $t \in [0,T]$.
(iii) The map $M : [0,T] \times \mathcal{B}(I\!\!R^d) \to \tilde{\mathcal{A}}$ is in $MM([0,T] \times \mathcal{B}, \tilde{\mathcal{A}})$.

5 QUANTUM STOCHASTIC PARTIAL DIFFERENTIAL EQUATIONS

An equation of the form

$$\begin{aligned}\frac{\partial X}{\partial t}(t,x) &= H\left(t,x,X,\frac{\partial X}{\partial x},\cdots,\frac{\partial^l X}{\partial x^l}\right)+E\left(t,x,X,\frac{\partial X}{\partial x},\cdots,\frac{\partial^k X}{\partial x^k}\right)\dot{\wedge}_\pi(t,x)\\ &+F\left(t,x,X,\frac{\partial X}{\partial k},\cdots,\frac{\partial^m X}{\partial x^m}\right)\dot{A}_\mu(t,x)+G\left(t,x,X,\frac{\partial X}{\partial x'},\cdots,\frac{\partial^n X}{\partial x^n}\right)\dot{A}^\dagger_\sigma(t,x)(5.1)\end{aligned}$$

$t\in[0,T]$, $x\in B$, a bounded subset of $I\!R^d$ where $\frac{\partial^a X}{\partial x^a}$ denotes $\frac{\partial^{a_1}\cdots a_d X}{\partial x^{a_1}\partial x^{a_2}\cdots\partial x^{a_d}}$ with $a=a_1+\cdots+a_d$ for some nonnegative integers $a_1,\ldots,a_d$ will be called a quantum stochastic partial differential equation (qspde) of order $\max\{l,k,m,n\}$. Furnished with appropriate constraints on E,F,G,H and some boundary conditions, the first problem is to make sense of this relation, after which one must tackle the issue of the existence and uniqueness of its solutions. As an application of the theory of martingale measures formulated above, we search for a solution, in the τ_s-topology, on $\mathcal{A}$, of the following semilinear quantum stochastic heat equation in one space dimension, associated with the quantum martingale measures, $A_\mu, A^\dagger_\sigma, \Lambda_\pi$, where E,F,G lie in $L^2_{ucloc}(I\!R_+,\tilde{\mathcal{A}})$ and satisfy some Lipschitz conditions

$$(\partial_t X)(t,x) = (\partial_{xx}X)(t,x)+E(t,X(t,x)\dot{\wedge}_\pi(t,x)+F(t,X(t,x))\dot{A}_\mu(t,x)+G(t,X(t,x))\dot{A}^+_\sigma(t,x) \quad (5.2)$$

$$X(0,t)=X_0(x),\ x\in[0,L];\ \ \partial_x X(t,0)=0=(\partial_x X)(t,L),\ \ t>0 \quad (5.3)$$

To this end, we will need the following well known (Walsh 1986) properties of the Green's function $\mathbf{G}(t;x,y)$ of equation (5.2).

G(i) $\int_0^t \mathbf{G}(s,x,y)\mathbf{G}(t;y,z)dy=\mathbf{G}(s+t;x,z)$;

G(ii) $\mathbf{G}(t;x,y)=\mathbf{G}(t;y,x)$;

G(iii) $\mathbf{G}(t;x,y)\le\frac{C_T}{\sqrt{t}}\exp\left(-\frac{(x-y)^2}{4t}-t\right),\ x,y\in[0,L],\ t\in(0,T]$, where C_T is some positive constant.

Proceeding formally and using (5.3) as well as some of the properties of the Green's function cited above, the QSPDE(5.2) may be re-written in the form

$$\begin{aligned}&\int_0^t X(t,x)\varphi(x)dx-\int_0^L X_0(x)\varphi(x)dx\\ =&\int_0^t\int_0^L X(s,x)\varphi''(x)dsdX+\int_0^t\int_0^L\varphi(x)[E(s,X(s,x))\wedge_\pi(ds,dX)\\ &+F(s,X(s,x))A_\mu(ds,dx)+G(s,X(s,x))A^\dagger_a(ds,dx)] \qquad (5.4)\end{aligned}$$

valid for all $\varphi\in C^\infty[(0,L])$ such that $\varphi'(0)=0=\varphi'(L)$. It follows that (5.4) holds whenever X solves (5.2) and (5.3). Moreover, adapting the reasoning in (Walsh, 1986), one checks that X solves (5.4) if and only if X is a solution of the stochastic integral equation

$$\begin{aligned}X(t,x) &= \int_0^L X_0(y)G(t;x,y)dy+\int_0^t\int_0^L G(t-s;x,y)([E(s,X(s,x))\wedge_\pi(ds,dx)\\ &+F(s,X(s,x))A_\mu(ds,dx)+G(s,X(s,x))A^\dagger_\sigma(ds,dx)] \qquad (5.5)\end{aligned}$$

In what follows, $L,T>0$ and $E,F,G\in L^2_{ucloc}(I\!R_+,\tilde{\mathcal{A}})$ are Lipschitzian, that is, for each $\xi\in I\!D\underline{\otimes}I\!E$, there is a locally bounded function $K_\xi:[0,T]\to I\!R_+$ such that

$$\max\{\|E(t,z_1)-E(t,z_2)\|_\xi,\ \|F(t,z_1)-F(t,z_2)\|_\xi,\|G(t,z_1)-G(t,z_2)\|_\xi\}$$

$$\le K_\xi(t)\|z_1-z_2\|_{\varepsilon(\xi)},z_1,\ z_2\in\mathcal{A},\ t\in[0,T],$$

for some self map $\varepsilon: I\!D\underline{\otimes}I\!E\to I\!D\underline{\otimes}I\!E$.

Notation 5.1. We will require the following notation.
If $\varepsilon: I\!D\underline{\otimes}I\!E\to I\!D\underline{\otimes}I\!E$, write $\varepsilon^n, n\ge1$, for the n-fold composition of ε with itself, i.e. $\varepsilon^n(\xi)=\varepsilon(\varepsilon^{n-1}(\xi)),\ \xi\in I\!D\underline{\otimes}I\!E,\ n=1,2,\ldots,$, with ε^0 the identity map on $I\!D\underline{\otimes}I\!E$.

For each $\xi\in I\!D\underline{\otimes}I\!E$, let $(I\!D\underline{\otimes}I\!E)_{\varepsilon,\xi}$ be the orbit of ξ under ε, i.e.

$$(I\!D\underline{\otimes}I\!E)_{\varepsilon,\xi}=\{\varepsilon^n(\xi):n=0,1,2,\ldots\}$$

Definition 5.2: By a solution of (5.2) and (5.3) in the τ_s-topology, we shall refer to a solution of the stochastic integral equation (5.5) that satisfies the following condition:

$$\sup_{\vartheta\in(I\!D\underline{\otimes}I\!E)_{\varepsilon,\xi}}\ \sup_{x\in[0,L]}\ \sup_{s\in[0,t]}\|X(t,X)\|^2_\vartheta<\infty,\ \xi\in I\!D\underline{\otimes}I\!E \quad (5.6)$$

5.1 Solutions in the τ_s-Topology

Theorem 5.3: Suppose that the following conditions are satisfied:

$$\max\{\|\mu(s,B)\|_\Upsilon, \|\sigma(s,B)\|_\Upsilon, \|\pi(s,B)\|_{B(\Upsilon)}\} \leq c_{\mu\sigma\pi}(s)\omega_{\mu\sigma\pi}(B). \tag{5.7}$$

for some nonnegative, locally bounded functions $c_{\mu\sigma\pi}$ on $\mathbb{R}_+$, σ-finite measure $\omega_{\mu\sigma\pi}$ on $(\mathbb{R}, \mathcal{B}(\mathbb{R})$, is absolutely continuous with respect to the Lebesgue measure on $(\mathbb{R},\ \mathcal{B}(\mathbb{R}))$, with locally bounded Radon-Nikodym derivative $\omega_{\mu\sigma\pi}$ and arbitrary $\alpha, \beta \in \mathcal{S}_\Upsilon,\ (s,B) \in \mathbb{R}_+ \times \mathcal{B}(\mathbb{R}^d)$.
Then the QSPDE(5.2), equipped with conditions (5.3) and (5.6) possesses a unique solution in the τ_s-topology.

Proof. Existence: This will be proved by means of a Picard's iteration process. Let $X_0(t,x) = X_0(x)$, $x \in [0,L]$, $t \in [0,T]$. Then, $t \longmapsto X_0(t,x)$, $t \in [0,T]$ is adapted, for each $x \in [0,L]$. Introduce the iterative scheme

$$\begin{aligned} X_{n+1}(t,x) \quad &= \quad \int_0^L X_0(y)G(t;x,y)dy + \int_0^t \int_0^L G(t-s;x,y)[E(X_n(s,x),s) \wedge_\pi (ds,dx) \\ &\quad +F(X_n(s,x),s)A_\mu(ds,dx) + G(X_n(s,x),s)A_\sigma^\dagger(ds,dx)],\ n = 0,1,2,\ldots \end{aligned}$$

From the adaptedness of X_0, it follows that $\{X_n(\cdot,x) : n = 1,2,\ldots\}$ is a sequence of adapted stochastic processes, for each $x \in [0,L]$. Define $Z_n(t,x) = X_{n+1}(t,x) - X_n(t,x),\ n = 0,1,2,\ldots$. Then

$$\begin{aligned} Z_n(t,x) = \int_0^t \int_0^L G(t-s,x,y)[(E(s,X_n(s,x)) - E(s,X_{n-1}(s,x))) \wedge_\pi (ds,dx) \\ +(F(s,X_n(s,x)) - F(s,X_{n-1}(s,x)))A_\mu(ds,dx) + (G(s,X_n(s,x)) - G(s,X_{n-1}(s,x)))A_\sigma^\dagger(ds,dx)], \end{aligned}$$

$n = 1,2,\ldots$; whence for arbitrary $\xi = u \otimes e(\beta) \in \mathbb{D}\underline{\otimes}\mathbb{E}$, we have

$$\begin{aligned} \|Z_n(t,x)\|_\xi^2 \quad &\leq \quad 3\int_0^t \int_0^L G(t-s;x,y)^2[\|(E(s,X_n(s,x)) - E(s,X_{n-1}(s,x)))\|_\xi^2 \times \\ &\qquad |\langle\beta(s), \pi(s,dx)\beta(s)\rangle| \\ &\qquad +\|(F(s,X_n(s,x)) - F(s,X_{n-1}(s,x)))\|_\xi^2|\langle\beta(s), \mu(s,dx)\rangle| \\ &\qquad +\|(G(s,X_n(s,x)) - G(s,X_{n-1}(s,x)))\|_\xi^2|\langle\beta(s), \sigma(s,dx)\rangle|]ds, \\ &\leq \quad C_{T\xi}\int_0^t \int_0^L G(t-s;x,y)^2\|Z_{n-1}(s,y)\|_{\varepsilon(\xi)}^2\omega_{\mu\sigma\pi}(dy)ds, \end{aligned}$$

where $C_{T\xi} = 9\left(\sup_{0\leq s\leq T}[k_\xi(s)C_{\mu\sigma\pi}(s)\max(\|\beta(s)\|^2,\ \|\beta(s)\|)]\right)$. Let

$$Z(t)_\xi^2 = \sup_{r\in(\mathbb{D}\underline{\otimes}\mathbb{E})_{\varepsilon,\xi}} \sup_{x\in[0,L]} \sup_{s\in[0,t]} \|Z_n(t,x)\|_r^2,\ t \in [0,T],\ \xi \in \mathbb{D}\underline{\otimes}\mathbb{E}$$

Then, from $G(i),\ G(iii)$ above, whence $\int_\infty^\infty G(t;x,y)^2dy \leq c_T/\sqrt{t}$, for all $x \in [0,L]$,
we get

$$Z_n(t)_\xi^2 \leq 3c_TC_{T\xi}\left(\sup_{0\leq s\leq T}[\omega_{\mu\sigma\pi}(s)]\right)\int_0^t \frac{1}{\sqrt{t-s}}Z_{n-1}(s)_\xi^2ds,\quad t \in [0,T].$$

Choose any $p \in (1,2)$ and estimate the integral, using Hölder's inequality, to obtain

$$Z_n(t)_\xi^{2q} \leq K_{T\xi}\int_0^t Z_{n-1}(s)_\xi^{2q}ds,\quad q = \frac{p}{p-1},\quad \text{with}\quad p \in (1,2),\ t \in [0,T]$$

where $K_{T\xi}$ is some positive constant. Applying a form of Gronwall's Lemma we get

$$Z_n(t)_\xi^{2q} \leq Z_0(t)_\xi^{2q}(K_{T\xi}t)^n/(n)!,\quad n = 0,1,2,\ldots,\ t \in [0,T].$$

Hence as $\left(\sqrt[2q]{K_{T\eta\xi}t}\right)^n / \sqrt[2q]{(n!)}$ is the general term of a convergent infinite series of nonnegative numbers, the last inequality implies $\sum_{n=1}^\infty Z_n(t)_\xi < \infty$, $t \in [0,T]$. From the definition of $Z_n(t)_\xi$ it follows that the sequence $X_n(t,x)$ is τ_s-convergent in $\mathcal{A}$ to some $X(t,x)$ for arbitrary $(t,x) \in [0,T] \times [0,L]$, whence the sequence

$$\begin{aligned} \int_0^t \int_0^L G(t-s;x,y)[E(s,X_n(s,x)) \wedge_\pi (ds,dx) + F(s,X_n(s,x))A_\mu(ds,dx) \\ +G(s,X_n(s,x))A_\sigma^\dagger(ds,dx)] \end{aligned}$$

also τ_s-converges to

$$\int_0^t \int_0^L G(t-s;x,y)[E(s,X(s,x)) \wedge_\pi (ds,dx) + F(s,X(s,x))A_\mu(ds,dx) + G(s,X(s,x))A_\sigma^\dagger(ds,dx)],$$

for arbitrary $(t,x) \in [0,T] \times [0,L]$, showing that X is a solution of (5.5). Finally, since each $t \longmapsto X_n(t,\cdot)$, $t \in [0,T]$, is adapted for each n, it follows that $t \longmapsto X(t,\cdot)$, $t \in [0,T]$ is adapted. This concludes the proof of the existence of a solution of (5.5) and hence of (5.2) and (5.3).

Uniqueness: Suppose X and Y both solve (5.5). Then, with $Z = X - Y$, we have

$$\begin{aligned} Z(t,x) \quad = \quad & \int_0^t \int_0^L G(t-s;x,y)[(E(s,X(s,x)) - E(s,Y(s,x))) \wedge_\pi (ds,dx) \\ & +(F(s,X(s,x)) - F(s,Y(s,x)))A_\mu(ds,dx) \\ & +(G(s,X(s,x)) - G(s,Y(s,x)))A_\sigma^\dagger(ds,dx)]. \end{aligned}$$

Let $\xi \in I\!D \underline{\otimes} I\!E$ be arbitrary and

$$Z(t)_\xi^2 = \sup_{\vartheta \in (I\!D \underline{\otimes} I\!E)_{\varepsilon,\xi}} \sup_{x\in[0,L]} \sup_{s\in[0,t]} \|Z(t,x)\|_\vartheta, \; t \in [0,T].$$

Then arguing as above, we get

$$Z(t)_\xi^{2q} \le K_{T\xi} \int_0^t Z(s)_\xi^{2q} ds, \quad q = \frac{p}{p-1}, \text{ with } p \in (1,2), \; t \in [0,T].$$

where $K_{T\xi}$ is some positive constant. Applying Gronwall's Lemma, it follows that $Z(t)_\xi = 0$, for arbitrary $\xi \in I\!D \underline{\otimes} I\!E$ and each $t \in [0,T]$, whence $X(t,x) = Y(t,x)$ for all $(t,x) \in [0,T] \times [0,L]$. This proves uniqueness.

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7 REFERENCES

[1] D. Applebaum. Spectral families of quantum stochastic integrals. *Communications in Mathematical Physics* **170** (1995), pp. 607-628.

[2] D. Applebaum. Quantum martingale measures and stochastic partial differential equations in Fock space. *Journal of Mathematical Physics* **39** (1998), pp. 3019 - 3030.

[3] E.O. Ayoola. On Convergence of one step schemes for weak solutions of quantum stochastic differential equations. Acta Appl. Math. 67(1), 19-58, (2001).

[4] E.O. Ayoola. Topological Properties of Solutions Sets of Lipschitzian Quantum Stochastic Differential Inclusions. Acta Appl. Math. 100(1), 15-37, (2008).

[5] H. Carmichael. *An open systems approach to quantum Optics*, Springer-Verlag, Berlin, Heidelberg, New York (1993).

[6] E. Davies. *The quantum theory of open systems.* Academic Press, London, New York, San Francisco (1976).

[7] G.O.S. Ekhaguere. Lipschitzian Quantum Stochastic Differential Inclusions. Int J. Theoretical Physics, 31(11), 20003-2034 (1992).

[8] A. Guichardet, *Symmetric Hilbert spaces and related topics*, Springer-Verlag, Berlin (1972).

[9] R.L. Hudson and K.R. Parthasarathy. Quantum Ito's Formular and Stochastic Evolutions. Commun, Math. Physics, 93, 301-324 (1984).

[10] K.R. Parthasarathy. *An introduction to quantum stochastic calculus, Birkhauser,* Basel (1992).

[11] G. Da Prato and L. Tubaro, *Stochastic partial differential equations and applications III*, Longman Scientific and Technical (1992).

[12] G. Lindblad. On the generators of quantum dynamical semigroups. Commun. Math. Physics 48 (1976) 119-130.

[13] J.B. Walsh. An introduction to stochastic partial differential equations. In P.L Hennequin (editor). *Ecole d'ete de probabilites de Saint-Flour XIV, 1984, Lecture Notes in Mathematics* **1180** (1986), pp 265-437.

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CARATHEODORY SOLUTION OF QUANTUM STOCHASTIC DIFFERENTIAL INCLUSIONS

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ABSTRACT. This work is concerned with the existence of solution of Quantum stochastic differential inclusions in the sense of Caratheodory. The multivalued stochastic process involved which is non-convex is Scorza-Dragoni lower semicontinuous (SD-l.s.c.) hence giving rise to a directionally continuous selection. The Quantum stochastic differential inclusion is driven by annihilation, creation and gauge operators.

Keywords and phrases: Lower semicontinuous multifunctions, Scorza Dragoni's property, quantum stochastic processes.
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1. INTRODUCTION

The vast applications of differential inclusions in control theory, economic model, evolution inclusions to mention a few, had made the study of differential inclusions of great interest [1], [8], [18]. Likewise, the quantum stochastic differential inclusions which is a multivalued generalization of quantum stochastic differential equation of Hudson and Parthasarathy has vast applications. This extension was first done in [9] in which the existence of solutions of Lipschitzian quantum stochastic differential inclusions was established. The study of solution set of this problem was done in [2], [3] and references cited there. The case of discontinuous quantum stochastic differential inclusions has application in the study of optimal quantum stochastic control [15]. The quantum stochastic calculus is driven by quantum stochastic processes called annihilation, creation and gauge arising from quantum field operators.
A multivalued map that is lower semicontinuous and convex-valued has continuous selection by Michael selection theorem, but if the convexity is dropped the continuous selection does not exist. But for a differential inclusion with lower semicontinuous multifunction that is not convex-valued, there is an analogue of Michael selection

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theorem called the directionally continuous selection [4] which gave rise to a class of discontinuous differential equations. A more general case of this selection for infinite dimensional space is found in [5].

The quantum stochastic differential inclusions considered in this work has its coefficients to be multivalued stochastic processes that have a special form of lower semicontinuity called Scorza-Dragoni lower semicontinuous case. It is noteworthy that the Scorza-Dragoni property is a multivalued generalization of Lusin property[14]. The directionally continuous selection of the Scorza-Dragoni of the multifunction gave rise to a class of quantum stochastic differential equations considered in [16] which have solutions in the sense of Caratheodory. Apart from the application of this work in quantum stochastic control, another motivation for the work is the application of the results in the study of non-convex quantum stochastic evolution inclusions which shall be considered in a later work.

In section 2 we give preliminaries which are essential for the work and we prove the main results in section 3.

2. PRELIMINARY

In what follows, if U is a topological space, we denote by $\mathrm{clos}(U)$, the collection of all non-empty closed subsets of U.

To each pair (D, H) consisting of a pre-Hilbert space D and its completion H, we associate the set $L_w^+(D, H)$ of all linear maps x from D into H, with the property that the domain of the operator adjoint contains D. The members of $L_w^+(D, H)$ are densely-defined linear operators on H which do not necessarily leave D invariant and $L_w^+(D, H)$ is a linear space when equipped with the usual notions of addition and scalar multiplication.

To H corresponds a Hilbert space $\Gamma(H)$ called the boson Fock space determined by H. A natural dense subset of $\Gamma(H)$ consists of linear space generated by the set of exponential vectors(Guichardet, [12]) in $\Gamma(H)$ of the form

$$e(f) = \bigoplus_{n=0}^{\infty} (n!)^{-\frac{1}{2}} \bigotimes^{n} f, \ f \in H$$

where $\bigotimes^0 f = 1$ and $\bigotimes^n f$ is the n-fold tensor product of f with itself for $n \geq 1$.

In what follows, $\mathbb{D}$ is some pre-Hilbert space whose completion is $\mathcal{R}$ and γ is a fixed Hilbert.

$L^2_\gamma(\mathbb{R}_+)$(resp. $L^2_\gamma([0,t))$, resp. $L^2_\gamma([t,\infty))$ $t \in \mathbb{R}_+$) is the space of

square integrable γ-valued maps on $\mathbb{R}_+$ (resp.$[0,t)$, resp.$[t,\infty)$).
The inner product of the Hilbert space $\mathcal{R}\otimes\Gamma(L^2_\gamma(\mathbb{R}_+))$ will be denoted by $\langle .,. \rangle$ and $\|\ .\ \|$ the norm induced by $\langle .,. \rangle$.
Let $\mathbb{E},\mathbb{E}_t$ and $\mathbb{E}^t$, $t>0$ be linear spaces generated by the exponential vectors in Fock spaces $\Gamma(L^2_\gamma(\mathbb{R}_+)),\Gamma(L^2_\gamma([0,t)))$ and $\Gamma(L^2_\gamma([t,\infty)))$ respectively ;

$$\begin{aligned}\mathcal{A}&\equiv L^+_w(\mathbb{D}\underline{\otimes}\mathbb{E},\mathcal{R}\otimes\Gamma(L^2_\gamma(\mathbb{R}_+)))\\ \mathcal{A}_t&\equiv L^+_w(\mathbb{D}\underline{\otimes}\mathbb{E}_t,\mathcal{R}\otimes\Gamma(L^2_\gamma([0,t))))\otimes\mathbb{I}^t\\ \mathcal{A}^t&\equiv\mathbb{I}_t\otimes L^+_w(\mathbb{E}^t,\Gamma(L^2_\gamma([t,\infty)))),\ t>0\end{aligned}$$

where $\underline{\otimes}$ denotes algebraic tensor product and $\mathbb{I}_t$(resp.$\mathbb{I}^t$) denotes the identity map on $\mathcal{R}\otimes\Gamma(L^2_\gamma([0,t))))$(resp.$\Gamma(L^2_\gamma([t,\infty))))$, $t>0$
For every $\eta,\xi\in\mathbb{D}\underline{\otimes}\mathbb{E}$ define

$$\|\ x\ \|_{\eta,\xi}=|\ \langle\eta,x\xi\rangle\ |,\ x\in\mathcal{A}$$

then the family of seminorms

$$\{\|\ .\ \|_{\eta\xi}:\eta,\xi\in\mathbb{D}\underline{\otimes}\mathbb{E}\}$$

generates a topology τ_w , weak topology .
The completion of the locally convex spaces $(\mathcal{A},\tau_w)$,$(\mathcal{A}_t,\tau_w)$ and $(\mathcal{A}^t,\tau_w)$ are respectively denoted by $\widetilde{\mathcal{A}}$,$\widetilde{\mathcal{A}}_t$ and $\widetilde{\mathcal{A}}^t$.
We define the Hausdorff topology on clos$(\widetilde{\mathcal{A}})$ as follows:
For $x\in\widetilde{\mathcal{A}}$, $\mathcal{M},\mathcal{N}\in$ clos$(\widetilde{\mathcal{A}})$ and $\eta,\xi\in\mathbb{D}\underline{\otimes}\mathbb{E}$, define

$$\rho_{\eta\xi}(\mathcal{M},\mathcal{N})\equiv\max(\delta_{\eta\xi}(\mathcal{M},\mathcal{N}),\delta_{\eta\xi}(\mathcal{N},\mathcal{M}))$$

where

$$\begin{aligned}\delta_{\eta\xi}(\mathcal{M},\mathcal{N})&\equiv\sup_{x\in\mathcal{M}}\mathbf{d}_{\eta\xi}(x,\mathcal{N})\text{ and}\\ \mathbf{d}_{\eta\xi}(x,\mathcal{N})&\equiv\inf_{y\in\mathcal{N}}\|\ x-y\ \|_{\eta\xi}\ .\end{aligned}$$

The Hausdorff topology which shall be employed in what follows, denoted by, τ_H , is generated by the family of pseudometrics $\{\rho_{\eta\xi}(.):\eta,\xi\in\mathbb{D}\underline{\otimes}\mathbb{E}\}$
Moreover, if $\mathcal{M}\in clos(\widetilde{\mathcal{A}})$, then $\|\ \mathcal{M}\ \|_{\eta\xi}$ is defined by

$$\|\ \mathcal{M}\ \|_{\eta\xi}\equiv\rho_{\eta\xi}(\mathcal{M},\{0\});$$

for arbitrary $\eta, \xi \in \mathbb{D}\underline{\otimes}\mathbb{E}$.
For $A, B \in clos(\mathbb{C})$ and $x \in \mathbb{C}$, a complex number, define

$$d(x, B) \equiv \inf_{y \in B} \mid x - y \mid$$
$$\delta(A, B) \equiv \sup_{x \in A} d(x, B)$$
$$\text{and } \rho(A, B) \equiv \max(\delta(A, B), \delta(B, A)).$$

Then ρ is a metric on $clos(\mathbb{C})$ and induces a metric topology on the space.
Let $I \subseteq \mathbb{R}_+$. A *stochastic process* indexed by I is an $\widetilde{\mathcal{A}}$-valued measurable map on I.
A stochastic process X is called *adapted* if $X(t) \in \widetilde{\mathcal{A}}_t$ for each $t \in I$.
We write Ad$(\widetilde{\mathcal{A}})$ for the set of all adapted stochastic processes indexed by I.
Definition 1: A member X of Ad$(\widetilde{\mathcal{A}})$ is called
(i) weakly absolutely continuous if the map $t \mapsto \langle \eta, X(t)\xi \rangle$, $t \in I$ is absolutely continuous for arbitrary $\eta, \xi \in \mathbb{D}\underline{\otimes}\mathbb{E}$
(ii) locally absolutely p-integrable if $\| X(.) \|_{\eta\xi}^p$ is Lebesgue - measurable and integrable on $[0, t) \subseteq I$ for each $t \in I$ and arbitrary $\eta, \xi \in \mathbb{D}\underline{\otimes}\mathbb{E}$.
We denote by Ad$(\widetilde{\mathcal{A}})_{wac}$(resp.$L^p_{loc}(\widetilde{\mathcal{A}})$) the set of all weakly, absolutely continuous(resp. locally absolutely p-integrable) members of Ad$(\widetilde{\mathcal{A}})$.
Stochastic integrators: Let $L^\infty_{\gamma,loc}(\mathbb{R}_+)$ [resp.$L^\infty_{B(\gamma),loc}(\mathbb{R}_+)$] be the linear space of all measurable , locally bounded functions from $\mathbb{R}_+$ to γ [resp. to B(γ) , the Banach space of bounded endomorphisms of γ]. If $f \in L^\infty_{\gamma,loc}(\mathbb{R}_+)$ and $\pi \in L^\infty_{B(\gamma),loc}(\mathbb{R}_+)$, then πf is the member of $L^\infty_{\gamma,loc}(\mathbb{R}_+)$ given by $(\pi f)(t) = \pi(t)f(t)$, $t \in \mathbb{R}_+$.
For $f \in L^2_\gamma(\mathbb{R})_+$ and $\pi \in L^\infty_{B(\gamma),loc}(\mathbb{R}_+)$; the annihilation , creation and gauge operators, $a(f), a^+(f)$ and $\lambda(\pi)$ in $L^+_w(\mathbb{D}, \Gamma(L^2_\gamma(\mathbb{R})_+))$ respectively, are defined as:

$$a(f)\mathbf{e}(g) = \langle f, g \rangle_{L^2_\gamma(\mathbb{R}_+)}\mathbf{e}(g)$$
$$a^+(f)\mathbf{e}(g) = \frac{d}{d\sigma}\mathbf{e}(g + \sigma f) \mid_{\sigma=0}$$
$$\lambda(\pi)\mathbf{e}(g) = \frac{d}{d\sigma}\mathbf{e}(e^{\sigma\pi} f) \mid_{\sigma=0}$$

$g \in L^2_\gamma(\mathbb{R})_+$
For arbitrary $f \in L^\infty_{\gamma,loc}(\mathbb{R}_+)$ and $\pi \in L^\infty_{B(\gamma),loc}(\mathbb{R}_+)$, they give rise

to the operator-valued maps A_f, A_f^+ and Λ_π defined by:

$$\begin{aligned}A_f(t) &\equiv a(f\chi_{[0,t)})\\ A_f^+(t) &\equiv a^+(f\chi_{[0,t)})\\ \Lambda_\pi(t) &\equiv \lambda(\pi\chi_{[0,t)})\end{aligned}$$

$t \in \mathbb{R}_+$, where χ_I denotes the indicator function of the Borel set $I \subseteq \mathbb{R}_+$. The maps A_f, A_f^+ and Λ_π are stochastic processes , called annihilation, creation and gauge processes , respectively, when their values are identified with their amplifications on $\mathcal{R}\otimes\Gamma(L^2_\gamma(\mathbb{R}_+))$. These are the stochastic integrators in Hudson and Parthasarathy[13] formulation of boson quantum stochastic integration.
For processes $p, q, u, v \in L^2_{loc}(\widetilde{\mathcal{A}})$, the quantum stochastic integral:

$$\int_{t_0}^{t} \big(p(s)d\Lambda_\pi(s) + q(s)dA_f(s) + u(s)dA_g^+(s) + v(s)ds\big),\ t_0, t \in \mathbb{R}_+$$

is interpreted in the sense of Hudson-Parthasarathy[13] The definition of Quantum stochastic differential Inclusions follows as in [9]. A relation of the form

$$\begin{aligned}dX(t) &\in E(t, X(t))d\Lambda_\pi(t) + F(t, X(t))dA_f(t)\\ &\qquad + G(t, X(t))dA_g^+(t) + H(t, X(t))dt \text{ almost all } t \in I \qquad (1)\\ X(t_0) &= x_0\end{aligned}$$

is called Quantum stochastic differential inclusions(QSDI) with coefficients E, F, G, H and initial data (t_0, x_0).
Equation(1) is understood in the integral form:

$$\begin{aligned}X(t) \in x_0 + \int_{t_0}^{t} &\big(E(s, X(s))d\Lambda_\pi(s) + F(s, X(s))dA_f(s)\\ &+G(s, X(s))dA_g^+(s) + H(s, X(s))ds\big),\ t \in I\end{aligned}$$

called a stochastic integral inclusion with coefficients E, F, G, H and initial data (t_0, x_0)
An equivalent form of (1) has been established in [9], Theorem 6.2

as :

$$(\mu E)(t,x)(\eta,\xi) = \{\langle \eta, \mu_{\alpha\beta}(t)p(t,x)\xi\rangle : p(t,x) \in E(t,x)\}$$
$$(\nu F)(t,x)(\eta,\xi) = \{\langle \eta, \nu_{\beta}(t)q(t,x)\xi\rangle : q(t,x) \in F(t,x)\}$$
$$(\sigma G)(t,x)(\eta,\xi) = \{\langle \eta, \sigma_{\alpha}(t)u(t,x)\xi\rangle : u(t,x) \in G(t,x)\}$$
$$\begin{aligned}\mathbb{P}(t,x)(\eta,\xi) &= (\mu E)(t,x)(\eta,\xi) + (\nu F)(t,x)(\eta,\xi) \\ &\quad + (\sigma G)(t,x)(\eta,\xi) + H(t,x)(\eta,\xi)\end{aligned} \tag{2}$$
$$\begin{aligned}H(t,x)(\eta,\xi) = \{&v(t,x)(\eta,\xi) : v(.,X(.)) \\ &\text{is a selection of } H(.,X(.))\forall\ X \in L^2_{loc}(\widetilde{\mathcal{A}})\}\end{aligned}$$

Then Problem (1) is equivalent to

$$\begin{aligned}\frac{d}{dt}\langle \eta, X(t)\xi\rangle &\in \mathbb{P}(t,X(t))(\eta,\xi) \\ X(t_0) &= x_0\end{aligned} \tag{3}$$

for arbitrary $\eta,\xi \in \mathbb{D}\underline{\otimes}\mathbb{E}$, almost all $t \in I$. Hence the existence of solution of (1) implies the existence of solution of (3) and vice-versa.

As explained in [9], for the map $\mathbb{P}$,

$$\mathbb{P}(t,x)(\eta,\xi) \neq \widetilde{\mathbb{P}}(t,\langle \eta, x\xi\rangle)$$

for some complex-valued multifunction $\widetilde{\mathbb{P}}$ defined on $I\times\mathbb{C}$ for $t \in I$, $x \in \widetilde{\mathcal{A}}$, $\eta,\xi \in \mathbb{D}\underline{\otimes}\mathbb{E}$.

Definition 2: For an arbitrary $\eta,\xi \in \mathbb{D}\underline{\otimes}\mathbb{E}$, let $M > 0$, we define a set $\Gamma^M_{\eta\xi}$, as

$$\Gamma^M_{\eta\xi} = \{(t,x) \in I \times \widetilde{\mathcal{A}} :\mid \langle \eta, x\xi\rangle \mid \leq Mt\}$$

Let $(t_0,x_0) \in I\times\widetilde{\mathcal{A}}$ and $\epsilon > 0$. For an arbitrary $\eta,\xi \in \mathbb{D}\underline{\otimes}\mathbb{E}$, $(t_0,x_0) \in I \times \widetilde{\mathcal{A}}$ and $\delta > 0$, the family of conical neighbourhoods;

$$\Gamma^M_{\eta\xi}((t_0,x_0),\delta) = \{(t,x) \in I \times \widetilde{\mathcal{A}} :\parallel x - x_0 \parallel_{\eta\xi} \leq M(t-t_0),$$
$$t_0 \leq t < t_0 + \delta\}$$

generates a topology, τ^+, which satisfies the following property:
(P) For every pair of sets $A \subset B$, with A closed and B open(in the original topology), there exists a set C, closed-open with respect to τ^+, such that $A \subset C \subset B$.
This topology follows from [5] and the references cited there.

Definition 3: (i) For an arbitrary pair $\eta,\xi \in \mathbb{D}\underline{\otimes}\mathbb{E}$ a map $\Phi : I \times \widetilde{\mathcal{A}} \to \widetilde{\mathcal{A}}$ will be said to be $\Gamma^M_{\eta\xi}$-continuous(directionally continuous or τ^+-continuous) at a point $(t_0, x_0) \in I \times \widetilde{\mathcal{A}}$, if for every $\epsilon > 0$ there exists $\delta > 0$ such that

$$\| \Phi(t,x) - \Phi(t_0,x_0) \|_{\eta\xi} \leq \epsilon \text{ if } t_0 \leq t \leq t_0 + \delta \text{ and } \| x - x_0 \|_{\eta\xi} \leq M(t - t_0)$$

(ii) For an arbitrary $\eta, \xi \in \mathbb{D}\underline{\otimes}\mathbb{E}$, $S \subset \widetilde{\mathcal{A}}$, a sesquilinear-form valued map $\Psi : S \to 2^{sesq(\mathbb{D}\underline{\otimes}\mathbb{E})^2}$ will be said to be lower semicontinuous on S if for every closed subset C of $\mathbb{C}$ the set $\{s \in S : \Psi(s)(\eta, \xi) \subset C\}$ is closed in S.

We remark that if E, F, G, H are lower semicontinuous on S, then the sesquilinear-form valued $\mathbb{P}$ is lower semicontinuous on S.

A multivalued generalization of Lusin property which is called Scorza - Dragoni property [14] employed in [6] is used to define the form of lower semicontinuity in this work. The well-known Lusin property is the following.

Definition 4:(Lusin's property)Let X and Y be two separable metric spaces and let $f : I \times X \to Y$ be function such that
(i)$t \to f(t, u)$ is measurable for every $u \in X$
(ii) $u \to f(t, u)$ is continuous for almost every $t \in I$, $I \subseteq \mathbb{R}_+$.
Then, for each $\epsilon > 0$, there exists a closed set $A \subseteq I$ such that $\lambda(I \setminus A) < \epsilon$, (λ is the Lebesgue measure on $\mathbb{R}$) and the restriction of f to $A \times X$ is continuous.

Definition 5: A sequilinear- form valued map $\Psi : [0, T] \times \widetilde{\mathcal{A}} \to 2^{sesq(\mathbb{D}\underline{\otimes}\mathbb{E})^2}$ is Scorza-Dragoni lower semicontinuous (SD-l.s.c.) on $[0, T] \times \widetilde{\mathcal{A}}$ if there exists a sequence of disjoint compact sets $J_n \subset [0, T]$, with $\text{meas}([0, T] \setminus \bigcup_{n\in\mathbb{N} J_n}) = 0$ such that Ψ is lower semicontinuous on each set $J_n \times \widetilde{\mathcal{A}}$.

If Ψ is lower semicontinuous and convex-valued then by Michael selection theorems, there exists continuous selection of Ψ. But if the convexity is removed and Ψ is not decomposable valued multifunction then the existence of continuous selection is not guaranteed. However, a non-convex analogue of Michael selection is Directional continuous selection result in [4] and for infinite dimensional space in [5]. We established in this work that such selection exists for SD-lsc multivalued stochastic process.

For an arbitrary $\eta, \xi \in \mathbb{D}\underline{\otimes}\mathbb{E}$, if $\Psi \in \mu E, \nu F, \sigma G, H$ appearing in (1) are SD-lsc then the map $(t, x) \to \mathbb{P}(t, x)(\eta, \xi)$ is SD-lsc.

A quantum stochastic differential inclusion will be said to be SD-lower semicontinuous if the coefficients are SD-lsc.

3. MAIN RESULTS

Theorem 1: For almost all $t \in I$, $\eta, \xi \in \mathbb{D}\underline{\otimes}\mathbb{E}$. Suppose the following holds:
(i) The maps $X \to \Psi(t,X)(\eta,\xi)$, $\Psi \in \{\mu E, \nu F, \sigma G, H\}$ are non-empty lower semicontinuous multivalued stochastic processes
(ii)The maps $t \to \Psi(t,X)(\eta,\xi)$ are closed
(iii) τ^+ is a topology on $I \times \widetilde{\mathcal{A}}$ with property (P).
Then the sesquilinear form valued multifunction, $(t, X(t)) \to \mathbb{P}(t, X(t))(\eta,\xi)$

$$\mathbb{P}(t,X(t))(\eta,\xi) = (\mu E)(t,X(t))(\eta,\xi) + (\nu F)(t,X(t))(\eta,\xi) + (\sigma G)(t,X(t))(\eta,\xi) + H(t,X(t))(\eta,\xi)$$

admits a τ^+-continuous selection.
Proof: $\mathbb{P}$ is non-empty , since each of $\Psi \in \{\mu E, \nu F, \sigma G, H\}$ is non-empty.
Therefore, $\mathbb{P}$ is a non-empty lower semicontinuous sesquilinear form-valued multifunction.
We shall employ a similar procedure as in the proof of Theorem 3.2 in [5] to construct a τ^+-continuous ϵ-approximate selections P_ϵ of $\mathbb{P}$, hence by inductive hypothesis we obtain a τ^+-continuous selection P of $\mathbb{P}$.
Let $\epsilon > 0$ be fixed , since $X \to \mathbb{P}(t,X)(\eta,\xi)$ is lower semicontinuous , for every $X(t) \in \widetilde{\mathcal{A}}$, we choose point $y_{\eta\xi,X}(t) \in \mathbb{P}(t,X(t))(\eta,\xi)$ and neighbourhood U_X of $X(t)$ such that

$$\inf_{y_{\eta\xi,\mathbb{P}}(t)\in\mathbb{P}(t,X(t'))(\eta,\xi)} \mid y_{\eta\xi,X}(t) - y_{\eta\xi,\mathbb{P}}(t) \mid < \epsilon \ \forall \ X(t') \in U_X \qquad (4)$$

Now , let $(V_\alpha)_{\alpha\in\beta^\epsilon}$ be a local finite open refinement of $(U_X)_{X(t)\in\widetilde{\mathcal{A}}}$, with $V_\alpha \subset U_{X_\alpha}$, and let $(W_\alpha)_{\alpha\in\beta^\epsilon}$ be another open refinement such that $cl(W_\alpha) \subset V_\alpha$ for all $\alpha \in \beta^\epsilon$. By property (P), for each α , we can choose a set Z_α , clopen w.r.t. τ^+, such that

$$cl(W_\alpha) \subset int(Z_\alpha) \subset cl(Z_\alpha) \subset V_\alpha \qquad (5)$$

Then $(Z_\alpha)_\alpha$ is a local finite τ^+ clopen covering of $\widetilde{\mathcal{A}}$. Let $\preceq$ be a well-ordering of the set β^ϵ, define for each $\alpha \in \beta^\epsilon$,

$$\Omega^\epsilon_\alpha = Z_\alpha \setminus (\bigcup_{\lambda<\alpha} Z_\lambda)$$

Set $\mathcal{O}^\epsilon = (\Omega^\epsilon_\alpha)$, $\alpha \in \beta^\epsilon$. By well-ordering , every $x \in \widetilde{\mathcal{A}}$ belongs to exactly one set $\Omega^\epsilon_{\overline{\alpha}}$ where $\overline{\alpha} = \min\{\alpha \in \beta^\epsilon : x \in Z_\alpha\}$. Hence , $\mathcal{O}^\epsilon$ is a partition of $\widetilde{\mathcal{A}}$. Moreover, since Z_α is locally finite(wrt τ and therefore wrt τ^+), the sets $\bigcup_{\lambda\leq\alpha} Z_\lambda$ are τ^+ clopen. Hence $\mathcal{O}^\epsilon$ is a τ^+ clopen disjoint covering of $\widetilde{\mathcal{A}}$ such that, $\{cl(\Omega^\epsilon_\alpha)\}$ refines $(V_\alpha)_\alpha$.

By setting $y^{\epsilon}_{\eta\xi,\alpha} = y_{\eta\xi,X_\alpha}$ and $P_\epsilon(t, X(t))(\eta, \xi) = y_{\eta\xi,X_\alpha}$, $\forall \alpha \in \beta^{\epsilon}$ we have τ^+ continuous function P_ϵ, which by (4), satisfies

$$\inf_{y_{\eta\xi,\mathbb{P}}(t) \in \mathbb{P}(t,X(t))(\eta,\xi)} \mid P_\epsilon(t, X(t))(\eta, \xi) - y_{\eta\xi,\mathbb{P}}(t) \mid < \epsilon$$

Therefore , there exists an ϵ-approximate selection P_ϵ of $\mathbb{P}$.
Since ϵ was arbitrarily chosen ,thus we have a τ^+-continuous selection P of $\mathbb{P}$. □

Theorem 2: Suppose the following holds for an arbitrary $\eta, \xi \in \mathbb{D}\underline{\otimes}\mathbb{E}$, $\Psi \in \{\mu E, \nu F, \sigma G, H\}$:
(i) $t \to \Psi(t, X(t))(\eta, \xi)$ are measurable for all $X \in \widetilde{\mathcal{A}}$
(ii) $X \to \Psi(t, X(t))(\eta, \xi)$ are SD-lower semicontinuous with respect to a seminorm $\| \,.\, \|_{\eta\xi}$, for almost all $t \in I$
(iii) Ψ are integrably bounded, that is, there exists $L^{\Psi}_{\eta\xi}(t) \in L^1(I)$ such that, a.e. $t \in I$, for all $X \in \widetilde{\mathcal{A}}$,

$$\inf_{y \in \Psi(t,x)(\eta,\xi)} \mid y \mid \leq L^{\Psi}_{\eta\xi}(t).$$

Then the SD-lower semicontinuous quantum stochastic differential inclusions

$$\frac{d}{dt}\langle \eta, X(t)\xi \rangle \in \mathbb{P}(t, X(t))(\eta, \xi) \qquad X(t_0) = x_0 \tag{6}$$

has an adapted weakly absolutely continuous solution in the sense of Caratheodory.

Proof: Since for arbitrary $\eta, \xi \in \mathbb{D}\underline{\otimes}\mathbb{E}$, $\Psi \in \mu E, \nu F, \sigma G, H$ are SD-lower semicontinuous then $\mathbb{P}(t, x)(\eta, \xi)$ is SD-lower semicontinuous, $\forall x \in \widetilde{\mathcal{A}}$, a.e. $t \in I$. The sequence of disjoint compact sets $J_n = \bigcap_\Psi J^{\Psi}_n$ and $meas(I \setminus \cup_{n \in \mathbb{N}} J_n) = 0$ such that $\mathbb{P}(.,.)(\eta, \xi)$ restricted to $\Omega_n = J_n \times \widetilde{\mathcal{A}}$ is lower semicontinuous, with respect to $\| \,.\, \|_{\eta\xi}$.
Also, suppose $L_{\eta\xi} = 5 \max L^{\Psi}_{\eta\xi}(t)$, then a.e. $t \in I$,

$$\inf_{y \in \mathbb{P}(t,x)(\eta,\xi)} \mid y \mid \leq L_{\eta\xi}(t),$$

for all $X \in \widetilde{\mathcal{A}}$
For each $n \geq 1$, we can apply Theorem (1) and obtain τ^+-continuous selections $P_n \in \mathbb{P}$.
For an arbitrary selection g from $\mathbb{P}$, if we define

$$P(t, X)(\eta, \xi) = \begin{cases} P_n(t, X)(\eta, \xi) & \text{if } t \in J_n, \\ g(t, X)(\eta, \xi) & \text{if } t \notin \cup_{n \in \mathbb{N}} J_n \end{cases}$$

then P is a τ^{+}-continuous selection of $\mathbb{P}$, such that $\mid P(t,x)(\eta,\xi) \mid \leq L_{\eta\xi}(t) < L_{n,\eta\xi}$, for every $(t,X) \in I \times \widetilde{\mathcal{A}}$, $\eta, \xi \in \mathbb{D}\underline{\otimes}\mathbb{E}$.
Then by applying Lusin's property to each bound of $L_{n,\eta\xi}$, $n \in \mathbb{N}$ the set of solutions of τ^{+}-continuous quantum stochastic differential equations is the solution set of (6) in the sense of Caratheodory. □

REFERENCES

[1] J.P. Aubin and A. Cellina, *Differential Inclusions*, Springer- Verlag, Berlin, 1984.
[2] E.O. Ayoola, Continuous selections of solution sets of Lipschitzian quantum stochastic differential inclusions.*Int. J. Theor.Phys.*43, 10, 2041-2059, 2004.
[3] E.O. Ayoola, Topological properties of solution sets of Lipschitzian quantum stochastic differential inclusions.*Acta Appl.Math*, 100, 15-37, 2008.
[4] A. Bressan and A. Cortesi, Directionally continuous selections in Banach spaces, *Non linear Analysis. Theory, Method and Applications* 13, 987-992, 1989.
[5] A. Bressan and G. Colombo, Selections and representations of multifunctions in paracompact spaces *Studia Mathematical* 102, 209-216, 1992.
[6] A. Cernea and V. Staicu, Directionally continuous selections and nonconvex evolution inclusions *Set-Valued Analysis* 11, 9-20, 2003.
[7] K. Deimling, *Non linear Functional Analysis*, Springer-Verlag, Berlin, 1985.
[8] K. Deimling, *Multivalued differential equations*, Walter de Gruyter , 1992.
[9] G.O.S. Ekhaguere, Lipschitzian quantum stochastic differential inclusions *Int. J. Theor. Phys.*, 31, 11, 2003-2034, 1992.
[10] G.O.S. Ekhaguere, Quantum stochastic evolutions *Int. J. Theor. Phys.*, 35, 9, 1909-1946, 1996.
[11] G.O.S. Ekhaguere, Topological solutions of Non commutative stochastic differential equations *Stoch. Anal. and Appl.* 25, 9, 961-993, 2007.
[12] A. Guichardet, *Symmetric Hilbert spaces and related topics* Lecture Notes in Mathematics, 261, Springer-Verlag, Berlin, 1972.
[13] R.L. Hudson and K.R. Parthasarathy, Quantum Ito's formula and stochastic evolutions*Comm. Math. Phys.*, 93, 3, 301-323, 1984.
[14] A. Kucia, Scorza-Dragoni type theorems ,*Fundamental Mathematical* 138, 197-203, 1991.
[15] M.O. Ogundiran and E.O. Ayoola, Mayer problem for Quantum stochastic control *J. Math. Phys.*51, 1, 023521-023521-8, 2010.
[16] M.O. Ogundiran and E.O. Ayoola, Directionally continuous quantum stochastic differential equations *Far East J. Appl. Maths.* 57, 1, 33-48, 2011.
[17] M. Reed and B. Simon, *Methods of Modern Mathematical Physics I. Functional Analysis*, 2nd edn., Academic press, New York-London, 1980.
[18] A. Tolstonogov, *Differential inclusions in a Banach space*, Mathematics and its applications, Kluwer Academics Publishers, New York, 2000.

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CONSTRUCTION OF APPROXIMATE ATTAINABILITY SETS FOR LIPSCHITZIAN QUANTUM STOCHASTIC DIFFERENTIAL INCLUSIONS

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ABSTRACT

We present a numerical method for constructing, with a specified accuracy attainability sets for Lipschitzian quantum stochastic differential inclusions. Results here generalize the Komarov-Pevchikh results concerning classical differential inclusions to the present noncommutative quantum setting involving unbounded linear operators on a Hilbert space.

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1. INTRODUCTION

There are many problems in quantum stochastic control theory and in the solutions of inclusions arising from regularization of discontinuous quantum

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stochastic differential equations that are concerned with the construction of attainability sets for quantum stochastic differential inclusions (QSDI). As in the classical cases attainability sets play prominent roles in the solutions of these problems. The computation of optimal control maps and the trajectories depends on the ability to evaluate or estimate the attainability sets of quantum controllable systems. Approximations and estimations of attainability sets for quantum stochastic differential inclusions are questions that have not been adequately addressed (if any at all) unlike their classical counterparts (see [10,11] and the references therein). Moreover in our bid to develop numerical schemes for quantum stochastic inclusions of Lipschitzian, hypermaximal monotone and evolution types (see [5,6,7]) estimates of their attainability sets are crucial for the task.The development of numerical schemes for quantum stochastic inclusions are very important since a sizeable number of equations arising from applications are discontinous but can be reformulated as regularized inclusions. Emphasis so far has been largely on numerical procedures for continuous quantum stochastic differential equations with high degree of smoothness of their matrix elements (see [2,3,4]). In this paper, we present a method for approximating attainability sets for QSDI with a given accuracy.This is accomplished by adapting the techniques and arguments of Komarov and Pevchikh [11] concerning classical inclusions to the present noncommutative quantum setting involving inclusions in certain locally convex spaces.

The rest of the paper is organised as follows:- Section 2 is devoted to some preliminary notations and statements of basic results of Ekhaguere [5] concerning quantum stochastic generalization of the Fillipov existence theorem for classical differential inclusions. The main result of this paper concerning the algorithm for constructing the attainability set is established in Section 3.

2. PRELIMINARY NOTATIONS AND STATEMENTS

2.1. Notations

In what follows, as in Ekhaguere [5], we employ the locally convex topological state space $\tilde{\mathcal{A}}$ of noncommutative stochastic processes whose topology is generated by the family of seminorms $\|x\|_{\eta\xi} = |\langle \eta, x\xi\rangle| \; x \in \tilde{\mathcal{A}}\,\eta, \xi \in I\!D \underline{\otimes} I\!E$. Moreover, we adopt the definitions and notations of spaces $Ad(\tilde{\mathcal{A}})$, $Ad(\tilde{\mathcal{A}})_{wac}$, $L^p_{loc}(\tilde{\mathcal{A}})$, $L^\infty_{\gamma,loc}(I\!R_+)$.

If A is a topological space,then clos(A) (resp. comp(A)) denotes the collection of nonvoid closed (resp. compact) subsets of A.

We employ the Hausdorff topology τ_H on clos$(\tilde{\mathcal{A}})$ determined by a family of pseudo-metrics $\{\rho_{\eta\xi}(\cdot), \eta, \xi \in I\!D \underline{\otimes} I\!E\}$ on clos$(\tilde{\mathcal{A}})$ as explained in Ekhaguere [5].

Similarly, for $A,\ B \in \text{clos}(\mathbb{C})$ and $x \in \mathbb{C}$, the complex numbers, let

$$\mathbf{d}(x, A) = \inf_{y\in A} |x - y|$$
$$\delta(A, B) = \sup_{x\in A} \mathbf{d}(x, B)$$

and

$$\rho(A, B) = \max(\delta(A, B), \delta(B, A))$$

Then we employ the metric topology on $clos(\mathbb{C})$ induced by ρ. The set-theoretic operations are adopted as usual.

2.2. Quantum Stochastic Differential Inclusions

A multivalued stochastic process indexed by $I \subseteq I\!R_+$ is a mutifunction on I with values in $\text{clos}(\tilde{\mathcal{A}})$. As in Ekhaguere [5], the set of all locally p-integrable multivalued stochastic processes will be denoted by $L^p_{loc}(\tilde{\mathcal{A}})_{mvs}, p \in (0, \infty)$ while $L^p_{loc}(I \times \tilde{\mathcal{A}})_{mvs}$ is the set of maps $\Phi : I \times \tilde{\mathcal{A}} \mapsto \text{clos}(\tilde{\mathcal{A}})$ such that the map $t \mapsto \Phi(t, X(t))$, $t \in I$ lies in $L^p_{loc}(\tilde{\mathcal{A}})_{mvs}$ for every $X \in L^p_{loc}(\tilde{\mathcal{A}})$.

For $f, g \in L^\infty_{\gamma,loc}(I\!R_+)$, $\pi \in L^\infty_{B(\gamma),loc}(I\!R_+)$, $\mathbf{1}$ is the identity map on $\mathcal{R} \otimes \Gamma(L^2_\gamma(I\!R_+))$ and M is any of the processes $A_f,\ A^+_g, \wedge_\pi$ and $s \mapsto s\mathbf{1}$, $s \in I\!R_+$, then the multivalued stochastic integral $\int_{t_0}^t \Phi(s, X(s))dM(s)$ is adopted as in Ekhaguere [5].

For E, F, G, H lying in $L^2_{loc}(I \times \tilde{\mathcal{A}})_{mvs}$, we consider the quantum stochastic integral inclusion given by

$$X(t) \in X_0 + \int_{t_0}^t (E(s, X(s))d \wedge_\pi (s) + F(s, X(s))dA_f(s) + G(s, X(s))dA^+_g(s) + H(s, X(s))ds), \quad t \in I, \tag{2.21}$$

with initial data (t_0, X_0) as in Ekhaguere [5].

For arbitrary $\eta, \xi \in I\!D \underline{\otimes} I\!E$, the inclusion (2.21) has been shown in [5, Theorem 6.3] to be equivalent to the following first order initial value nonclassical ordinary differential inclusion of nonclassical type

$$\frac{d}{dt}\langle \eta, X(t)\xi\rangle \in P(t, X(t))(\eta, \xi)$$
$$X(t_0) = X_0, \quad t \in [t_0, T] \tag{2.22}$$

Where $P(t, x)(\eta, \xi)$ is a multivalued sesquilinear form on $I\!D \underline{\otimes} I\!E$ with values in $\text{clos}(\mathbb{C})$ (see [5, section 6] for details).

Next we summarise the result of Ekhaguere [5, Theorem 8.2] which generalize the Fillipov existence theorem concerning classical differential inclusion to the noncommutative quantum setting.

Theorem 2.21 [5]. *Assume that the following conditions hold*

(a) *$Z : I \mapsto \tilde{\mathcal{A}}$ lies in $Ad(\tilde{\mathcal{A}})_{wac}$ such that there exists positive function $W_{\eta\xi}(t)$ satisfying*

$$\mathbf{d}\left(\frac{d}{dt}\langle \eta, Z(t)\xi\rangle, P(t, Z(t)(\eta, \xi)\right) \le W_{\eta\xi}(t)$$

(b) *Each of the maps E, F, G, H is Lipschitzian from $Q_{Z,\theta}$ to $(\mathrm{clos}(\tilde{\mathcal{A}}), \tau_H)$ where*

$$Q_{Z,\theta} = \{(t, x) \in I \times \tilde{\mathcal{A}} : \|x - Z(t)\|_{\eta\xi} \le \theta, \forall \eta, \xi \in I\!D \underline{\otimes} I\!E \text{ and } \|x_0 - Z(t_0)\|_{\eta\xi} \le \theta\}$$

(c) *For arbitrary $\eta, \xi \in I\!D \underline{\otimes} I\!E$, $t \in I$,*

$$E_{\eta\xi}(t) = \|x_0 - Z(t_0)\|_{\eta\xi} \exp\left(\int_{t_0}^{t} ds K^P_{\eta\xi}(s)\right) + \int_{t_0}^{t} ds W_{\eta\xi}(s) \times \exp\left(\int_{t_0}^{t} dr K^P_{\eta\xi}(r)\right)$$

If in addition, E, F, G, H are continuous from $I \times \tilde{\mathcal{A}}$ to $(clos(\tilde{\mathcal{A}}), \tau_H)$, then there exists a solution Φ of inclusion (2.21) *such that*

$$\|\Phi(t) - Z(t)\|_{\eta\xi} \le E_{\eta\xi}(t), ‘ \quad t \in J \tag{2.23}$$

and

$$\left|\frac{d}{dt}\langle \eta, \Phi(t)\xi\rangle - \frac{d}{dt}\langle \eta, Z(t)\xi\rangle\right| \le K^P_{\eta\xi}(t)E_{\eta\xi}(t) + W_{\eta\xi}(t)$$

for almost all $t \in J$ where $J = \{t \in I : E_{\eta\xi}(t) \le \theta\}$ and $K^P_{\eta\xi} : I \mapsto (o, \infty)$ is the Lipschitz function for P lying in $L^1_{loc}(I)$.

3. THE ALGORITHM FOR APPROXIMATING THE ATTAINABILITY SETS

Let $\eta, \xi \in I\!D \underline{\otimes} I\!E$ be arbitrary. For simplicity of notations, we consider the autonomous version of inclusion (2.22) given by

$$\begin{aligned} &\frac{d}{dt}\langle \eta, X(t)\xi\rangle \in P(X(t)(\eta, \xi)) \\ &X(t_0) = X_0 \end{aligned} \tag{3.1}$$

In connection with subsequent results, we list the following statements

(a) The multivalued sesquilinear form $P(x)(\eta,\xi)$ is Lipschitzian with Lipschitz constant $K_{\eta\xi}$, i.e

$$\rho(P(x)(\eta,\xi),\ P(y)(\eta,\xi)) \leq K_{\eta\xi}\|x-y\|_{\eta\xi} \quad \forall x, y \in \tilde{\mathcal{A}}$$

(b) For each $x \in \tilde{\mathcal{A}}, \eta, \xi \in I\!D \underline{\otimes} I\!E$, $P(x)(\eta,\xi)$ is convex and compact in $\mathbb{C}$, the complex field and contained in a sphere of sufficiently large radius r and centre at the origin.

(c) For $z \in \tilde{\mathcal{A}}, \eta, \xi \in I\!D \underline{\otimes} I\!E$ we denote by $z_{\eta\xi} := \langle \eta, z\xi\rangle$ and $S_r(z_{\eta\xi}) = \{x \in \mathbb{C} : |x - z_{\eta\xi}| \leq r\}$, the sphere of radius r centre at $z_{\eta\xi}$ in $\mathbb{C}$.

Definition. 3.1. Let $X_0 \in \text{Comp}(\tilde{\mathcal{A}}), \eta\,\xi \in I\!D \underline{\otimes} I\!E$, we denote by $X(\tau, X_0)(\eta,\xi) := X(\tau)(\eta,\xi)$ the set in $\mathbb{C}$ formed at the instance $\tau \geq 0$ by the endpoints of the matrix element $\langle \eta, \Phi(\cdot)\xi\rangle$ of solutions $\Phi(\cdot)$ of differential inclusions (3.1) that starts in $X_{0,\eta,\xi} := \{\langle \eta, x_0\xi\rangle / x_0 \in X_0\}$. $X(\tau)(\eta,\xi)$ is referred to as the attainability set of (3.1) at the instant τ.

Next, we describe an algorithm for the numerical approximation of the attainability set $X(\tau, X_0)(\eta,\xi)$ in a given interval $[0, T]$ and with a given accuracy. Our method, which is an adaptation of the techniques and arguments in [11], consists of constructing a multivalued map $Y_{\eta\xi} : [0, T] \mapsto \text{Comp}(\mathbb{C})$ that satisfies the inequality

$$\rho(X(t, X_0)(\eta,\xi), Y_{\eta\xi}(t)) \leq \epsilon$$

$\forall\ t \in [0, T]$ and for arbitrary small number ϵ, for each pair $\eta, \xi \in I\!D \underline{\otimes} I\!E$.

We fix an arbitrary q-net $\{X_q\}$ in $\tilde{\mathcal{A}}$ so that $\{\langle \eta, X_q\xi\rangle\}$ is a q-net in $\mathbb{C}$, the complex field such that any bounded set in $\mathbb{C}$ contains only a finite number of points belonging to the q-net $\{\langle \eta, X_q\xi\rangle\}$, $q \in I\!R_+$.

We obtain an N equal partition of the interval $[0, T] = \{[t_i, t_{i+1}] i = 0, 1, 2 \ldots N-1\}$ of length $h = \frac{T}{N}$ so that $t_i = i\frac{T}{N}$, $t_0 = 0$

We shall approximate $X(t)(\eta,\xi)$ by a mapping $t \mapsto Y_{\eta\xi}(t)$ such that at the instants t_i, $i = 0, 1, 2..N$, the set $Y_{i,\eta\xi} := Y_{\eta\xi}(t_i)$ are chosen from the q-net $\{\langle \eta, X_q\xi\rangle\}$ and for $s \in [t_i, t_{i+1}]$ they are given by

$$Y_{\eta\xi}(s) = \frac{1}{h}[Y_{i+1,\eta\xi}(s - t_i) + Y_{i,\eta\xi}(t_{i+1} - s)] \tag{3.2}$$

The algorithm for constructing the sets $Y_{i,\eta\xi}$, $i = 0, 1.2..N$ is as follows:
$Y_{0,\eta\xi}$ is an approximation of the initial set $X_{0,\eta\xi} = \{\langle \eta, x_0\xi\rangle / x_0 \in X_0\}$ by points from the q-net $\{\langle \eta, X_q\xi\rangle\}$ given by

$$Y_{0,\eta\xi} = \{u_{\eta\xi} \in \{\langle \eta, X_q\xi\rangle\} / \mathbf{d}(u_{\eta\xi}, X_{0,\eta\xi}) \leq q\} \tag{3.3}$$

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The sets $Y_{i,\eta\xi}$, $i = 1, 2 \ldots N$ are given by the equality

$$Y_{i,\eta\xi} = \bigcup_{y_{\eta\xi} \in Y_{i-1,\eta\xi}} W(y_{\eta\xi}) \tag{3.4}$$

and

$$W(y_{\eta\xi}) = \{p_{\eta\xi} \in \{\langle \eta, X_q \xi \rangle\} \bigcap S_{q+hr}(y_{\eta\xi}) / \mathbf{d}(y_{\eta\xi}, p_{\eta\xi} - hP(p)(\eta, \xi)) \leq \gamma$$

Where $p_{\eta\xi} = \langle \eta, p\xi \rangle$ $y_{\eta\xi} = \langle \eta, y\xi \rangle$ for some $p, y \in \{X_q\} \subset \tilde{\mathcal{A}}$ and $\gamma = 2q + hK_{\eta\xi}q + h^2 r K_{\eta,\xi}$.

The formulae (3.2)–(3.4) define the map $t \mapsto Y_{\eta\xi}(t)$ in the interval [0,T]. We have the following result.

Theorem 3.2. *Let the parameters q and N be chosen such that*

$$N \geq \frac{4rT \exp(TK_{\eta\xi})}{\epsilon} \tag{3.5}$$

$$q \leq \frac{K_{\eta\xi} r T^2}{N^2} \tag{3.6}$$

Then the inequality

$$\rho(X(t, X_0)(\eta, \xi), \quad Y_{\eta\xi}(t)) \leq \epsilon$$

$\forall t \in [0, T]$ *holds.*

Proof: The proof will be accomplished in two parts. First we prove that

$$Y_{N,\eta\xi} \subset X(T)(\eta, \xi) + S_\epsilon(0) \tag{3.7}$$

by taking an arbitrary point $y_{N,\eta\xi} \in Y_{N,\eta\xi}$ and show that a trajectory $\Phi(\cdot)$ of the inclusion (3.1) exists such that

$$\|\Phi(T) - y_N\|_{\eta\xi} \leq \epsilon$$

where $y_{N,\eta\xi} = \langle \eta, y_N \xi \rangle$ for some member y_N of the q-net $\{X_q\} \subset \tilde{\mathcal{A}}$.

From $y_{N,\eta\xi}$, we construct a sequence of elements $y_{i,\eta\xi}$, $i = 0, 1 \ldots N - 1$ such that $y_{i,\eta\xi} \in Y_{i,\eta\xi}$ as follows:
If $y_{i,\eta\xi}$ is already known, then we take an arbitrary element from the set

$$y_{i,\eta\xi} - hP(y_i)(\eta, \xi) + S_\gamma(0)$$

as $y_{i-1,\eta\xi}$.

It follows from the definition of $Y_{i,\eta\xi}$ that there exists at least one point $y_{i-1,\eta\xi} \in Y_{i-1,\eta\xi}$ since

$$(y_{i,\eta\xi} - hP(y_i)(\eta, \xi) + S_\gamma(0)) \bigcap Y_{i-1,\eta\xi} \neq \emptyset.$$

By the definition of $Y_{i,\eta\xi}$, $y_{0,\eta\xi} \in Y_{0,\eta\xi}$ implies that

$$\mathbf{d}(y_{0,\eta\xi}, X_{0,\eta\xi}) \leq q.$$

The distance between $y_{i,\eta\xi}$, and $y_{i+1,\eta\xi}$ can be estimated as follows:
Let $y_{i,\eta\xi} \in Y_{i,\eta\xi}$ then
$y_{i,\eta\xi} \in W(y'_{\eta\xi})$ for some $y'_{\eta\xi} \in Y_{i-1,\eta\xi}$

$$\text{i.e} \quad |y_{i,\eta\xi} - y'_{\eta\xi}| < q + hr \tag{3.8}$$

and $y_{i+1,\eta\xi} \in Y_{i+1,\eta\xi}$ implies that $y_{i+1,\eta\xi} \in W(y''_{\eta\xi})$ for some $y''_{\eta\xi} \in Y_{i,\eta\xi}$

$$\text{i.e} \quad |y_{i+1,\eta\xi} - y''_{\eta\xi}| < q + hr \tag{3.9}$$

setting $y_{i,\eta\xi} = y''_{\eta\xi} \in Y_{i,\eta\xi}$, then from (3.9)

$$|y_{i+1,\eta\xi} - y_{i,\eta\xi}| \leq q + hr$$

$$\text{i.e} \quad \|y_{i+1} - y_i\|_{\eta\xi} \leq q + hr < \gamma + hr \quad \forall i.$$

We now consider a complex valued map defined by

$$Z_{\eta\xi}(s) = \frac{1}{h}[(t_{i+1} - s)y_{i,\eta\xi} + (s - t_i)y_{i+1,\eta\xi}]$$

for $s \in [t_i, t_{i+1}]$, $i = 0, 1, 2 \ldots N-1$. Clearly, $Z_{\eta\xi}(t_i) = y_{i,\eta\xi}$ and for $s \in [t_i, t_{i+1})$

$$\frac{d}{ds}Z_{\eta\xi}(s) = \frac{1}{h}(y_{i+1,\eta\xi} - y_{i,\eta\xi})$$

Again, since

$$y_{i,\eta\xi} \in y_{i+1,\eta\xi} - hP(y_{i+1})(\eta, \xi) + S_\gamma(0)$$

then,

$$\frac{d}{ds}Z_{\eta\xi}(s) = \frac{1}{h}(y_{i+1,\eta\xi} - y_{i,\eta\xi}) \in P(y_{i+1})(\eta, \xi) + \frac{\gamma}{h}S_1(0) \tag{3.10}$$

for some map $Z : I \mapsto \tilde{\mathcal{A}}$. It is clear that Z is weakly absolutely continuous by definition.

Inclusion (3.10) implies that

$$\mathbf{d}\left(\frac{d}{ds}\langle \eta, Z(s)\xi\rangle, P(Z(t_{i+1}))(\eta, \xi)\right) \leq \frac{\gamma}{h} \tag{3.11}$$

By the noncommutative quantum version of the Filipov Theorem (2.21), there exists a solution Φ of (3.1) with the initial condition $\Phi(0) = x_0$ where $\|x_0 - y_0\|_{\eta\xi} \leq q$ such that for all $t \in [0, T]$,

$$\|\Phi(t) - Z(t)\|_{\eta\xi} \leq E(t)$$

where

$$E(t) \leq qe^{K_{\eta\xi}t} + \int_0^t \mathbf{d}\left(\frac{d}{ds}\langle \eta, Z(s)\xi\rangle, P(Z(s))(\eta, \xi)\right)e^{K_{\eta\xi}(t-s)}ds \tag{3.12}$$

We now estimate

$$\mathbf{d}\left(\frac{d}{ds}\langle \eta, Z(s)\xi\rangle, P(Z(s))(\eta, \xi)\right).$$

First we claim that

$$\|y_{i+1} - Z(s)\|_{\eta\xi} \leq \gamma + hr.$$

This follows since by definition of $Z_{\eta\xi}(s)$ above

$$Z(s) = \frac{1}{h}[(t_{i+1} - s)y_i + (s - t_i)y_{i+1}]$$

so that

$$y_{i+1} - Z(s) = \left(1 - \frac{s - t_i}{h}\right)y_{i+1} - \left(\frac{t_{i+1} - s}{h}\right)y_i$$

Hence

$$\|y_{i+1} - Z(s)\|_{\eta\xi} \leq \|y_{i+1} - y_i\|_{\eta\xi} \leq \gamma + hr$$

Now we have

$$\begin{aligned}
&\mathbf{d}\left(\frac{d}{ds}\langle \eta, Z(s)\xi\rangle, P(Z(s))(\eta, \xi)\right)\\
&\quad\leq \mathbf{d}\left(\frac{d}{ds}\langle \eta, Z(s)\xi\rangle, P(y_{i+1})(\eta, \xi)\right) + \rho(P(y_{i+1})(\eta, \xi), P(Z(s))(\eta, \xi))\\
&\quad\leq \mathbf{d}\left(\frac{d}{ds}\langle \eta, Z(s)\xi\rangle, P(y_{i+1})(\eta, \xi)\right) + K_{\eta\xi}\|y_{i+1} - Z(s)\|_{\eta\xi}\\
&\quad\leq \frac{\gamma}{h} + K_{\eta\xi}(\gamma + hr) := D_{\eta\xi},
\end{aligned}$$

by using (3.11) and the fact that $P(x)(\eta, \xi)$ is Lipschitzian.

From (3.12), we obtain the estimate

$$\|\Phi(t) - Z(t)\|_{\eta\xi} \leq qe^{K_{\eta\xi}t} + \int_0^t D_{\eta\xi}e^{K_{\eta\xi}(t-s)}ds$$

so that by evaluating the integral in the last inequality, we obtain

$$\|\Phi(t) - Z(t)\|_{\eta\xi} \leq e^{K_{\eta\xi}t}\left(q + \frac{D_{\eta\xi}}{K_{\eta\xi}}\right) - \frac{D_{\eta\xi}}{K_{\eta\xi}}.$$

The last inequality holds in particular for $t = T$.

If the parameters N and q are chosen such that inequality (3.5) and (3.6) are satisfied then

$$\|\Phi(T) - Z(T)\|_{\eta\xi} \leq \epsilon$$

i.e. expression (3.7) holds.

Similar constructions can also be carried out for all $t_i, i = 1, 2 \dots N-1$ and so the inclusion

$$Y_{\eta\xi}(t_i) \subset X(t_i)(\eta, \xi) + S_\epsilon(0)$$

holds for all i.

Again, following Theorem (2.21) and the choice of q and N, this inclusion is also true for all $s \in [t_i, t_{i+1}]$.

To complete the proof, we need to show that the opposite inclusion

$$X(t)(\eta, \xi) \subset Y_{\eta\xi}(t) + S_\epsilon(0) \tag{3.13}$$

holds $\forall t \in [0, T]$.

To this end we employ the following Lemma as in [11].

Lemma 3.3. *If* $X_{0,\eta\xi} \subset Y_{0,\eta\xi} + S_q(0)$ *then* $X(t_i)(\eta, \xi) \subset Y_{\eta\xi}(t_i) + S_q(0) \forall i = 1, 2 \dots N$.

Proof: Let the assertion of the Lemma hold for all $i = 0, 1, 2 \dots l$. Then we shall show that it holds for $i = l+1$ as well and so holds for all $i = 0, 1, 2 \dots N$. We fix an arbitrary point $x_{l+1,\eta\xi}$ in $X(t_{l+1})(\eta, \xi)$. In the q-neighbourhood of this point in $\mathbb{C}$, there is a point $y_{l+1,\eta\xi}$ belonging to the q-net $\{\langle \eta, x_q \xi \rangle\}$ such that $y_{l+1,\eta\xi} = \langle \eta, y_{l+1}\xi \rangle$ for some y_{l+1} in the q-net $\{x_q\} \subset \tilde{\mathcal{A}}$. We shall show that $y_{l+1,\eta\xi} \in Y_{l+1,\eta\xi}$. As a result, this will establish the assertion of the Lemma.

We denote by $x_{\eta\xi}(\cdot)$ the matrix element of the weakly absolutely continuos trajectory $x(\cdot)$ of the inclusion (3.1) that leads to the point $x_{l+1,\eta\xi}$ in $X(t_l)(\eta, \xi)$. Let $x_{\eta\xi}(\cdot)$ start at some point $x_{l,\eta\xi}$ in $X(t_l)(\eta, \xi)$. In the q-neighbourhood of $x_{l,\eta\xi}$ there is also a point $y_{l,\eta\xi}$ from the net $\{\langle \eta, X_q \xi \rangle\}$ such that $y_{l,\eta\xi} = \langle \eta, y_l \xi \rangle$ for some member $y_l \in \{X_q\}$.

To prove that $y_{l+1,\eta\xi} \in Y_{l+1,\eta\xi}$ it suffices to show that the inequality

$$\mathbf{d}(y_{l+1,\eta\xi} - hP(y_{l+1})(\eta, \xi), y_{l,\eta\xi}) \leq \gamma$$

is satisfied. We have

$$\begin{aligned}
\mathbf{d}(y_{l,\eta\xi}, y_{l+1,\eta\xi} - hP(y_{l+1})(\eta, \xi)) &= \min_{f_{\eta\xi} \in P(y_{l+1})(\eta,\xi)} |y_{l+1,\eta\xi} - hf_{\eta\xi} - y_{l,\eta\xi}| \\
= \min_{f_{\eta\xi} \in P(y_{l+1})(\eta,\xi)} &|y_{l+1,\eta\xi} - x_{l+1,\eta\xi} + x_{l+1,\eta\xi} - hf_{\eta\xi} - y_{l,\eta\xi} + x_{l,\eta\xi} \\
&- x_{l,\eta\xi}| \leq 2q + \min_{f_{\eta\xi} \in P(y_{l+1})(\eta,\xi)} |x_{l+1,\eta\xi} - hf_{\eta\xi} - x_{l,\eta\xi}|
\end{aligned}$$

Since $x_{l,\eta\xi} \in X(t_l)(\eta,\xi)$ then by our hypothesis and the construction of the sequence of elements of $Y_{\eta\xi}(t_l)$, we have

$$x_{l,\eta\xi} = x_{l+1,\eta\xi} - hf^0_{\eta\xi} + d_0$$

where $f^0_{\eta\xi} \in P(x_{l+1})(\eta,\xi)$ and $d_0 \in S_{h^2 r K_{\eta\xi}}(0) \subset S_\gamma(0)$.

We finally obtain

$$\begin{aligned}&\mathbf{d}(y_{l,\eta\xi}, y_{l+1,\eta\xi} - hP(y_{l+1})(\eta,\xi))\\ &\quad\le h\rho(P(y_{l+1})(\eta,\xi), P(x_{l+1})(\eta\xi)) + 2q + h^2 r K_{\eta\xi}\\ &\quad\le 2q + hK_{\eta\xi}|y_{l+1,\eta\xi} - x_{l+1,\eta\xi}| + h^2 r K_{\eta\xi}\\ &\quad\le 2q + hqK_{\eta\xi} + h^2 r K_{\eta\xi} = \gamma\end{aligned}$$

Since $x_{l+1,\eta\xi} \in X(t_{l+1})(\eta,\xi)$ is arbitrary, it follows that

$$X(t_{l+1})(\eta,\xi) \subset Y_{\eta\xi}(t_{l+1}) + S_q(0)$$

thus proving Lemma (3.3).

Since $q < \epsilon$, it follows from the Lemma that the inclusion (3.13) holds $\forall\, t_i,\quad i = 0, 1, 2 \ldots N$.

It follows from Theorem (2.21) that (3.13) holds for intermediate values of t.

This complete the proof of Theorem (3.2). □

REFERENCES

1. J.P. Aubin, and H. Frankowska, Set-Valued Analysis, *Birkhauser*, 1990.
2. E.O. Ayoola, On Numerical Procedures For Solving Lipschitzian Quantum Stochastic Differential Equations. *Ph.D Thesis (1998), University of Ibadan, Nigeria*.
3. E.O. Ayoola, Solutions of Lipschizian Quantum Stochastic Differential Equations In A Locally Convex Topological Space. *To Appear In Journal of Science Research, vol.5 (1) 1999*.
4. E.O. Ayoola, Converging Multistep Methods For Weak Solutions Of Quantum Stochastic Differential Equations. *Stochastic Analysis And Applications vol.18, no.4, (2000) 525-554*.
5. G.O.S. Ekhaguere, Lipschitzian quantum stochastic differential inclusions, *Internat. J. Theoret. Phys.* Vol. 31, No. 11, (1992) 2003-2034.
6. G.O.S. Ekhaguere, Quantum stochastic differential inclusions of hypermaximal monotone type, *Internat. J. Theoret. Phys.* Vol. 34, no. 3, (1995), 323-353.
7. G.O.S. Ekhaguere, Quantum stochastic evolutions. *Internat. J. Theoret. Phys.* Vol. 35, no. 9, (1996), 1909-1946.

8. Filipov, A.F.
(a) On the existence of solutions of multivalued differential equation Mat. Zametki,10 (1971), 307-313.
(b) Classical Solutions of differential equations with multivalued right hand side, SIAM J. Control,5 (1967) 609-621.
9. R.L. Hudson, and K.R. Parthasarathy, Quantum Ito's formulae and stochastic evolutions, *Commun. Math. Phys. 93,* (1984) 301-324.
10. V.A. Komarov, Estimates of the attainability sets of differential inclusions. Mat. Zametki,37,6,(1985) 916-925.
11. V.A. Komarov, and K.E. Pevchikh, A method of approximating attainability sets for differential inclusions with a specified accuracy, Comput. Maths. Math. Phys, Vol 31 No 1 (1991) 109-112.
12. G.N. Konstantinov, Normalization of interactions on dynamical systems, *Izd. IGU Irkutsk* (1983)
13. A.B Kurzhanskii, Control and observation under conditions of indeterminacy, Nauka, Moscow, (1977).

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Viable solutions of lower semicontinuous quantum stochastic differential inclusions

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Abstract

We establish the existence and some properties of viable solutions of lower semicontinuous quantum stochastic differential inclusions within the framework of the Hudson–Parthasarathy formulations of quantum stochastic calculus. The main results here are accomplished by establishing a major auxiliary selection result. The results here extend the classical Nagumo viability theorems,valid on finite dimensional Euclidean spaces, to the present infinite dimensional locally convex space of non-commutative stochastic processes.

Keywords Lower semicontinuous · Nagumo viability · Tangent cone · Fock spaces

Mathematics Subject Classification 60H10 · 60H20 · 81S25

1 Introduction

This paper continues our previous works in [4–9,12–16,19,20]. On this occasion, the existence and some properties of viable solutions of lower semicontinouos quantum stochastic differential inclusions (QSDI) are established. In our previous considerations, existence of solutions were sought and established globally in the locally convex space of solutions. In this work, the global requirement are removed by restricting the solution space to a subset of the entire space satisfying some topological conditions. By employing the multivalued analogue of quantum stochastic calculus developed by Hudson and Parthasarathy [17], in the framework of [13,19] the main results of this paper are established.

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It is well known that classical differential inclusions could be solved by reducing them to differential equations through selection theorems. By employing a similar idea, this paper employs a non commutative generalization of Michael selection result established in [20] to transform the lower semicontinuous quantum stochastic differential inclusions under consideration to a quantum stochastic differential equation.

The existence of viable solutions of differential equations and inclusions defined on finite dimensional Euclidean spaces have been well studied, see, for example [1–3,10,11,18,21,22]. However, similar classes of problems have not been well studied for QSDI. This is a major motivation for this work. In the classical finite dimensional setting, a necessary and sufficient condition for the existence of viable solutions was established by Nagumo [22] in which the closed subset K, which is the viability subset is bounded and satisfy the tangential condition. Other researchers have similarly worked on further developments, and the extensions of Nagumo theorems and applications see [10,11,18,22].

However, for the Nagumo-type fixed point results to work in the present non commutative settings, this paper first established an auxiliary result by circumventing certain difficulties using the unique properties of the family of seminorms that defines the topology for the underling locally convex space of non commutative stochastic processes.

The rest of the paper is organized as follows: Sect. 2 is devoted to the preliminaries and some notations. The main results on viability of solutions including the convergence of approximate solutions are established in Sect. 3.

2 Preliminaries

Let D be an inner product space and H, the completion of D. We denote by $L^+(D, H)$, the set $\{X : D \to H : X$ is a linear map satisfying $Dom\ X^* \supseteq D$, where X^* is the operator adjoint of $X\}$.
We remark that $L^+(D, H)$ is a linear space under the usual notions of addition and scalar multiplication of operators.
In what follows, $\mathbb{D}$ is some inner product space with R as its completion, and γ is some fixed Hilbert space.

For each $t \in \mathbb{R}_+$, we write $L^2_\gamma(\mathbb{R}_+)$, (resp. $L^2_\gamma([0,t))$ resp. $L^2_\gamma([t,\infty))$), for the Hilbert spaces of square integrable, γ-valued maps on $\mathbb{R}_+ \equiv [0,\infty)$, (resp. $[0,t)$; resp. $[t,\infty)$). Then we introduce the following spaces:

(i) $\mathcal{A} \equiv L^+(\mathbb{D}\underline{\otimes}\, \mathbb{E},\ \mathcal{R} \otimes \Gamma(L^2_\gamma(\mathbb{R}_+))$.
(ii) $\mathcal{A}_t \equiv L^+(\mathbb{D}\underline{\otimes}\, \mathbb{E}_t, \mathcal{R} \otimes \Gamma(L^2_\gamma([0,t))) \otimes 1^t$.
(iii) $\mathcal{A}^t \equiv 1_t \otimes L^+(\mathbb{D}\underline{\otimes}\, \mathbb{E}^t, \mathcal{R} \otimes \Gamma(L^2_\gamma([t,\infty))), t > 0$

where $\underline{\otimes}$ denotes algebraic tensor product and 1_t (resp. 1^t) denotes the identity map on $\mathcal{R} \otimes \Gamma(L^2_\gamma([0,t)))(\text{resp.}\Gamma(L^2_\gamma([t,\infty)))$, $t > 0$. We note that $\mathcal{A}^t$ and $\mathcal{A}_t$, $t > 0$, may be naturally identified with subspaces of $\mathcal{A}$. For $\eta, \xi \in \mathbb{D}\underline{\otimes}\, \mathbb{E}$, define $\|\cdot\|_{\eta\xi}$ on $\mathcal{A}$ by $\|x\|_{\eta\xi} = |\langle \eta, x\xi\rangle|$, $x \in \mathcal{A}$. Then $\{\|\cdot\|_{\eta\xi}, \eta, \xi \in \mathbb{D}\underline{\otimes}\, \mathbb{E}\}$ is a family of seminorms on $\mathcal{A}$; we write τ_w for the locally convex Hausdorff topology on $\mathcal{A}$ determined by this family. We denote by $\tilde{\mathcal{A}}$, $\tilde{\mathcal{A}}_t$ and $\tilde{\mathcal{A}}^t$ the completions of the locally convex topological spaces $(\mathcal{A}, \tau_w)$, $(\mathcal{A}_t, \tau_w)$ and $(\mathcal{A}^t, \tau_w)$, $t > 0$, respectively.

We define the Hausdorff topology on clos($\widetilde{\mathcal{A}}$) as follows: For $x \in \widetilde{\mathcal{A}}$, $\mathcal{M}, \mathcal{N} \in$ clos($\widetilde{\mathcal{A}}$) and $\eta, \xi \in \mathbb{D}\underline{\otimes}\mathbb{E}$, define

$$\rho_{\eta\xi}(\mathcal{M}, \mathcal{N}) \equiv \max(\delta_{\eta\xi}(\mathcal{M}, \mathcal{N}), \delta_{\eta\xi}(\mathcal{N}, \mathcal{M}))$$

where

$$\delta_{\eta\xi}(\mathcal{M}, \mathcal{N}) \equiv \sup_{x\in\mathcal{M}} \mathbf{d}_{\eta\xi}(x, \mathcal{N}) \text{ and}$$
$$\mathbf{d}_{\eta\xi}(x, \mathcal{N}) \equiv \inf_{y\in\mathcal{N}} \| x - y \|_{\eta\xi} .$$

The Hausdorff topology which shall be employed in what follows, denoted by, τ_H , is generated by the family of pseudometrics $\{\rho_{\eta\xi}(.) : \eta\xi \in \mathbb{D}\underline{\otimes}\mathbb{E}\}$.
Moreover, if $\mathcal{M} \in clos(\widetilde{\mathcal{A}})$, then $\| \mathcal{M} \|_{\eta\xi}$ is defined by

$$\| \mathcal{M} \|_{\eta\xi} \equiv \rho_{\eta\xi}(\mathcal{M}, \{0\});$$

for arbitrary $\eta, \xi \in \mathbb{D}\underline{\otimes}\mathbb{E}$.
For $A, B \in clos(C)$ *and* $x \in C$, a complex number, define

$$\mathbf{d}(x, \mathcal{B}) \equiv \inf_{y\in\mathcal{B}} | x - y |$$
$$\delta_{\xi}(\mathcal{A}, \mathcal{B}) \equiv \sup_{x\in\mathcal{A}} \mathbf{d}_{\xi}(x, \mathcal{B}) \text{ and}$$
$$\rho(\mathcal{A}, \mathcal{B}) \equiv max(\delta(\mathcal{A}, \mathcal{B}), \delta(\mathcal{A}, \mathcal{B}))$$

Then ρ is a metric on $clos(C)$ and induces a metric topology on the space.
We define the Hausdorff topology on clos($\widetilde{\mathcal{A}}$) as follows: For $x \in \widetilde{\mathcal{A}}$, $\mathcal{M}, \mathcal{N} \in$ clos($\widetilde{\mathcal{A}}$) and $\eta, \xi \in \mathbb{D}\underline{\otimes}\mathbb{E}$, define

$$\rho_{\eta\xi}(\mathcal{M}, \mathcal{N}) \equiv \max(\delta_{\eta\xi}(\mathcal{M}, \mathcal{N}), \delta_{\eta\xi}(\mathcal{N}, \mathcal{M}))$$

where

$$\delta_{\eta\xi}(\mathcal{M}, \mathcal{N}) \equiv \sup_{x\in\mathcal{M}} \mathbf{d}_{\eta\xi}(x, \mathcal{N}) \text{ and}$$
$$\mathbf{d}_{\eta\xi}(x, \mathcal{N}) \equiv \inf_{y\in\mathcal{N}} \| x - y \|_{\eta\xi} .$$

The Hausdorff topology which shall be employed in what follows, denoted by, τ_H , is generated by the family of pseudometrics $\{\rho_{\eta\xi}(.) : \eta\xi \in \mathbb{D}\underline{\otimes}\mathbb{E}\}$.

Definition 1 Let $I \subseteq \mathbb{R}_+$,

(i) A map $X : I \to \tilde{\mathcal{A}}$ is called a stochastic process indexed by I.
(ii) A stochastic process X is called adapted if $X(t) \in \tilde{\mathcal{A}}_t$ for each $t \in I$.
We denote by $Ad(\tilde{\mathcal{A}})$ the set of all adapted stochastic processes indexed by I.
(iii) A member X of $Ad(\tilde{\mathcal{A}})$ is called

(a) weakly absolutely continuous if the map $t \rightarrow \langle \eta, X(t)\xi \rangle,\ t \in I$, is absolutely continuous for arbitrary $\eta, \xi \in \mathbb{D}\underline{\otimes}\ \mathbb{E}$. We denote this subset of $Ad(\tilde{\mathcal{A}})$ by $Ad(\tilde{\mathcal{A}})_{wac}$.

(b) locally absolutely p-integrable if $\|X(\cdot)\|^p_{\eta\xi}$ is Lebesgue measurable and integrable on $[t_0, t) \subseteq I$ for each $t \in I$, $p \in (0, \infty)$ and arbitrary $\eta, \xi \in \mathbb{D}\underline{\otimes}\ \mathbb{E}$. We denote this subset of $Ad(\tilde{\mathcal{A}})$ by $L^p_{loc}(\tilde{\mathcal{A}})$.

Stochastic Integrators: Let $B(\gamma)$ denote the Banach space of bounded endomorphisms of γ and let the spaces $L^\infty_{\gamma,loc}(\mathbb{R}_+)$ and $L^\infty_{B(\gamma),loc}(\mathbb{R}_+)$ be defined by: $L^\infty_{\gamma,loc}(\mathbb{R}_+) = \{f : \mathbb{R}_+ \longrightarrow \gamma | f$ is linear, measurable and locally bounded function on $\mathbb{R}_+\}$. $L^\infty_{B(\gamma),loc}(\mathbb{R}_+) = \{\pi : \mathbb{R}_+ \longrightarrow B(\gamma) | \pi$ is linear, measurable and locally bounded function on $\mathbb{R}_+\}$.

For $f \in L^\infty_{\gamma,loc}(\mathbb{R}_+)$ and $\pi \in L^\infty_{B(\gamma),loc}(\mathbb{R}_+)$, we define $\pi f \in L^\infty_{\gamma,loc}(\mathbb{R}_+)$ by $(\pi f)(t) = \pi(t) f(t),\ t \in \mathbb{R}_+$. Also, for $f \in L^2_\gamma(\mathbb{R}_+)$ and $\pi \in L^\infty_{B(\gamma),loc}(\mathbb{R}_+)$, we define the operators $a(f)$, $a^+(f)$ and $\lambda(\pi)$ in $L^+(\mathbb{D}, \Gamma(L^2_\gamma(\mathbb{R}_+)))$ as follows:

$$\begin{aligned} a(f)e(g) &= \langle f, g \rangle L^2_\gamma(\mathbb{R}_+) e(g) \\ a^+(f)e(g) &= \frac{d}{d\sigma} e(g + \sigma f)|\sigma = 0 \\ \lambda(\pi)e(g) &= \frac{d}{d\sigma} e(e^{\sigma\pi} f)|\sigma = 0 \end{aligned}$$

for $g \in L^2_\gamma(\mathbb{R}_+)$.

The operators $a(f)$, $a^+(f)$ and $\lambda(\pi)$ for arbitrary $f \in L^\infty_{\gamma,loc}(\mathbb{R}_+)$ and $\pi \in L^\infty_{B(\gamma),loc}(\mathbb{R}_+)$ give rise to the operator-valued maps A_f, A^+_f and Λ_π defined by

$$A_f(t) \equiv a(f\chi[0, t))$$

$$A^+_f(t) \equiv a^+(f\chi[0, t))$$

$$\Lambda_\pi(t) \equiv \lambda(\pi\chi[0, t))$$

$t \in \mathbb{R}_+$ where χ_I denotes the indicator function of the Borel set $I \subseteq \mathbb{R}_+$.

The operators $a(f)$, $a^+(f)$ and $\lambda(\pi)$ are the annihilation, creation and gauge operators of quantum field theory. The maps A_f, A^+_f and Λ_π are stochastic processes, called the annihilation, creation and gauge processes, respectively when their values are identified with their ampliations on $\mathcal{R}\underline{\otimes}\ \Gamma(L^2_\gamma(\mathbb{R}_+))$. These are the stochastic integrators in the Hudson and Parthasarathy [17] formulation of Boson quantum stochastic integration which we adopt in the sequel.

2.1 Quantum stochastic differential inclusion

Definition 2 (1) By a multivalued stochastic process indexed by $I \subseteq R_+$, we mean a multifunction on I with values in clos$(\tilde{\mathcal{A}})$.

(2) If Φ is a multivalued stochastic process indexed by $I \subseteq R_+$, then a selection of Φ is a stochastic process $X : I \to \tilde{\mathcal{A}}$ with the property that $X(t) \in \Phi(t)$ for almost all $t \in I$.
(3) A multivalued stochastic process Φ will be called (i) adapted if $\Phi(t) \subseteq \tilde{\mathcal{A}}_t$ for each $t \in R_+$; (ii) measurable if $t \mapsto d_{\eta\xi}(x, \Phi(t))$ is measurable for arbitrary $x \in \tilde{\mathcal{A}}, \eta, \xi \in D\underline{\otimes}\,\mathbb{E}$; (iii) locally absolutely p-integrable if $t \mapsto \|\Phi(t)\|_{\eta\xi}, t \in R_+$, lies in $L^p_{loc}(I)$ for arbitrary $\xi \in D\underline{\otimes}\,\mathbb{E}$

The set of all locally absolutely p-integrable multivalued stochastic processes will be denoted by $L^p_{loc}(\tilde{\mathcal{A}})_{mvs}$. For $p \in (0, \infty)$ and $I \subseteq R_+$, $L^p_{loc}(I \times \tilde{\mathcal{A}})_{mvs}$ is the set of maps $\Phi : I \times \tilde{\mathcal{A}} \to clos(\tilde{\mathcal{A}})$ such that $t \mapsto \Phi(t, X(t)), t \in I$, lies in $L^p_{loc}(\tilde{\mathcal{A}})_{mvs}$ for every $X \in L^p_{loc}(\tilde{\mathcal{A}})$. If $\Phi \in L^p_{loc}(I \times \tilde{\mathcal{A}})_{mvs}$, then

$$L_p(\Phi) \equiv \{\varphi \in L^p_{loc}(\tilde{\mathcal{A}}) : \varphi \text{ is a selection of } \Phi\}$$

Let $f, g \in L^\infty_{\Upsilon,loc}(R_+), \pi \in L^\infty_{B(\Upsilon),loc}(R_+)$, 1 is the identity map on , $\mathcal{R} \otimes \Gamma(L^2_\gamma(R_+))$, and M is any of the stochastic processes A_f, A^+_g, Λ_π , and $s \mapsto s1, s \in R_+$. We introduce stochastic integral (resp. differential) expressions as follows. If $\Phi \in L^2_{loc}(I \times \tilde{\mathcal{A}})_{mvs}$ and $(t, X) \in I \times L^2_{loc}(I \times \tilde{\mathcal{A}})$, then we make the definition

$$\int_{t0}^{t} \Phi(s, X(s))dM(s) \equiv \left\{\int_{t0}^{t} \varphi(s)\, dM(s) : \varphi \in L_2(\Phi)\right\}$$

This leads to the following definition.

Definition 3 Let $E, F, G, H \in L^2_{loc}(I \times \tilde{\mathcal{A}})_{mvs}$ and (t_0, x_0) be a fixed point of $I \times \tilde{\mathcal{A}}$. Then, a relation of the form

$$\begin{aligned} dX(t) \in +E(t, X(t)d\Lambda_\pi(t) + F(t, X(t)dA_f(t) \\ +G(t, X(t))dA^+_g(t) + H(t, X(t))dt, \quad t \in I, \end{aligned} \tag{1}$$

is called quantum stochastic differential inclusions(QSDI) with coefficients in E,F,G, H and initial data (t_0, x_0).

Equation (1) is understood in the integral form:

$$\begin{aligned} X(t) \in x_0 + \int_{t_0}^{t} (E(s, X(s))d\Lambda_\pi(s) + F(s, X(s))dA_f(s)) \\ +G(s, X(s))dA^+_g(s) + H(s, X(s))ds), \quad t \in I, \end{aligned}$$

called a stochastic integral inclusion with coefficients E,F,G,H and initial data (t_0, x_0) An equivalent form of (1) was established in [13], Theorem 6.2 as follows: For $\eta, \xi \in D\underline{\otimes}\,\mathbb{E}.\alpha, \beta \in L^2_\gamma(R_+)$ with $\eta = c \otimes e(\alpha)$, $\xi = d \otimes e(\beta)$,define the following complex-valued functions:

$$\mu_{\alpha\beta}, \upsilon_\beta, \sigma_\alpha : I \longrightarrow C, I \subset \mathcal{R}_+, by$$

$$\mu_{\alpha\beta}(t) = \langle \alpha(t), \pi(t)\beta(t)\rangle_\gamma$$
$$v_\beta(t) = \langle f(t), \beta(t)\rangle_\gamma$$
$$\sigma_\alpha(t) = \langle \alpha(t), g(t)\rangle_\gamma$$

$t \in I$, $f, g \in L^2_{\gamma,loc}(R_+)$, $\pi \in L^\infty_{B(\gamma),loc}$. To these functions we associate the maps $\mu E, vF, \sigma G, P$ from $I \times \tilde{A}$ into the set of sesquilinear forms on $D\underline{\otimes}\,\mathbb{E}$ define by

$$\begin{aligned}
(\mu E)(t,x)(\eta,\xi) &= \left\{\langle \eta, \mu_{\alpha\beta}(t)p(t,x)\xi\rangle : p(t,x) \in E(t,x)\right\} \\
(vF)(t,x)(\eta,\xi) &= \left\{\langle \eta, v_\beta(t)q(t,x)\xi\rangle : q(t,x) \in F(t,x)\right\} \\
(\sigma G)(t,x)(\eta,\xi) &= \{\langle \eta, \sigma_\alpha(t)u(t,x)\xi\rangle : u(t,x) \in G(t,x)\} \\
P_{\alpha\beta}(t,x) &= \mu_{\alpha\beta}(t)E(t,x) + v_\beta(t)F(t,x) + \sigma_\alpha(t)G(t,x) + H(t,x) \\
\mathbb{P}(t,x)(\eta,\xi) &= \langle \eta, P_{\alpha\beta}(t,x)\xi\rangle, i.e \\
P(t,x)(\eta,\xi) &= (\mu E)(t,x)(\eta,\xi) + (vF)(t,x)(\eta,\xi) \\
&\quad + (\sigma G)(t,x)(\eta,\xi) + H(t,x)(\sigma,\xi) \\
H(t,x)(\eta,\xi) &= \left\{v(t,x)(\eta,\xi) : v(.,X(.)) \text{ is a selection of} H(.,X(.)) \forall X \in L^2_{loc}(\tilde{A})\right\}
\end{aligned} \tag{2}$$

Then problem (1) is equivalent to

$$\begin{aligned}
&\frac{d}{dt}\langle \eta, X(t)\xi\rangle \in \mathbb{P}(t, X(t))(\eta,\xi) \\
&\langle \eta, X(t_0)\xi\rangle = \langle \eta, x_0\xi\rangle
\end{aligned} \tag{3}$$

for arbitrary $\eta, \xi \in \mathbb{D}\underline{\otimes}\,\mathbb{E}$ and almost all $t \in I$. Hence the existence of solution of (1) implies the existence of solution of (3) and vice-versa. As explained in [13], the sesquilinear form valued map P:

$$\mathbb{P}(t,x)(\eta,\xi) \neq \tilde{\mathbb{P}}(t, \langle \eta, x\xi\rangle)$$

For some complex-valued multifunction $\tilde{P}$ defined on $I \times C$ for $t \in I, x \in \tilde{A}, \eta, \xi \in D\underline{\otimes}\,\mathbb{E}$.

Before proceeding to the proof of the main result in this work , we make use of a result in [20] in which the multifunction $(t,x) \longrightarrow \mathbb{P}(t,x)(\eta,\xi)$ is lower semi continuous with respect to the seminorm $\|.\|_{\eta\xi}$, closed and convex.

then there exists a continuous selection, $P : I \times \tilde{A} \longrightarrow sesq(\mathbb{D}\underline{\otimes}\mathbb{E})$ of $\mathbb{P}$ such that

$$\begin{aligned}
&\frac{d}{dt}\langle \eta, X(t)\xi\rangle = P(t, X(t))(\eta,\xi) \\
&\langle \eta, X(t_0)\xi\rangle = \langle \eta, x_0\xi\rangle \; a.e \;\; t \in I
\end{aligned} \tag{4}$$

for arbitrary pair $\eta, \xi \in \mathbb{D}\underline{\otimes}\mathbb{E})$,$(t,x) \longrightarrow P(t,x)(\eta,\xi)$ is continuous.

3 Viability theory

Definition 4 Let $P : I \times \tilde{\mathcal{A}} \longrightarrow sesq(\mathbb{D}\underline{\otimes}\,\mathbb{E})^2$ be a sesqulinear valued funtion, then the subset K of $\tilde{\mathcal{A}}$ is viable with respect to P if for every $(t_0, x_0) \in I \times K$ there exists $T \in I, T > t_0$ such that Eq. (4) have at least one solution K.

Definition 5 Let $K \subseteq \tilde{\mathcal{A}}$, A subset $K \in clos(\tilde{\mathcal{A}})$ is locally closed if $K(\eta,\xi)$ is a closed subset with values in $\mathbb{C}$ then $K(\eta,\xi)$ is locally closed if for each $x_{\eta\xi} \in K(\eta,\xi)$, there exists $\rho > 0$ such that $D(x_{\eta\xi},\rho) \cap K(\eta,\xi)$ is closed for arbitrary pair $\eta,\xi \in \mathbb{D}\underline{\otimes}\mathbb{E}$.

We now define tangent cone as it applies to our non commutative settings. We make use of Bouligand–Severi tangency concept in [11].

Definition 6 Let $K \subset \tilde{\mathcal{A}}$, $E \subset \tilde{\mathcal{A}}$ such that $K(\eta,\xi) \subset \mathbb{C}$, $E(\eta,\xi) \subset \mathbb{C}$ and $x \in K$ such that $x_{\eta\xi} \in K(\eta,\xi) \subset \mathbb{C}$. Then the set $E(\eta,\xi)$ is tangent to the set $K(\eta,\xi)$ at the point $x_{\eta\xi}$ if

$$\liminf_{h\mapsto 0} \frac{1}{h}\mathbf{d}(x_{\eta\xi} + hE(\eta,\xi); K(\eta,\xi)) = 0$$

We denote by $\mathcal{T}_{K(\eta,\xi)}$ the class of all sets which are tangent to $K(\eta,\xi)$ at the point $x_{\eta\xi}$ for arbitrary $\eta,\xi \in D\underline{\otimes}\,\mathbb{E}$.

Proposition 1 *The set $\mathcal{T}_{K(\eta,\xi)}(x_{\eta\xi})$ of all vectors which are tangent to the set $K(\eta,\xi)$ at the point $x_{\eta\xi}$ is a closed cone.*

Proof Let $(x_{\eta\xi}) \in K(\eta,\xi)$ According to definition 6, $E(\eta,\xi) \in \mathcal{T}_{K(\eta,\xi)}(x_{\eta\xi})$ if

$$\liminf_{t\mapsto 0} \frac{1}{t}\mathbf{d}(x_{\eta\xi} + tE(\eta,\xi); K(\eta,\xi)) = 0$$

Let $s > 0$, we observe that

$$\begin{aligned}\liminf_{t\mapsto 0} \frac{1}{t}\mathbf{d}(x_{\eta\xi} + tsE(\eta,\xi); K(\eta,\xi)) &= s\liminf_{t\mapsto 0} \frac{1}{ts}\mathbf{d}(x_{\eta\xi} + tsE(\eta,\xi); K(\eta,\xi))\\ &= s\liminf_{\tau\mapsto 0} \frac{1}{\tau}\mathbf{d}(x_{\eta\xi} + \tau E(\eta,\xi); K(\eta,\xi))\end{aligned}$$

Hence, $sE(\eta,\xi) \in \mathcal{T}_K(x_{\eta\xi})$ To complete the proof, we need to show that $\mathcal{T}_K(x_{\eta\xi})$ is a closed set.

Let $\mathbb{N}^*$ be the set of strictly positive natural numbers. Let $(E_n(\eta,\xi))_{n\in\mathbb{N}^*}$ be a sequence of elements in $\mathcal{T}_K(x_{\eta\xi})$, convergent to $E(\eta,\xi)$ then we have

$$\begin{aligned}\frac{1}{t}\mathbf{d}(x_{\eta\xi} + tE(\eta,\xi); K(\eta,\xi)) &\leq \frac{1}{t}\,|t(E(\eta,\xi) - E_n(\eta,\xi))|\\ &\quad+\frac{1}{t}\mathbf{d}(x_{\eta\xi} + tE_n(\eta,\xi); K(\eta,\xi))\\ &= |E(\eta,\xi) - E_n(\eta,\xi)|\\ &\quad+\frac{1}{t}\mathbf{d}(x_{\eta\xi} + tE_n(\eta,\xi); K(\eta,\xi))\end{aligned}$$

for every $n \in \mathbb{N}^*$. So

$$\liminf_{t\mapsto 0} \frac{1}{t}\mathbf{d}\left(x_{\eta\xi} + tE(\eta,\xi); K(\eta,\xi)\right) \leq |E(\eta,\xi) - E_n(\eta,\xi)|$$

for every $n \in \mathbb{N}^*$. Since $\lim_{n\to\infty} |E(\eta,\xi) - E_n(\eta,\xi)| = 0$, it follows that

$$\liminf_{t\mapsto 0} \frac{1}{t}\mathbf{d}(x_{\eta\xi} + tE(\eta,\xi); K(\eta,\xi)) = 0.$$

which shows that the set $\mathcal{T}_{K(\eta,\xi)}(x_{\eta\xi})$ is a closed cone and this achieves the proof. □

Proposition 2 *A set $E(\eta,\xi) \in \mathbb{C}$ belongs to the cone $\mathcal{T}_{K(\eta,\xi)}(x_{\eta\xi})$ if and only if for every $\epsilon > 0$ there exists $h \in (0,\epsilon)$ and $q_{\eta\xi,h} \in D_{\eta\xi}(0,\epsilon)$ with the property*

$$x_{\eta\xi} + h\left(E(\eta,\xi) + q_{\eta\xi,h}\right) \in K(\eta,\xi)$$

Proof We see that $E(\eta,\xi) \in \mathcal{T}_{K(\eta,\xi)}(x_{\eta\xi})$ if and only if for every $\epsilon > 0$ there exists $h \in (0,\epsilon)$ and $p_{\eta\xi,h} \in K(\eta,\xi)$ such that

$$\frac{1}{h}|x_{\eta\xi} + hE(\eta,\xi) - p_{\eta\xi,h}| \leq \epsilon.$$

let

$$q_{\eta\xi,h} = \frac{1}{h}(p_{\eta\xi,h} - x_{\eta\xi} - hE(\eta,\xi)),$$

and we have both $|q_{\eta\xi,h}| \leq \epsilon$ and $x_{\eta\xi} + h(E(\eta,\xi) + q_{\eta\xi,h}) = p_{\eta\xi,h} \in K(\eta,\xi)$. □

3.1 Main result

In this section, we establish the quantum generalization of Nagumo viability result.

Existence of Approximate Solutions : Let $(t_0, x_{0,\eta\xi}) \in I \times K(\eta,\xi)$, then there exists $\rho > 0$, such that $D(x_{0,\eta\xi},\rho) \cap K(\eta,\xi)$ be closed, then there exists $M_{\eta\xi} > 0$, such that

$$|P(t,x)(\eta,\xi)| \leq M_{\eta\xi} \tag{5}$$

for every $t \in [t_0, T]$ and $x \in D_{\eta\xi}(x,\rho) \cap K \subset \tilde{\mathcal{A}}$ and $x_{\eta\xi} \in D(x_{0,\eta\xi},\rho) \cap K(\eta,\xi)$ and

$$(T - t_0)(M_{\eta\xi} + 1) \leq \rho \tag{6}$$

The existence of these three numbers will be made possible because $K(\eta,\xi)$ is locally closed and by the continuity of P which implies its boundedness on

$[t_0, T] \times D(x_{0,\eta\xi}, \rho)$, and so the existence of $M_{\eta\xi} > 0$, and the fact that $T \in I$, $T > t_0$, is chosen very close to t_0. The following lemma concerns the existence of family of approximate solutions for the problem defined on interval $[t_0, c]$.

Lemma 1 *Suppose $K \subset \tilde{\mathcal{A}} \neq \emptyset$ satisfying the following*

(i) K *is locally closed.*
(ii) $P : I \times \tilde{\mathcal{A}} \longrightarrow sesq(\mathbb{D}\underline{\otimes}\mathbb{E})^2$ *is continuous.*
(iii) $P(t_0, x_0)(\eta, \xi) \in \mathcal{T}_{\mathcal{K}(\equiv,\sim)}(x_{0,\eta\xi})$ *for each* $(t_0, x_0) \in I \times K$.

Then, for each $\epsilon \in (0, 1)$*, there exist: a non decreasing function*

$$\sigma : [t_0, T] \longrightarrow [t_0, T]$$

and two stochastic processes

$$g : [t_0, T] \longrightarrow \tilde{\mathcal{A}} \tag{7}$$

and

$$\varphi : [t_0, T] \longrightarrow \tilde{\mathcal{A}}$$

lying in $Ad(\tilde{\mathcal{A}})_{wac} \cap L^2_{loc}(\tilde{\mathcal{A}})$ *such that the corresponding sesquilinear form valued maps associated with any pair of* $\eta, \xi \in \mathbb{D}\underline{\otimes}\mathbb{E})$ *given by*

$$g : [t_0, T] \longrightarrow sesq(\mathbb{D}\underline{\otimes}\mathbb{E})^2$$

such that

$$g(t)(\eta, \xi) = \langle \eta, g(t)\xi \rangle$$

and

$$\varphi : [t_0, T] \longrightarrow sesq(\mathbb{D}\underline{\otimes}\mathbb{E})^2$$

such that

$$\varphi(t)(\eta, \xi) = \langle \eta, \varphi(t)\xi \rangle$$

satisfy the followin

(i) $t - \epsilon \leq \sigma(t) \leq t$ *for every* $t \in [t_0, T]$
(ii) $|g_{\eta\xi}(t)| \leq \epsilon$ *for every* $t \in [t_0, T]$
(iii) $\varphi_{\eta\xi}(\sigma(t)) \in D(x_{0,\eta\xi}, \rho) \cap K(\eta, \xi)$ *for every* $t \in [t_0, T]$ *and* $\varphi_{\eta\xi}(T) \in D(x_{0,\eta\xi}, \rho) \cap K(\eta, \xi)$

(iv) φ *satisfies*

$$\langle \eta, \varphi(t)\xi \rangle = \langle \eta, x_0\xi \rangle + \int_{t_0}^{t} P(\sigma(s), \varphi(\sigma(s))(\eta, \xi)ds + \int_{t_0}^{t} g(s)(\eta, \xi)ds$$

for every $t \in [t_0, T]$

A pair of the triple (σ, g, φ) *as in Eq. (7) with the associated* $(\sigma, g_{\eta\xi}, \varphi_{\eta\xi})$ *satisfying (i), (ii), (iii) and (iv) above is called an* ϵ*- approximate solution to the problem (4) on the interval* $[t_0, T]$.

Proof Let $t_0 \in I$, $x_{0,\eta\xi} \in K(\eta, \xi)$ and let $\rho > 0$, $M > 0$ and $T > t_0$ be as above. Let $\epsilon \in (0, 1)$. We first show the existence of an ϵ- approximate solution on an interval $[t_0, c]$ with $c \in (t_0, T]$.
Since for every $(t_0, x_{0,\eta\xi}) \in I \times K(\eta, \xi)$, $P(t_0, x_0)(\eta, \xi) \in \mathcal{T}_{\mathcal{K}(\equiv,\sim)}(x_{0,\eta\xi})$, from Proposition (2), it follows that there exists $c \in (t_0, T]$, $c - t_0 \leq \epsilon$ and $q_{\eta\xi,h}$ has values in $\mathbb{C}$ with $| \ q_{\eta\xi,h} \ | \leq \epsilon$ such that

$$x_{0,\eta\xi} + (c - t_0)P(t_0, x_0)(\eta, \xi) + (c - t_0)q_{\eta\xi}, h \in K(\eta, \xi)$$

Let $I_c = [t_0, c]$, we now define the functions $\sigma : [t_0, c] \longrightarrow [t_0, c]$, and stochastic processes $g : [t_0, c] \longrightarrow \tilde{A}$ and $\varphi : [t_0, c] \longrightarrow \tilde{A}$ lying in $Ad(\tilde{A})_{wac} \cap L^2_{loc}(\tilde{A})$ satisfying

$$\begin{cases} \sigma(t) = t_0 & \text{for } t \in [t_0, c], , \\ g_{\eta\xi}(t) = q_{\eta\xi} & \text{for } t \in [t_0, c], \\ \varphi_{\eta\xi}(t) = x_{0,\eta\xi} + (t - t_0)P(t_0, x_0)(\eta, \xi) + (t - t_0)q_{\eta\xi} & \text{for } t \in [t_0, c]. \end{cases}$$

The triple $(\sigma, g_{\eta\xi}, \varphi_{\eta\xi})$ is an ϵ approximate solution to the problem (4) on the interval $[t_0, c]$. This shows that conditions (i), (ii) and (iv) are satisfied, we now show that condition (iii) is also satisfied using (5), (6) and (i). From (i) $\sigma(t) = t_0$ and $\langle \eta, X(t_0)\xi \rangle = \langle \eta, x_0\xi \rangle$,then $\langle \eta, \varphi(\sigma(t))\xi \rangle = \langle \eta, x_0\xi \rangle$, therefore we have $\varphi(\sigma(t))(\eta, \xi) \in D(x_{0,\eta\xi}, \rho) \cap K(\eta, \xi)$ for every $t \in [t_0, c]$. Therefore, $\varphi(c)(\eta, \xi) \in K(\eta, \xi)$. However, by (5) and (6),we have

$$\begin{aligned} |\varphi(c)(\eta, \xi) - \varphi_0(\eta, \xi)| &\leq \ (c - t_0) \, |P(t_0, \varphi_0)(\eta, \xi)| + (c - t_0) \, |q| \\ &\leq (T - t_0)(M_{\eta\xi} + 1) \leq \rho \end{aligned}$$

For every $t \in [t_0, c]$. Thus (iii) is also satisfied.

We now define the $\epsilon-$ approximate solution on the whole interval I. We make use of Brezis–Browder theorem in [8]. Let $\mathcal{S}$ be the set of all ϵ—approximate solutions to the problem (4) defined on the interval $[t_0, c]$ with $c \in (t_0, T]$. On $\mathcal{S}$ we define the relation " $\preceq''$ by $(\sigma_1, g_{1,\eta\xi}, \varphi_{1,\eta\xi}) \preceq (\sigma_2, g_{2,\eta\xi}, \varphi_{2,\eta\xi})$ if the domain of definition $[t_0, c_1]$ of the first triple is included in the domain of definition $[t_0, c_2]$ of the second triple, and the two ϵ—approximate solutions coincide on the common part of the domains. Then, " $\preceq''$is a pre-order relation on $\mathcal{S}$. Firstly, we show that each increasing sequence $((\sigma_m, g_{m,\eta\xi}, \varphi_{m,\eta\xi}))_m$ is bounded from above . Let $((\sigma_m, g_{m,\eta\xi}, \varphi_{m,\eta\xi}))_m$ be an increasing sequence, and let $c^* = lim_m c_m$ where $[t_0, c_m]$ denotes the domain of definition of $(\sigma_m, g_{m,\eta\xi}, \varphi_{m,\eta\xi})$. Then $c^* \in (t_0, T]$.

We will show that there exists at least one element, $(\sigma^*, g^*_{\eta\xi}, \varphi^*_{\eta\xi}) \in \mathcal{S}$, defined on $[t_0, c^*]$ and satisfying $(\sigma_m, g_{m,\eta\xi}, \varphi_m) \preceq (\sigma^*, g^*_{\eta\xi}, x^*_{\eta\xi})$ for each $m \in \mathbb{N}$. In order to do this, we first prove that there exists $\lim_m \varphi_m(c_m)(\eta, \xi)$.

For each $m, n \in \mathbb{N}$, $m \leq n$ we have $u_m(s) = u_n(s)$ for all $s \in [t_0, c_m]$. Taking into account (iii), (iv) and (5), we have

$$\begin{aligned} \mid \varphi_m(c_m)(\eta, \xi) - \varphi_n(c_n)(\eta, \xi) \mid &\leq \int_{c_m}^{c_n} \mid P(\sigma_n(s), \varphi_n(\sigma_n(s)))(\eta, \xi) \mid ds \\ &\quad + \int_{c_m}^{c_n} \mid g_n(s)(\eta, \xi) \mid ds \\ &\leq (M_{\eta\xi} + \epsilon) \mid c_n - c_m \mid \end{aligned}$$

for every $m, n \in \mathbb{N}$, which shows that there exists

$$\lim_{m \mapsto \infty} \varphi_m(c_m)(\eta, \xi) \in D(x_{0,\eta\xi}, \rho) \cap K(\eta, \xi)$$

Furthermore, because all the functions in the set $\{\sigma_m : m \in \mathbb{N}\}$ are non decreasing, with values in $[t_0, c^*]$, and satisfy

$\sigma_m(c_m) \leq \sigma_p(c_p)$ for every $m, p \in \mathbb{N}$, there exists $\lim_{m\mapsto\infty} \sigma_m(c_m)$, then the limit exists and belongs to $[t_0, c^*]$. We now define a triple function $(\sigma^*, g^*_{\eta\xi}, \varphi^*_{\eta\xi}) : [t_0, c^*] \longrightarrow [t_0, c^*] \times \mathbb{C} \times \mathbb{C}$ by

$$\sigma^*(t) = \begin{cases} \sigma_m(t) & \text{for } t \in [t_0, c_m], m \in \mathbb{N}, \\ \lim\limits_{m\mapsto\infty} \sigma_m(c_m) & \text{for } t = c^*, \end{cases}$$

$$g^*_{\eta\xi}(t) = \begin{cases} g_m(t)(\eta, \xi) & \text{for } t \in [t_0, c_m], m \in \mathbb{N}, \text{for all } \eta, \xi \in \mathbb{D}\underline{\otimes}\mathbb{E}), \\ 0 & \text{for } t = c^*, \end{cases}$$

$$\varphi^*_{\eta\xi}(t) = \begin{cases} \varphi_m(t)(\eta, \xi) & \text{for } t \in [t_0, c_m], m \in \mathbb{N}, \text{for all } \eta, \xi \in \mathbb{D}\underline{\otimes}\mathbb{E}), \\ \lim\limits_{m\mapsto\infty} \varphi_m(c_m)(\eta, \xi) & \text{for } t = c^*, \end{cases}$$

This shows that $(\sigma^*, g^*_{\eta\xi}, \varphi^*_{\eta\xi})$ is an ϵ- approximate solution which is an upper bound for $((\sigma_m, g_{m,\eta\xi}, \varphi_{m,\eta\xi}))_m$. Applying (ii) of Brezis–Browder theorem, we define the function

$\mathcal{M} : \mathcal{S} \longrightarrow \mathbb{R} \cup \{+\infty\}$. Then, for each $\zeta_0 \in \mathcal{S}$ there exists an $\mathcal{M}$- maximal element $\bar{\zeta} \in \mathcal{S}$ satisfying $\zeta_0 \preceq \bar{\zeta}$. This shows that $\mathcal{M}((\sigma, g_{\eta\xi}, \varphi_{\eta\xi})) = c$ where $[t_0, c]$ is the domain of definition of $(\sigma, g_{\eta\xi}, \varphi_{\eta\xi})$. Then $\mathcal{M}$ satisfies the hypothesis of Brezis–Browder theorem . Then, $\mathcal{S}$ contains at least one $\mathcal{M}$- maximal element $(\bar{\sigma}, \bar{g}_{\eta\xi}, \bar{\varphi}_{\eta\xi})$ defined on $[t_0, \bar{c}]$. In other words, if $(\tilde{\sigma}, \tilde{g}_{\eta\xi}, \tilde{\varphi}_{\eta\xi}) \in \mathcal{S}$, defined on $[t_0, \tilde{c}]$, satisfies $(\bar{\sigma}, \bar{g}_{\eta\xi}, \bar{\varphi}_{\eta\xi}) \preceq (\tilde{\sigma}, \tilde{g}_{\eta\xi}, \tilde{\varphi}_{\eta\xi})$, then we necessarily have $\bar{c} = \tilde{c}$. We will show next that $\bar{c} = T$. we assume by contradiction that $\bar{c} < T$. Then, taking into account the fact that $\bar{\varphi}_{\eta\xi}(\bar{c}) \in D(x_{0,\eta\xi}, \rho) \cap K(\eta, \xi)$, we have

$$
\begin{aligned}
&| \bar{\varphi_{\eta\xi}}(\bar{c}) - x_{0,\eta\xi} | \\
&\leq \int_{t_0}^{\bar{c}} | P(\bar{\sigma}(s), \bar{\varphi}(\bar{\sigma}(s)))(\eta, \xi) | ds + \int_{t_0}^{\bar{c}} | \bar{g}(\eta, \xi)(s) | ds \\
&\leq (\bar{c} - t_0)(M_{\eta\xi} + \epsilon) \\
&\leq (\bar{c} - t_0)(M_{\eta\xi} + 1) < (T - t_0)(M_{\eta\xi} + 1) \leq \rho
\end{aligned}
$$

Then, as $\bar{\varphi}_{\eta\xi}(\bar{c}) \in K(\eta, \xi)$ and $P(\bar{c}, \bar{\varphi}(\bar{c}))(\eta, \xi) \in \mathcal{T}_K(\bar{\varphi}(\bar{c}))(\eta, \xi)$, there exists $\delta(0, T - \bar{c})$, $\delta \leq \epsilon$ and $q_{\eta\xi} \in \mathbb{C}$ such that $| q_{\eta\xi} | \leq \epsilon$ and

$$
\bar{\varphi}_{\eta\xi}(\bar{c}) + \delta P(\bar{c}, \bar{\varphi}(\bar{c}))(\eta, \xi) + \delta q_{\eta\xi} \in K(\eta, \xi)
$$

From the inequality above we have

$$
| \bar{\varphi}(\bar{c})(\eta, \xi) + \delta[P(\bar{c}, \bar{\varphi}(\bar{c}))(\eta, \xi) + q_{\eta\xi}] - \varphi_0(\eta, \xi) | \leq \rho
$$

We now define the functions $\sigma : [t_0, \bar{c} + \delta] \longrightarrow [t_0, \bar{c} + \delta]$ and $g : [t_0, \bar{c} + \delta] \longrightarrow \mathbb{C}$ by

$$
\sigma(t) = \begin{cases} \bar{\sigma}(t) & \text{for } t \in [t_0, \bar{c}], \\ \bar{c} & \text{for } t \in [\bar{c}, \bar{c} + \delta], \end{cases}
$$

$$
g_{\eta\xi}(t) = \begin{cases} \bar{g}_{\eta\xi}(t) & \text{for } t \in [t_0, \bar{c}], \text{ and for any } \eta, \xi \in \mathbb{D}\underline{\otimes}\mathbb{E}), \\ q & \text{for } t \in [\bar{c}, \bar{c} + \delta], \end{cases}
$$

so $| g_{\eta\xi}(t) | \leq \epsilon$ for every $t \in [t_0, \bar{c} + \delta]$. In addition, for every $t \in [t_0, \bar{c} + \delta]$, $\sigma(t) \in [t_0, \bar{c}]$ and therefore $\bar{\varphi}(\sigma(t))$ is well-defined and belongs to the set $D(x_{0,\eta\xi}, \rho) \cap K$. Accordingly, we can define $\varphi_{\eta\xi} : [t_0, \bar{c} + \delta] \longrightarrow \mathbb{C}$ by

$$
\langle \eta, \varphi(t)\xi \rangle = \langle \eta, \varphi_0 \xi \rangle + \int_{t_0}^{t} P(\sigma(s), \bar{\varphi}(\sigma(s)))(\eta, \xi) ds + \int_{t_0}^{t} g(\eta, \xi)(s) ds
$$

for every $t \in [t_0, \bar{c} + \delta]$. clearly, $\varphi_{\eta,\xi}$ coincides with $\bar{\varphi_{\eta,\xi}}$ on $[t_0, \bar{c}]$ since the domain $[t_0, \bar{c}]$ is included in the domain of $[\bar{c}, \bar{c} + \delta]$ and then it readily follows that $\varphi_{\eta,\xi}$, σ and $g_{\eta,\xi}$ satisfy all the conditions in (i) and (ii). In order to prove (iii) and (iv) we observe that

$$
\varphi_{\eta\xi}(t) = \begin{cases} \bar{\varphi}_{\eta\xi}(t) & \text{for } t \in [t_0, \bar{c}]. \\ \varphi_{\eta\xi}(\bar{c}) + (t - \bar{c}) P(\bar{c}, \bar{\varphi}(\bar{c}))(\eta\xi) + (t - \bar{c}) q & \text{for } t \in [\bar{c}, \bar{c} + \delta] \end{cases}
$$

Then $\varphi_{\eta\xi}$ satisfies the equation in (iv). since

$$
\varphi_{\eta\xi}(\sigma(t)) = \begin{cases} \bar{\varphi}_{\eta\xi}(\bar{\sigma}(t) & \text{for } t \in [t_0, \bar{c}]. \\ \bar{\varphi}_{\eta\xi}(\bar{c}) & \text{for } t \in [t_0, \bar{c} + \delta] \end{cases}
$$

it follows that $\varphi_{\eta\xi}(\sigma(t)) \in D(x_{0,\eta\xi}, \rho) \cap K$.

Furthermore, from the choice of δ and q, we have both $\varphi_{\eta\xi}(\bar{c}+\delta) = \bar{\varphi_{\eta\xi}}(\bar{c})(\eta,\xi) + \delta P(\bar{c}, \bar{\varphi}(\bar{c}))(\eta,\xi) + \delta q \in K(\eta,\xi)$ and

$$\begin{aligned} &| \varphi(\bar{c}+\delta)(\eta,\xi) - x_0(\eta,\xi) | \\ &\quad = | \bar{\varphi}(\bar{c})(\eta,\xi) + \delta P(\bar{c}, \bar{\varphi}(\bar{c}))(\eta,\xi) + \delta q - x_0(\eta,\xi) | \\ &\quad \le \rho \end{aligned}$$

and consequently, $\varphi_{\eta\xi}$ satisfies (iii). Thus $(\sigma, g_{\eta\xi}, \varphi_{\eta\xi}) \in \mathcal{S}$. Furthermore, since $(\bar{\sigma}, \bar{g}_{\eta\xi}, \bar{\varphi}_{\eta\xi}) \preceq (\sigma, g_{\eta\xi}, \varphi_{\eta\xi})$ and $\bar{c} < \bar{c}+\delta$, it follows that $(\bar{\sigma}, \bar{g}_{\eta\xi}, \bar{\varphi}_{\eta\xi})$ is not a $\mathcal{M}$- maximal element. But this is absurd, we can eliminate this contradiction, only if each maximal element in the set $\mathcal{S}$ is defined on $[t_0, T]$. Hence $\bar{c} = T$.

Theorem 1 *Let $K \subset \tilde{\mathcal{A}}$ Assume that the following conditions hold:*

(i) The map $(t,x) \longrightarrow P(t,x)(\eta,\xi)$ associated with the right hand-side of equation (4) is continuous.
(ii) $K(\eta,\xi)$ is non empty and locally closed
(iii) There exists $M_{\eta\xi} > 0$ such that $|P(t,x)(\eta,\xi)| \le M_{\eta\xi}$ for every $t \in [t_0, T]$ and $x \in D(x_{0,\eta\xi}, \rho) \cap K(\eta,\xi)$
(iv) $(T - t_0)(M_{\eta\xi} + 1) \le \rho$

Then $K(\eta,\xi)$ is viable with respect to P if and only if for every $(t_0, x_0) \in I \times K$ we have $P(t_0, x_0)(\eta,\xi) \in \mathcal{T}_{\mathcal{K}(\equiv,\sim)}(x_{0,\eta\xi})$

Proof The proof is divided into two parts; We proceed as follows: **If Part**: Suppose $K(\eta,\xi)$ is viable with respect to P, then there exists a solution φ that satisfy Eq. (4).

Let $(t_0, x_{0,\eta\xi}) \in I \times K(\eta,\xi)$, We prove that

$$\lim_{h \mapsto 0} \frac{1}{h} \mathbf{d}(x_{0,\eta\xi} + hP(t_0, x_0)(\eta,\xi); K(\eta,\xi)) = 0.$$

then, there exists $T \in I, T > t_0$, and a stochastic process $\varphi \in K$ with $\langle \eta, \varphi(t_0)\xi \rangle \in K(\eta,\xi)$ satisfying Eq. (4).

$$\begin{aligned} &\mathbf{d}(x_{0,\eta\xi} + hP(t_0, x_0)(\eta,\xi); K(\eta,\xi)) \\ &\quad \le \left| x_{0,\eta\xi} + hP(t_0, x_0)(\eta,\xi) - \varphi(t_0)(\eta,\xi) \right| \\ &\quad = \lim_{h \mapsto 0} \frac{1}{h} \left| x_{0,\eta\xi} + hP(t_0, x_0)(\eta,\xi) - \langle \eta, \varphi(t_0+h)\xi \rangle \right| \\ &\quad = \lim_{h \mapsto 0} \left| P(t_0, \varphi(t_0))(\eta,\xi) - \frac{\langle \eta, (\varphi(t_0+h) - \varphi(t_0)\xi \rangle}{h} \right| \\ &\quad = \left| P(t_0, \varphi(t_0))(\eta,\xi) - \lim_{h \mapsto 0} \frac{\langle \eta, (\varphi(t_0+h) - \varphi(t_0)\xi \rangle}{h} \right| \\ &\quad = \left| P(t_0, \varphi(t_0))(\eta,\xi) - \frac{d}{dt} \langle \eta, \varphi(t)\xi \rangle \right|_{t=t_0} \\ &\quad = 0 \end{aligned}$$

This shows that the stochastic process φ is a solution to Eq. (4) and belongs to K.

Only If Part Suppose $P(t_0, x_0)(\eta, \xi) \in \mathcal{T}_{\mathcal{K}(\equiv,\sim)}$ then we prove that P is viable to K.

This concerns the existence and convergence of approximate solutions.

The proof is divided into two steps. The first step is concerned with the proof of existence of a family of "approximate solutions" for the problem defined on interval $[t_0, c]$ with $c \in I$ and later showed that the problem above admits such approximate solutions, all defined on an interval $[t_0, T]$ independent of the "approximate order". The proof of the approximate solution is given by lemma 1 Finally, in the second step, we shall prove the uniform convergence on $[t_0, T]$ of a sequence of such approximate solutions to a solution of the problem (4). □

3.2 Convergence of approximate solutions

Let $(\epsilon_k)_{k\in\mathbb{N}}$ be a sequence from (0, 1) decreasing to 0 and let $(\sigma_k, g_{\eta\xi,k}, \varphi_{\eta\xi,k})_{k\in\mathbb{N}}$ be a sequence of ϵ_k- approximate solutions defined on $[t_0, T]$.
From (i) and (ii), it follows that

$$\lim_{k\mapsto\infty} \sigma_k(t) = t \text{ and } \lim_{k\mapsto\infty} g_{\eta\xi,k}(t) = 0 \tag{8}$$

uniformly on $[t_0, T]$. On the other hand, from (iii), (iv) and (6) we have

$$\begin{aligned}
&|\langle \eta, \varphi_k(t)\xi\rangle| \\
&\quad \le |\langle \eta, (\varphi_k(t) - \varphi_0)\xi\rangle| + |\langle \eta, \varphi_0\xi\rangle| \\
&\quad \le \int_{t_0}^{T} \mid P(\sigma_k(s), \varphi_k(\sigma_k(s)))(\eta, \xi) \mid ds + \int_{t_0}^{T} |g_k(s)(\eta, \xi)|ds + |\varphi_{0,\eta\xi}| \\
&\quad \le (T - t_0)(M_{\eta\xi} + 1) + |\varphi_0|_{\eta\xi} \le \rho + |\varphi_{0,\eta\xi}|
\end{aligned}$$

for every $k \in \mathbb{N}$ and every $t \in [t_0, T]$. Hence, the sequence $(\varphi_k)_{k\in\mathbb{N}}$ is uniformly bounded on $[t_0, T]$. Again from (iv), we have

$$\begin{aligned}
&|\langle \eta, \varphi_k(t) - \varphi_k(s)\xi\rangle| \\
&\quad \le |\int_s^t \frac{d}{dt}\langle \eta, \varphi_k(t)\xi\rangle dt_0| + |\int_s^t |\langle \eta, g_k(t_0)\xi\rangle|dt_0 \,| \\
&\quad \le| P(\sigma_k(s), \varphi_k(\sigma_k(s)))(\eta, \xi) \mid ds + |g_k(s)(\eta, \xi)|ds \\
&\quad \le (M_{\eta\xi} + 1)\,|t - s|
\end{aligned}$$

for every $t, s \in [t_0, T]$. Consequently the sequence $(\varphi_k)_{k\in\mathbb{N}}$ is equicontinuous on $[t_0, T]$. However from Arzela - Ascolis theorem there exists at least a subsequence of $(\varphi_{\eta\xi,k})_{k\in\mathbb{N}}$ that is uniformly convergent to some point $\varphi_{\eta\xi}$. i.e there exists a stochastic process $\varphi : [t_0, T] \longrightarrow \tilde{\mathcal{A}}$ lying in$Ad(\tilde{\mathcal{A}})_{wac} \cap L^2_{loc}(\tilde{\mathcal{A}})$ such that $\varphi_{\eta\xi} = \langle \eta, \varphi\xi\rangle$ and $\varphi_{\eta\xi,k} = \langle \eta, \varphi_k\xi\rangle$ then,

$$\lim_{k\to\infty} \langle \eta, \varphi_k\xi\rangle = \langle \eta, \lim_{k\to\infty} \varphi_k\xi\rangle$$

$$= \langle \eta, \varphi \xi \rangle$$

Now using (iii), (8) and of the fact that $D(x_{0,\eta\xi}, \rho) \cap K(\eta, \xi)$ is closed, we conclude that $\varphi(t)(\eta, \xi) \in D(x_{0,\eta\xi}, \rho) \cap K(\eta, \xi)$ for every $t \in [t_0, T]$.

$$\langle \eta, \varphi_k(t)\xi \rangle = \langle \eta, \varphi_0 \xi \rangle + \int_{t_0}^{t} P(\sigma_k(t_0), \varphi_k(\sigma_k(s)))(\eta, \xi)ds + \int_{t_0}^{t} g_k(s)(\eta, \xi)ds$$

now, taking the limit of the above and using (8), we have that

$$\langle \eta, \varphi(t)\xi \rangle = \langle \eta, \varphi_0 \xi \rangle + \int_{t_0}^{t} P(s, \varphi(s))(\eta\xi)ds$$

for every $t \in [t_0, T]$, which gives the proof of the theorem.

Compliance with ethical standards

Conflict of interest The authors declare that there is no conflict of interest.

References

1. Aubin, J.: Viability Theory. Birkhauser, Basel (1991)
2. Aubin, J., Bayen, A., Pierre, P.: Viability Theory New Directions. Springer, Berlin (2011)
3. Aubin, J.P., Cellina, A.: Differential Inclusions; Set-Valued Maps and Viability Theory, pp. 139–171. Springer, Berlin (1984)
4. Ayoola, E.O.: Lipschitzian quantum stochastic differential equations and the associated Kurzweil equations. Stoch. Anal. Appl. **19**(4), 581–603 (2001)
5. Ayoola, E.O.: On convergence of one-step schemes for weak solutions of quantum stochastic differential equations. Acta Appl. Math. **67**(1), 19–58 (2001)
6. Ayoola, E.O.: Continuous selections of solution sets of Lipschitzian quantum stochastic differential inclusions. Int. J. Theor. Phys. **43**(10), 2041–2059 (2004)
7. Ayoola, E.O.: Topological properties of solution sets of Lipschitzian quantum stochastic differential inclusions. Acta Appl. Math. **100**(1), 15–37 (2008)
8. Ayoola, E.O.: Error estimates for discretized quantum stochastic differential inclusions. Stoch. Anal. Appl. **21**(6), 1215–1230 (2003)
9. Ayoola, E.O.: Exponential formula for the reachable sets of quantum stochastic differential inclusions. Stoch. Anal. Appl. **21**(3), 515–543 (2003)
10. Carja, O., Necula, M., Vrabie, I.I.: Viability, Invariance and Applications, vol. 207. Elsevier, Amsterdam (2007)
11. Cârja, O., Vrabie, I.I.: Differential equations on closed sets. In: Canada, A., Drabek, P., Fonda, A. (eds.) Handbook of Differential Equations, vol. 2, pp. 147–238. Elsevier, North Holland (2005)
12. Ekhaguere, G.: The functional Ito formula in quantum stochastic calculus. J. Math. Phys. **31**(12), 2921–2929 (1990)
13. Ekhaguere, G.: Lipschitzian quantum stochastic differential inclusions. Int. J. Theor. Phys. **31**(11), 2003–2027 (1992)
14. Ekhaguere, G.: Quantum stochastic differential inclusions of hypermaximal monotone type. Int. J. Theor. Phys. **34**(3), 323–353 (1995)
15. Ekhaguere, G.: Quantum stochastic evolutions. Int. J. Theor. Phys. **35**(9), 1909–1946 (1996)
16. Ekhaguere, G.: Topological solutions of noncommutative stochastic differential equations. Stoch. Anal. Appl. **25**(5), 961–993 (2007)
17. Hudson, R.L., Parthasarathy, K.R.: Quantum Ito's formula and stochastic evolutions. Commun. Math. Phys. **93**(3), 301–323 (1984)

18. Marques, M.D.M., et al.: Viability results for nonautonomous differential inclusions. J. Convex Anal. **7**(2), 437–443 (2000)
19. Ogundiran, M.O., Ayoola, E.O.: Upper semicontinuous quantum stochastic differential inclusions via Kakutani-fan fixed point theorem. Dyn. Syst. Appl. **21**, 121–132 (2012)
20. Ogundiran, M.O., Ayoola, E.O.: Lower semicontinuous quantum stochastic differential inclusions. Eur. J. Math. Sci. **2**(1), 1–16 (2013)
21. Smirnov, G.V.: Introduction to the Theory of Differential Inclusions, vol. 41. American Mathematical Society, Providence (2002)
22. Vrabie, I.I.: A nagumo type viability theorem. An. Stiint. Univ. Al. I. Cuza Iasi. Mat. (NS) **51**, 293–308 (2005)

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Further results on the existence, uniqueness and stability of strong solutions of quantum stochastic differential equations

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Abstract

Under a more general Lipschitz condition on the coefficients than our consideration in [E.O. Ayoola, Existence and stability results for strong solutions of quantum stochastic differential equations, Stochastic Anal. Appl. 20 (2) (2002) 263–281], we establish the existence, uniqueness and stability of strong solutions of quantum stochastic differential equations (QSDE). This enables us to exhibit a class of Lipschitzian QSDE whose coefficients are continuous on the locally convex space of solution.

Keywords: QSDE; Fock space; Exponential vectors; Lipschitzian; Quantum stochastic processes

1. Introduction

This paper continues our previous work in [1] concerning the investigation of the properties of strong solutions of quantum stochastic differential equations (QSDE) in integral form given by

$$X(t) = X_0 + \int_0^t (E(s, X(s))\mathrm{d}\wedge_\pi(s) + F(s, X(s))\mathrm{d}A_f(s) + G(s, X(s))\mathrm{d}A_g^+(s) + H(s, X(s))\mathrm{d}s), \; t \in [0, T]. \tag{1.1}$$

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Eq. (1.1) is understood in the framework of the Hudson–Parthasarathy [2] formulation of quantum stochastic calculus. The stochastic integrators $\wedge_\pi, A_f, A_g^+$ are the usual gauge, annihilation and creation processes associated with the basic field operators of quantum field theory, defined for π, f, g belonging to appropriate function spaces. On this occasion, we consider a more general class of Lipschitzian coefficients E, F, G, H. Under the present Lipschitz condition, we establish the existence, uniqueness and stability of strong solution of Eq. (1.1). An immediate consequence of our result is that we are able to exhibit a wider class of Lipschitzian QSDE (1.1) whose coefficients are only continuous on the space of our quantum stochastic processes. This extends our previous results in [1] achieved by employing a method of successive approximations in the same way as in [3]. Our previous works [1,4–7] have focussed on some qualitative aspects and approximations of the weak and strong solutions of (1.1) and the associated quantum stochastic differential inclusions considered elsewhere.

We employ the notations and structures introduced in [1]. Details of the various spaces employed in this paper can be found in the reference. We employ strong topology in this paper and refer to the solution of (1.1) as strong compared with the weak topology employed in Refs. [4–8] leading to weak solutions of the associated QSDE. We refer the reader to Refs. [1,2,4–10] for some interesting accounts of the Hudson–Parthasarathy quantum stochastic calculus.

The rest of the paper is organised as follows. Section 2 is devoted to some fundamental results, notations and assumptions. Our main results concerning the existence, uniqueness and stability of QSDE (1.1) are established in Section 3.

2. Some fundamental results and assumptions

As outlined in [1], we shall adopt the following notations and spaces in what follows. $\mathbb{D}$ is some inner product space with $\mathcal{R}$ as its completion and γ is some fixed Hilbert space. We denote by $L^2_\gamma(\mathbb{R}_+)$ the Hilbert space of square integrable, γ-valued maps on $\mathbb{R}_+ := [0, \infty)$. Furthermore, we let $\mathbb{E}$ denote the linear space generated by the exponential vectors in $\Gamma(L^2_\gamma(\mathbb{R}_+))$, the Boson Fock space determined by the space $L^2_\gamma(\mathbb{R}_+)$. Let $\mathcal{B} = L^+_W(\mathbb{D} \underline{\otimes} \mathbb{E}, \mathcal{R} \otimes \Gamma(L^2_\gamma(\mathbb{R}_+)))$ denote the linear space of all linear operators from $\mathbb{D} \underline{\otimes} \mathbb{E}$ into $\mathcal{R} \otimes \Gamma(L^2_\gamma(\mathbb{R}_+))$ with the property that the domain of the operator adjoint contains $\mathbb{D} \underline{\otimes} \mathbb{E}$. We shall denote by $\tilde{\mathcal{B}}$ the completion of the topological space $(\mathcal{B}, \tau)$, where τ is the topology generated by the family of seminorms $\|x\|_\xi = \|x\xi\|, \xi \in \mathbb{D} \underline{\otimes} \mathbb{E}$. Here, $\|\cdot\|$ is the norm of the space $\mathcal{R} \otimes \Gamma(L^2_\gamma(\mathbb{R}_+))$.

In the formulations of this paper, quantum stochastic processes are $\tilde{\mathcal{B}}$-valued maps defined on the interval $[0, T]$. As in [1], we shall denote by $L^p_{loc}(\tilde{\mathcal{B}})$, $p \in (0, \infty)$, the set of adapted, locally absolutely p-integrable stochastic processes, and by $Ad(\tilde{\mathcal{B}})_{ac}$ the set of adapted absolutely continuous stochastic processes.

In the proofs of our main results, we shall extensively employ the following results due to Hudson and Parthasarathy [2].

Theorem 2.1. (a) *Let $p, q, u, v \in L^2_{loc}(\tilde{\mathcal{B}})$ and let M be their stochastic integral. If $\eta, \xi \in \mathbb{D} \underline{\otimes} \mathbb{E}$ with $\eta = c \otimes e(\alpha), \xi = d \otimes e(\beta), \alpha, \beta \in L^\infty_{\gamma,loc}(\mathbb{R}_+)$ and $t \geq 0$, then*

$$<\eta, M(t)\xi> = \int_0^t <\eta, \{<\alpha(s), \pi(s)\beta(s)>_\gamma\ p(s) + <f(s), \beta(s)>_\gamma\ q(s) + <\alpha(s), g(s)>_\gamma\ u(s) + v(s)\}\xi> \mathrm{d}s. \tag{2.1}$$

(b) *Let* $K(T) = \sup_{0\le s\le T} \max\{|\langle\beta(s), \pi(s)\beta(s)\rangle|, |\langle f(s), \beta(s)\rangle|, |\langle\beta(s), g(s)\rangle|, \|\pi(s)\beta(s)\|^2, \|g(s)\|^2\}$. *Then for* $T > 0$ *and* $0 \le t \le T$,

$$\|M(t)\xi\|^2 \le 6K(T)^2 \int_0^T \mathrm{e}^{t-s}\{\|p(s)\xi\|^2 + \|q(s)\xi\|^2 + \|u(s)\xi\|^2 + \|v(s)\xi\|^2\}\mathrm{d}s. \tag{2.2}$$

(c) *Let* $0 \le s \le t \le T$. *Then*

$$\|(M(t) - M(s))\xi\|^2 \le 6K(T)^2 \int_s^t \mathrm{e}^{t-\tau}\{\|p(\tau)\xi\|^2 + \|q(\tau)\xi\|^2 + \|u(\tau)\xi\|^2 + \|v(\tau)\xi\|^2\}\mathrm{d}\tau. \tag{2.3}$$

In particular, M is absolutely continuous and thus belongs to the space $L^2_{loc}(\tilde{\mathcal{B}})$.

Definition 2.1. (a) Let $\mathrm{Fin}(\mathbb{D}\underline{\otimes}\mathbb{E})$ denote the set of all finite subsets of $\mathbb{D}\underline{\otimes}\mathbb{E}$. If $x \in \tilde{\mathcal{B}}$, and $\Theta \in \mathrm{Fin}(\mathbb{D}\underline{\otimes}\mathbb{E})$, define $\|x\|_\Theta$ by $\|x\|_\Theta = \max_{\xi\in\Theta}\|x\|_\xi$. Then, the set $\{\|\cdot\|_\Theta : \Theta \in \mathrm{Fin}(\mathbb{D}\underline{\otimes}\mathbb{E})\}$ is a family of seminorms on $\tilde{\mathcal{B}}$. We denote by $\tilde{\tau}$ the topology generated by this family of seminorms on $\tilde{\mathcal{B}}$.

(b) Let $I = [0, T] \subseteq \mathbb{R}_+$. A map $\Phi : I \times \tilde{\mathcal{B}} \longrightarrow \tilde{\mathcal{B}}$ will be called Lipschitzian if, for each $\xi \in \mathbb{D}\underline{\otimes}\mathbb{E}$, the map satisfies an estimate of the type

$$\|\Phi(t, x) - \Phi(t, y)\|_\xi \le K_\xi^\Phi(t)\|x - y\|_{\Theta_\Phi(\xi)} \tag{2.4}$$

for all $x, y \in \tilde{\mathcal{B}}$ and almost all $t \in I$ and where $K_\xi^\Phi : I \longrightarrow (0,\infty)$ lies in $L^1_{loc}(I)$ and Θ_Φ is a map from $\mathbb{D}\underline{\otimes}\mathbb{E}$ into $\mathrm{Fin}(\mathbb{D}\underline{\otimes}\mathbb{E})$.

Remark. Let $L(\tilde{\mathcal{B}})$ denote the linear space of all continuous endomorphisms of $\tilde{\mathcal{B}}$. Then the above definition enables us to exhibit a class of Lipschitzian maps as follows.

Theorem 2.2. *Let* $A : \mathbb{R}_+ \to L(\tilde{\mathcal{B}})$ *and* $F : \mathbb{R}_+ \times \tilde{\mathcal{B}} \to \tilde{\mathcal{B}}$ *be given by* $F(t, x) = A(t)x$, *for* $x \in \tilde{\mathcal{B}}, t \in \mathbb{R}_+$. *Then* F *is Lipschitzian.*

Proof. Let $x, y \in \tilde{\mathcal{B}}, t \in \mathbb{R}_+$, then

$$\|F(t, x) - F(t, y)\|_\xi = \|A(t)x - A(t)y\|_\xi = \|A(t)(x - y)\|_\xi \le C_\xi^A(t)\|x - y\|_{\Theta_A(\xi)},$$

where $C_\xi^A(t)$ is a positive function depending on A, t, ξ, and Θ_A is a map from $\mathbb{D}\underline{\otimes}\mathbb{E}$ into $\mathrm{Fin}(\mathbb{D}\underline{\otimes}\mathbb{E})$. □

Remark. (a) Theorem 2.2 demonstrates that all continuous linear maps of $\tilde{\mathcal{B}}$ into itself are automatically Lipschitzian in the sense of this paper.

(b) Since Θ is a finite set, we see that $\|x\|_\Theta = \|x\|_{\xi'}$, for some $\xi' \in \Theta$. Using the foregoing fact, we employ in the proof of our main results below the fact that a map $\Phi : I \times \tilde{\mathcal{B}} \to \tilde{\mathcal{B}}$ is Lipschitzian in the sense of (2.4) if, given any $\xi \in \mathbb{D}\underline{\otimes}\mathbb{E}$, there corresponds $\xi' \in \mathbb{D}\underline{\otimes}\mathbb{E}$ such that $\|\Phi(t, x) - \Phi(t, y)\|_\xi \le K_\xi^\Phi(t)\|x - y\|_{\xi'}$ for all $x, y \in \tilde{\mathcal{B}}$ and almost all $t \in I$.

(c) Using the definition in (b), we see that, if $P : \mathbb{R}_+ \to \tilde{\mathcal{B}}$ and $\xi_0 \in \mathbb{D}\underline{\otimes}\mathbb{E}$ is a fixed point, then the map F defined by $F(t, x) = \|x\xi_0\|P(t)$ is Lipschitzian. This can be shown as follows: for any $t \in \mathbb{R}_+, x, y, \in \tilde{\mathcal{B}}$,

$$\begin{aligned}\|F(t, x) - F(t, y)\|_\xi &= \|(\|x\xi_0\| - \|y\xi_0\|)P(t)\|_\xi \le |\|x\xi_0\| - \|y\xi_0\||\,\|P(t)\|_\xi \\ &\le \|P(t)\|_\xi\|x - y\|_{\xi_0}. \quad \square\end{aligned}$$

3. Existence, uniqueness and stability of solution

The main results of this paper are established in this section. We recall here that by a solution of Eq. (1.1), we mean an absolutely continuous stochastic process $\phi \in L^2_{loc}(\tilde{\mathcal{B}})$ satisfying Eq. (1.1). We present the following theorem.

Theorem 3.1. *Suppose that the coefficients E, F, G, H appearing in Eq. (1.1) are Lipschitzian and belong to $L^2_{loc}(I \times \tilde{\mathcal{B}})$. Then for any fixed point X_0 of $\tilde{\mathcal{B}}$, there exists a unique adapted and absolutely continuous solution Φ of quantum stochastic differential equation (1.1) satisfying $\Phi(0) = X_0$.*

Proof. As in [1], we will establish the theorem by constructing a Cauchy sequence $\{\Phi_n\}_{n\geq 0}$ of successive approximations of Φ in $\tilde{\mathcal{B}}$. In what follows, let $\xi \in \mathbb{D}\underline{\otimes}\mathbb{E}$ be arbitrary. Let $t \in [0, T]$ and define $\Phi_0(t) = X_0$, and for $n \geq 0$

$$\Phi_{n+1}(t) = X_0 + \int_0^t (E(s, \Phi_n(s))\mathrm{d}\wedge_\pi(s) + F(s, \Phi_n(s))\mathrm{d}A_f(s) + G(s, \Phi_n(s))\mathrm{d}A_g^+(s) + H(s, \Phi_n(s))\mathrm{d}s). \tag{3.1}$$

It has been established in [1] that each $\Phi_n(t), n \geq 1$ defines an adapted absolutely continuous process in $L^2_{loc}(\tilde{\mathcal{B}})$. We now consider the convergence of the successive approximations. We have

$$\begin{aligned}\|\Phi_{n+1}(t) - \Phi_n(t)\|_\xi = \Bigg\| \int_0^t &(E(s, \Phi_n(s)) - E(s, \Phi_{n-1}(s)))\mathrm{d}\wedge_\pi(s) + (F(s, \Phi_n(s)) \\ &- F(s, \Phi_{n-1}(s)))\mathrm{d}A_f(s) + (G(s, \Phi_n(s)) - G(s, \Phi_{n-1}(s)))\mathrm{d}A_g^+(s) \\ &+ (H(s, \Phi_n(s)) - H(s, \Phi_{n-1}(s)))\mathrm{d}s \Bigg\|_\xi .\end{aligned} \tag{3.2}$$

By Theorem 2.1, we have

$$\begin{aligned}\|\Phi_{n+1}(t) - \Phi_n(t)\|_\xi^2 \leq 6K(T)^2 \int_0^t \mathrm{e}^{t-s}\{&\|E(s, \Phi_n(s)) - E(s, \Phi_{n-1}(s))\|_\xi^2 \\ &+ \|F(s, \Phi_n(s)) - F(s, \Phi_{n-1}(s))\|_\xi^2 + \|G(s, \Phi_n(s)) - G(s, \Phi_{n-1}(s))\|_\xi^2 \\ &+ \|H(s, \Phi_n(s)) - H(s, \Phi_{n-1}(s))\|_\xi^2\}\mathrm{d}s.\end{aligned} \tag{3.3}$$

By the Lipschitz condition (2.4) satisfied by the coefficients of (1.1), we have for each $M \in \{E, F, G, H\}$, $\|M(s, \Phi_n(s)) - M(s, \Phi_{n-1}(s))\|_\xi \leq K_\xi^M(s)\|\Phi_n(s) - \Phi_{n-1}(s)\|_{\Theta_M(\xi)}$. Consequently, there exists $\xi_M^1 \in \Theta_M(\xi)$ satisfying

$$\|\Phi_n(s) - \Phi_{n-1}(s)\|_{\Theta_M(\xi)} = \|\Phi_n(s) - \Phi_{n-1}(s)\|_{\xi_M^1}. \tag{3.4}$$

We now put

$$\|\Phi_n(s) - \Phi_{n-1}(s)\|_{\xi_1} = \max_{M\in\{E,F,G,H\}} \|\Phi_n(s) - \Phi_{n-1}(s)\|_{\xi_M^1} \tag{3.5}$$

and

$$C(T) = 6K(T)^2, L_\xi = \operatorname{ess}\sup_{s\in[0,T]} \left[K_\xi(s) = \sum_{M\in\{E,F,G,H\}} K_\xi^M(s)^2\right]. \tag{3.6}$$

Then from (3.3),

$$\|\Phi_{n+1}(t) - \Phi_n(t)\|_\xi^2 \le C(T)L_\xi \int_0^t \mathrm{e}^{t-s}\|\Phi_n(s) - \Phi_{n-1}(s)\|_{\xi_1}^2 \mathrm{d}s$$
$$= C(T)L_\xi \mathrm{e}^t \int_0^t \mathrm{e}^{-s}\|\Phi_n(s) - \Phi_{n-1}(s)\|_{\xi_1}^2 \mathrm{d}s. \tag{3.7}$$

Iterating again, there exists an element $\xi_2 \in \mathbb{D}\underline{\otimes}\mathbb{E}$ satisfying estimate of the form (3.5) such that

$$\|\Phi_{n+1}(t) - \Phi_n(t)\|_\xi^2 \le C(T)^2 L_\xi L_{\xi_1} \mathrm{e}^t \int_0^t \int_0^s \mathrm{e}^{-s'}\|\Phi_{n-1}(s') - \Phi_{n-2}(s')\|_{\xi_2}^2 \mathrm{d}s'\mathrm{d}s. \tag{3.8}$$

At the nth iteration, we have positive real numbers $L_{\xi_j}, j = 0, 1, \dots, n-1$ and elements $\xi, \xi_1, \xi_2, \dots, \xi_n \in \mathbb{D}\underline{\otimes}\mathbb{E}$, $\xi = \xi_0$, such that

$$\|\Phi_{n+1}(t) - \Phi_n(t)\|_\xi^2 \le C(T)^n M(\xi)^n \mathrm{e}^t \int_0^t \mathrm{d}s_1 \int_0^{s_1} \mathrm{d}s_2 \cdots \int_0^{s_{n-2}} \mathrm{d}s_{n-1}$$
$$\times \int_0^{s_{n-1}} \mathrm{e}^{-s_n}\|\Phi_1(s_n) - \Phi_0(s_n)\|_{\xi_n}^2 \mathrm{d}s_n, \tag{3.9}$$

where $M_n(\xi) = \max\{L_{\xi_j}, j = 0, 1, 2, \dots, n-1\}$ and $M(\xi) = \sup_{n\in\mathbb{N}}\{M_n(\xi)\}$.

By the continuity of the map $s \to \|\Phi_1(s) - X_0\|_\eta$ on $[0, T]$ for any $\eta \in \mathbb{D}\underline{\otimes}\mathbb{E}$, we have

$$R_{\xi_n} = \sup_{s\in[0,T]} \|\Phi_1(s) - X_0\|_{\xi_n} < \infty. \tag{3.10}$$

Putting $R_\xi = \sup_{n\in\mathbb{N}}\{R_{\xi_n}\}$, then from (3.9), we get

$$\|\Phi_{n+1}(t) - \Phi_n(t)\|_\xi^2 \le [C(T)M(\xi)]^n \mathrm{e}^T \frac{T^n}{n!} R_\xi^2, \qquad \forall n = 0, 1, 2, \dots. \tag{3.11}$$

Therefore, for any $n > k$,

$$\|\Phi_{n+1}(t) - \Phi_{k+1}(t)\|_\xi = \left\|\sum_{m=k+1}^{n} (\Phi_{m+1}(t) - \Phi_m(t))\right\|_\xi \le \sum_{m=k+1}^{n} \|\Phi_{m+1}(t) - \Phi_m(t)\|_\xi$$
$$\le \mathrm{e}^{\frac{T}{2}} R_\xi \sum_{m=k+1}^{n} \left(\frac{[C(T)M(\xi)]^m T^m}{m!}\right)^{\frac{1}{2}}$$
$$\le \mathrm{e}^{\frac{T}{2}} R_\xi \sum_{m=k+1}^{n} \left(\frac{[C(T)M(\xi)]^m T^m}{m!}\right)^{\frac{1}{2}} < \infty.$$

This shows that the sequence $\{\Phi_n(t)\}$ is Cauchy in $\tilde{\mathcal{B}}$ and converges uniformly to some $\Phi(t)$. Since each $\Phi_n(t)$ is adapted and absolutely continuous, the same is true of $\Phi(t)$. Next, we show that $\Phi(t)$ satisfies Eq. (1.1). Clearly, $\Phi(t_0) = X_0$. Again, by Eq. (3.7), there exists $\eta \in \mathbb{D}\underline{\otimes}\mathbb{E}$ such that

$$\left\|\int_0^t (E(s, \Phi_n(s)) - E(s, \Phi(s)))\mathrm{d}\wedge_\pi(s) + (F(s, \Phi_n(s)) - F(s, \Phi(s)))\mathrm{d}A_f(s)\right.$$
$$\left. + (G(s, \Phi_n(s)) - G(s, \Phi(s)))\mathrm{d}A_g^+(s) + (H(s, \Phi_n(s)) - H(s, \Phi(s)))\mathrm{d}s\right\|_\xi^2$$
$$\le C(T)L_\xi \mathrm{e}^t \int_0^t \mathrm{e}^{-s}\|\Phi_n(s) - \Phi(s)\|_\eta^2 \mathrm{d}s \to 0 \text{ as } n \to \infty.$$

Thus

$$\begin{aligned}\Phi(t) &= \lim_{n\to\infty} \Phi_{n+1}(t)\\ &= X_0 + \int_0^t (E(s, \Phi(s))\mathrm{d}\wedge_\pi(s) + F(s, \Phi(s))\mathrm{d}A_f(s) + G(s, \Phi(s))\mathrm{d}A_g^+(s) + H(s, \Phi(s))\mathrm{d}s).\end{aligned}$$

That is, $\Phi(t), t \in [0, T]$ is a solution of Eq. (1.1).

Uniqueness

Suppose that $Y(t), t \in [0, T]$ is another adapted absolutely continuous solution of (1.1) with $Y(0) = X_0$. Then, in the same way as in the proof of existence of solution, we obtain the estimate

$$\| \Phi(t) - Y(t)\|_\xi^2 \le [C(T)M(\xi)]^n \mathrm{e}^T \frac{T^n}{n!} \sup_{0\le t\le T} \| \Phi(t) - Y(t)\|_{\xi_n}^2. \tag{3.12}$$

Since the right-hand side of (3.12) is finite for each $n \in \mathbb{N}$, the sequence converges to zero as $n \to \infty$. Consequently, $\| \Phi(t) - Y(t)\|_\xi = 0, \forall \xi \in \mathbb{D}\underline{\otimes}\mathbb{E}$, and so $\Phi(t) = Y(t)$ on $\mathbb{D}\underline{\otimes}\mathbb{E}$, $t \in [0, T]$. □

Stability

As in [1], we show under our present Lipschitz condition that the solutions to the stochastic differential equation (1.1) are stable. By this stability, we mean that small changes in the initial condition lead to small changes in the solution over a given finite time interval and for arbitrary elements $\xi \in \mathbb{D}\underline{\otimes}\mathbb{E}$. To this end, we make the following notations and statements.

(a) The coefficients E, F, G, H and the integrators $\wedge_\pi, A_f, A_g^+$ and the Lebesgue measure remain as in Theorem 3.1 above. Let $X(t), Y(t), t \in [0, T]$ be the solutions to the QSDE (1.1) corresponding to the initial conditions $X(t_0) = X_0$ and $Y(t_0) = Y_0$, respectively, where $X_0, Y_0 \in \tilde{\mathcal{B}}$.

(b) We define the function

$$K_\xi(s) = \sum_{M\in\{E,F,G,H\}} (K_\xi^M(s))^2, \tag{3.13}$$

and the constants

$$L_\xi = \operatorname{ess}\sup_{s\in[0,T]} K_\xi(s), \qquad C(T) = 12K(T)^2, \tag{3.14}$$

where $K(T)$ remains as in Theorem 2.1. The solution $X(t)$ is stable under the changes in $X(t_0) = X_0$ in the following sense.

Theorem 3.2. *Given $\epsilon > 0$, there exists $\partial > 0$ such that if $\|X_0 - Y_0\|_\xi < \partial$, for all $\xi \in \mathbb{D}\underline{\otimes}\mathbb{E}$, then $\|X(t) - Y(t)\|_\xi < \epsilon$ for all $t \in [0, T]$.*

Proof. As in the proof of Theorem 3.1, let $X_n(t)$, for $n = 0, 1, \ldots$ and $Y_n(t)$, for $n = 0, 1, \ldots$ be the iterates corresponding to initial conditions X_0 and Y_0 respectively, so that $X_0(t) = X_0$ and $Y_0(t) = Y_0$ for all $0 \le t \le T$. Then we obtain the following inequalities.

$$\|X_{n+1}(t) - Y_{n+1}(t)\|_\xi \leq \|(X_0 - Y_0)\|_\xi + \bigg\| \int_0^t (E(s, X_n(s) - E(s, Y_n(s)))\mathrm{d}\wedge_\pi(s) + (F(s, X_n(s)) - F(s, Y_n(s)))\mathrm{d}A_f + (G(s, X_n(s)) - G(s, Y_n(s)))\mathrm{d}A_g^+(s) + (H(s, X_n(s)) - H(s, Y_n(s)))\mathrm{d}s) \bigg\|_\xi . \quad (3.15)$$

Therefore, by employing the estimates in Theorem 2.1, we have

$$\begin{aligned}\|X_{n+1}(t) - Y_{n+1}(t)\|_\xi^2 &\leq 2\|X_0 - Y_0\|_\xi^2 + 2\bigg\| \int_0^t (E(s, X_n(s)) - E(s, Y_n(s)))\mathrm{d}\wedge_\pi(s) \\ &\quad + (F(s, X_n(s)) - F(s, Y_n(s)))\mathrm{d}A_f(s) + (G(s, X_n(s)) \\ &\quad - G(s, Y_n(s)))\mathrm{d}A_g^+(s) + (H(s, X_n(s)) - H(s, Y_n(s)))\mathrm{d}s \bigg\|_\xi^2 \\ &\leq 2\|X_0 - Y_0\|_\xi^2 + C(T)\int_0^t \mathrm{e}^{t-s}\{\|E(s, X_n(s)) - E(s, Y_n(s))\|_\xi^2 \\ &\quad + \|F(s, X_n(s)) - F(s, Y_n(s))\|_\xi^2 + \|G(s, X_n(s)) - G(s, Y_n(s))\|_\xi^2 \\ &\quad + \|H(s, X_n(s)) - H(s, Y_n(s))\|_\xi^2\}\mathrm{d}s. \end{aligned} \quad (3.16)$$

By the Lipschitz hypothesis on the coefficients E, F, G, H, there exist elements $\xi_{M,1} \in \Theta_M(\xi) \in \mathrm{Fin}(\mathbb{D} \underline{\otimes} \mathbb{E})$ for each $M \in \{E, F, G, H\}$ such that

$$\begin{aligned}\|X_{n+1}(t) - Y_{n+1}(t)\|_\xi^2 &\leq 2\|X_0 - Y_0\|_\xi^2 + C(T) \\ &\quad \times \int_0^t \mathrm{e}^{t-s_1}\left[\sum_{M\in\{E,F,G,H\}} K_\xi^M(s_1)^2\|X_n(s_1) - Y_n(s_1)\|_{\xi_{M,1}}^2\right]\mathrm{d}s_1 \\ &\leq 2\|X_0 - Y_0\|_\xi^2 + C(T)L_\xi \mathrm{e}^t \int_0^t \mathrm{e}^{-s_1}\|X_n(s_1) - Y_n(s_1)\|_{\xi_1}^2\mathrm{d}s_1, \end{aligned} \quad (3.17)$$

where $\xi_1 \in \{\xi_{M,1} : M \in \{E, F, G, H\}\}$ satisfying

$$\|X_n(s_1) - Y_n(s_1)\|_{\xi_1}^2 = \max_{M\in\{E,F,G,H\}}\{\|X_n(s_1) - Y_n(s_1)\|_{\xi_{M,1}}^2\}, s_1 \in [0, T]. \quad (3.18)$$

Similarly, there exists $\xi_2 \in \mathbb{D} \underline{\otimes} \mathbb{E}$ such that

$$\|X_n(s_1) - Y_n(s_1)\|_{\xi_1}^2 \leq 2\|X_0 - Y_0\|_{\xi_1}^2 + C(T)L_{\xi_1}\int_0^{s_1} \mathrm{e}^{s_1-s_2}\|X_{n-1}(s_2) - Y_{n-1}(s_2)\|_{\xi_2}^2\mathrm{d}s_2. \quad (3.19)$$

On account of (3.17), we have for $0 \leq t \leq T$,

$$\begin{aligned}\|X_{n+1}(t) - Y_{n+1}(t)\|_\xi^2 &\leq 2\|X_0 - Y_0\|_\xi^2 + 2C(T)\|X_0 - Y_0\|_{\xi_1}^2 L_\xi \mathrm{e}^t \int_0^t \mathrm{e}^{-s_1}\mathrm{d}s_1 \\ &\quad + C(T)^2 L_\xi L_{\xi_1}\mathrm{e}^t \int_0^t \int_0^{s_1} \mathrm{e}^{-s_2}\|X_{n-1}(s_2) - Y_{n-1}(s_2)\|_{\xi_2}\mathrm{d}s_2\mathrm{d}s_1. \end{aligned} \quad (3.20)$$

Continuing the iteration, we have

$$\begin{aligned}\|X_{n+1}(t)-Y_{n+1}(t)\|_\xi^2 \le\; & 2\|X_0-Y_0\|_\xi^2 \mathrm{e}^T + 2C(T)\|X_0-Y_0\|_{\xi_1}^2 L_\xi \mathrm{e}^T t \\ & + 2C(T)^2\|X_0-Y_0\|_{\xi_2}^2 L_\xi L_{\xi_1}\mathrm{e}^T \int_0^t\int_0^{s_1} \mathrm{d}s_2\mathrm{d}s_1 \\ & + 2C(T)^3\|X_0-Y_0\|_{\xi_3}^2 L_\xi L_{\xi_1} L_{\xi_2}\mathrm{e}^T \int_0^t\int_0^{s_1}\int_0^{s_2} \mathrm{d}s_3\mathrm{d}s_2\mathrm{d}s_1 \\ & + \cdots + C(T)^{n+1}\mathrm{e}^T L_\xi L_{\xi_1} L_{\xi_2}\cdots L_{\xi_n}\int_0^t\int_0^{s_1}\cdots \\ & \times \int_0^{s_n} \|X_0(s_{n+1})-Y_0(s_{n+1})\|_{\xi_{n+1}}^2 \mathrm{d}s_1\mathrm{d}s_2\mathrm{d}s_3\cdots\mathrm{d}s_{n+1}.\end{aligned}$$

Finally, by putting $\mathbf{L}(\xi) = \sup_{n\in\mathbb{N}}\{L_\xi, L_{\xi_1}, L_{\xi_2}, \ldots, L_{\xi_n}\}$ and $\eta_n \in \{\xi, \xi_1, \xi_2, \ldots, \xi_n, \xi_{n+1}\}$ such that $\|X_0-Y_0\|_{\eta_n} = \max\{\|X_0-Y_0\|_{\xi_j}, j = 0, 1, 2, \ldots, n+1\}$. Then we have,

$$\begin{aligned}\|X_{n+1}(t)-Y_{n+1}(t)\|_\xi^2 \le\; & 2\mathrm{e}^T\|X_0-Y_0\|_{\eta_n}^2 \sum_{m=0}^{n+1}[C(T)\mathbf{L}(\xi)]^m \frac{T^m}{m!} \\ \le\; & 2\|X_0-Y_0\|_{\eta_n}^2 \mathrm{e}^{(C(T)\mathbf{L}(\xi)T+T)}.\end{aligned} \tag{3.21}$$

We now take the square root of both sides of (3.21), apply the condition that $\|X_0 - Y_0\|_\eta < \partial$ for all $\eta \in \mathbb{D}\underline{\otimes}\mathbb{E}$, and conclude, by letting $n \to \infty$, that $\|X(t) - Y(t)\|_\xi \le \epsilon$, where $\partial = \epsilon \left[2\mathrm{e}^{(C(T)\mathbf{L}(\xi)T+T)}\right]^{-\frac{1}{2}}$. □

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References

[1] E.O. Ayoola, Existence and stability results for strong solutions of quantum stochastic differential equations, Stochastic Anal. Appl. 20 (2) (2002) 263–281.
[2] R.L. Hudson, K.R. Parthasarathy, Quantum Ito's formulae and stochastic evolutions, Commun. Math. Phys. 93 (1984) 301–324.
[3] C. Barnett, R.F. Streater, I.F. Wilde, The Ito–Clifford integral II-stochastic differential equations, J. London Math. Soc. 27 (2) (1983) 373–384.
[4] E.O. Ayoola, On convergence of one step schemes for weak solutions of quantum stochastic differential equations, Acta Appl. Math. 67 (1) (2001) 19–58.
[5] E.O. Ayoola, Lipschitzian quantum stochastic differential equations and the associated Kurzweil equations, Stochastic Anal. Appl. 19 (4) (2001) 581–603.
[6] E.O. Ayoola, Lagrangian quadrature schemes for computing weak solutions of quantum stochastic differential equations, SIAM J. Numer. Anal. 39 (6) (2002) 1835–1864.
[7] E.O. Ayoola, Converging multistep methods for weak solutions of quantum stochastic differential equations, Stochastic Anal. Appl. 18 (4) (2000) 525–554.
[8] G.O.S. Ekhaguere, Lipschitzian quantum stochastic differential inclusions, Internat. J. Theoret. Phys. 31 (11) (1992) 2003–2034.

[9] P. Glockner, Quantum stochastic differential equations on $*$-bialgebra, Math. Proc. Cambridge Philos. Soc. 109 (1991) 571–595.
[10] K.R. Parthasarathy, An Introduction to Quantum Stochastic Calculus, Monographs in Mathematics, vol. 85, Birkhauser Verlag, 1992.

MATHEMATICAL ANALYSIS OF BASIC REPRODUCTION NUMBER FOR THE SPREAD AND CONTROL OF MALARIA MODEL WITH NON-DRUG COMPLIANT HUMANS

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Abstract: Malaria arises when there is an infection of a host by Plasmodium falciparum that causes malaria in humans. Non-drug compliance results from not taking medication as prescribed by doctors. Previous research had concentrated on mathematical modeling of transmission dynamics of malaria without considering some infectious humans who do not comply to drug. This study is therefore designed to model transmission dynamics of malaria taking into consideration some infectious humans who do not comply to drug. The model is formulated using nonlinear ordinary differential equations. The human population is partitioned into Susceptible human (S_H), Exposed Human E_H, Infectious human (I_H), Non-drug compliant human I_{NH} and Recovered human (R_H). Using next generation matrix, the reproduction number R_0 is obtained. This is used to analyse the global stability of the disease-free equilibria and local stability of the endemic equilibria of the model. The global stability of the disease-free equilibria and the local stability of the endemic equilibrium of the model are established through the construction of suitable Lyapunov function and analysis of characteristic equation. It is shown that the disease-free equilibrium is globally asymptotically stable whenever $R_o < 1$. It is also shown that the endemic equilibrium becomes stable through the Routh-Hurwitz stability criteria. Reproduction number

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1 Introduction

Malaria is a complex parasitic disease. It is mostly confined to tropical and subtropical regions of Africa and Asia because of rainfall, warm temperatures, stagnant water and poor sanitation that pave way for the provision of conducive environment for mosquito breeding [1, 17, 16]. Although, there were tremendous progresses in the fight against malaria. According to the World Health Organization's records for the year 2013, there were 207 million malaria cases worldwide with 627,000 deaths in 2012 [19, 25].

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Malaria infection is characterized by high fever, chills, sweating, fatigue, headache and nausea. If it is left untreated, it can cause acute anemia, organ failure or brain damage among the problems. Malaria is common and life-threatening public health problem in many tropical and sub-tropical areas of the world. It is currently endemic in over hundred countries. Each year, approximated three hundred million people fall ill with malaria and one million deaths are recorded. It is transmitted by female anopheles mosquitoes who bite mainly between sunset and sunrise [3, 26].

Human malaria is caused by five different species of the parasite belonging to genius Plasmodium: Plasmodium falciparum (the most deadly), Plasmodium vivax, Plasmodium knowlesi, Plasmodium malariae and Plasmodium ovale. The last two are fairly uncommon. Plasmodium knowlesi causes malaria in animal but can also infect humans and may be fatal. Animal malaria does not spread to humans [11].

Malaria symptoms appear seven days or more (usually 7-15 days) after being bitten by infectious mosquitoes. Malaria is preventable and curable. It can be treated in just 48 hours through the use of Artemisinin-based Combination Therapy (ACT) with drug compliance. But it can result into complication if it is diagnosed and treated lately. It can be prevented by using insecticides, treated bed nets, spraying with residual insecticides e.t.c.

Over the years, mathematical modeling of the spread of malaria has become an important tool in understanding the transmission dynamics of the diseases, predicting and controlling the spread of malaria in the future. Bakary et al.,[5] formulated a mathematical model of non-autonomous ordinary differential equations describing the dynamics of malaria transmission with age structure for the vector population. They obtained the basic reproduction number, R_o and proved that the disease-free equilibrium is locally asymptotically stable for $R_o < 1$. They performed numerical simulations to illustrate their analytical results. They concluded that malaria transmission can be controlled by fighting against the proliferation of the mosquitoes namely, by reducing available breeder sites. Ousmane et al., [20] presented a mathematical model of malaria transmission by considering two models: a model of vector population and a model of virus transmission. They applied Lyapunov principle to study the stability of equilibrium points. They determined the basic reproduction number using the next generation matrix. Their numerical simulations revealed that malaria management is concerned firstly by lowering the mosquito threshold parameters to a value less than unity. Chitnis et al., [7] formulated a mathematical model for the spread of malaria in human and mosquito population where they found that the disease-free equilibrium is locally asymptotically stable when $R_o < 1$ and unstable when $R_o > 1$. Their numerical simulations showed that for larger values of the disease-induced death rate, a subcritical (backward) bifurcation is possible at $R_o = 1$. Wedajo et al., [24] formulated and analyzed SIR model of malaria that included infected immigrants. The reproduction number R_o of their model was calculated using the next generation matrix method. They established the global stability of the equilibrium points using the Lyapunov function and LaSalle Invariance Principle. They simulated their analytical results and concluded that the infected immigrants will contribute positively and increase the disease in the population.

We modified and extended a model developed by Wedajo et al., [24] by incorporating a new class of non-drug compliant human compartment into the human population. These are the people who are given medication by their doctors but do not take it as prescribed. These include those who fail to take the correct dosage and those who do not complete their medication, that is, those who stop taking their medication as soon as they think that they feel better after few days of starting treatment. Using stability theory of nonlinear ordinary differential equations, global dynamics of the model is analyzed. Also, local stability of the endemic equilibrium solution of the model is established.

In addition to the introductory section, the paper has three more sections. Section two shows the mathematical formulation of the model. In section three, transformation of the model is presented. In section four, stability analysis of the model is carried out. Section five discusses the results and concludes the modeling work.

2 Materials and Methods

In this section, a model for the spread of malaria in the human population and mosquito vector population is formulated. A malaria model incorporating some infectious humans who do not comply with drug, is introduced. The total human population denoted by N_H is sub-divided into five classes namely; the susceptible humans S_H, the exposed humans E_H, the infectious humans I_H, the non-drug compliant humans I_{NH} and the recovered humans R_H so that $N_H = S_H + E_H + I_H + I_{NH} + R_H$. Also, the total mosquito vector population denoted by N_V, is sub-divided into two classes namely; the susceptible mosquito vector, S_V and the infected mosquito vector I_H. Thus the total population N_H and N_V for human and mosquito population is given by $N_H = S_H + E_H + I_H + I_{NH} + R_H$ and $N_V = S_V + I_V$.

2.1 Nomenclature/Values of Parameters Involved in the Model

a = average biting rate on man by a single mosquito (infection rate) 0.29 [10]
b = the proportion of bites on man that produces infection 0.75 [10]
p = probability that a mosquito becomes infected 0.75 [10]
θ= fraction of infectious who comply with drug 0.8 [Assumed]
$(1-\theta)\tau$= fraction of infectious who do not comply with drug 0.2 [Assumed]
τ = drug efficacy 0.01-0.7 [Assumed]
δ = death rate due to malaria 0.333 [22]
μ_N = death due to non-drug compliance 0.05 [Assumed]
ν = recovery rate 0.0022 [9]
π_h = natural birth rate of humans 0.0015875 [9]
π_v = natural birth rate of mosquitoes 0.071 [9]

α = progression rate of exposed humans 0.0588 [6]
μ_h = natural death rate of humans 0.00004 [8]
μ_v = natural death rate of mosquitoes 0.05 [9]
r=education on drug use 0.5 [Assumed]
γ=loss of immunity rate 0.000017 [4, 10]
m=$\frac{N_V}{N_H}$ the number of female mosquitoes per human host [2, 23]

2.2 Assumptions of the Model

The following assumptions were made in order to formulate the equations of the model:
(a) The exposed humans recover and return to susceptible population if their immunity is able to combat the dormant parasites
(b) The exposed humans progress to become infectious if their immunity is unable to combat the dormant parasites
(c) The exposed humans are those who have dormant parasites in them i.e they cannot yet infect a susceptible mosquito
(d) All humans are born susceptible and there is no vertical transmission
(e) Some infectious human hosts who are given medication by their doctors and comply with drug (i.e they take the correct dosage and complete treatment) get treated fully and move to the recovered human host compartment.
(f) Some infectious human hosts who do not comply with drug get treated partially and move to non-drug compliant human compartment.
(g) Proportion of active parasites are still in the blood of non-drug compliant humans
(h) When a susceptible mosquito bites the non-drug compliant humans, it becomes infected
(j) Susceptible humans progress to become exposed.
(k) Recovered humans have some immunity that can be lost and again susceptible.

The population of susceptible humans is generated either by birth or immigration at a constant rate π_h. The interaction of humans and female mosquitoes is modelled by standard incidence [23], with the terms $\dfrac{abS_HI_V}{N_H}$, which denotes the rate at which susceptible humans S_H get infected by infected mosquitoes I_V. The population increases at the rate ν due to the recovery rate of the exposed humans (if the immune system of the exposed humans is able to combat the dormant plasmodium parasite because at the exposed stage, plasmodium parasites are still dormant in the liver). It increases again due to loss of immunity of recovered humans at the rates γ. The population also decreases when the susceptible humans die naturally at the rate μ_h. Putting all these together gives the following equation for the rate of change of the susceptible population:

$$\frac{dS_H}{dt} = \pi_h N_H - \frac{abS_HI_V}{N_H} + \nu E_H + \gamma R_H - \mu_h S_H$$

The population of exposed humans is generated as a result of progression of the susceptible humans who are infected with plasmodium falciparum by the infected mosquitoes but have not started displaying symptoms, i.e., they are infected but not yet infectious, with the terms

$\frac{abS_H I_V}{N_H}$. It decreases as a result of recovery of the exposed humans and the progression of the exposed humans to become infectious(the dormant parasites undergo nuclear division and thousands of them move down to the blood stream, if the immune system is unable to combat the parasites at the exposed stage) at the rates ν and αab. It diminishes due to natural death at the rate μ_h. Thus,

$$\frac{dE_H}{dt} = \frac{abS_H I_V}{N_H} - \nu E_H - \frac{\alpha ab s_H I_V}{N_H} - \mu_h E_H$$

The population of infectious humans is generated by the progression rate of the exposed humans at the rate αab. It diminishes due to drug efficacy τ, death due to malaria δ and natural death μ_h. Thus we have

$$\frac{dI_H}{dt} = \frac{\alpha abS_H I_V}{N_H} - \frac{\tau abS_H I_V}{N_H} - \delta I_H - \mu_h I_H$$

The population of non-drug compliant humans is generated by a fraction $(1-\theta)\tau$ of infectious humans who do not comply with drug. The population reduces due to non-drug compliance, education on drug use and natural death at the rates μ_N, $r\tau$ and μ_h so that

$$\frac{dI_{NH}}{dt} = \frac{(1-\theta)\tau abS_H I_V}{N_H} - \mu_N I_{NH} - r\tau I_{NH} - \mu_h I_{NH}$$

The population of recovered humans is generated by the fraction $\theta\tau$ of infectious humans who comply with drug. It reduces due to loss of immunity of the recovered humans and natural death at the rates γ and μ_h. It again increases due to education on drug use at the rate r τ. Thus,

$$\frac{dR_H}{dt} = \frac{\theta\tau abS_H I_V}{N_H} - \gamma R_H + r\tau I_{NH} - \mu_h R_H$$

In a similar way, the population of mosquito vector changes so that we have the following:

$$\frac{dS_V}{dt} = \pi_V N_V - \frac{apS_V(I_H + I_{NH})}{N_H} - \mu_V S_V$$

$$\frac{dI_V}{dt} = \frac{apS_V(I_H + I_{NH})}{N_H} - \mu_V I_V$$

Putting everything together, we have the following system of ordinary differential equations:

$$\frac{dS_H}{dt} = \pi_h N_H - \frac{abS_H I_V}{N_H} + \nu E_H + \gamma R_H - \mu_h S_H \quad (2.1)$$

$$\frac{dE_H}{dt} = \frac{abS_H I_V}{N_H} - \nu E_H - \frac{\alpha ab s_H I_V}{N_H} - \mu_h E_H \quad (2.2)$$

$$\frac{dI_H}{dt} = \frac{\alpha abS_H I_V}{N_H} - \frac{\tau abS_H I_V}{N_H} - \delta I_H - \mu_h I_H \quad (2.3)$$

$$\frac{dI_{NH}}{dt} = \frac{(1-\theta)\tau abS_H I_V}{N_H} - \mu_N I_{NH} - r\tau I_{NH} - \mu_h I_{NH} \quad (2.4)$$

$$\frac{dR_H}{dt} = \frac{\theta\tau abS_H I_V}{N_H} - \gamma R_H + r\tau I_{NH} - \mu_h R_H \quad (2.5)$$

$$\frac{dS_V}{dt} = \pi_V N_V - \frac{apS_V(I_H + I_{NH})}{N_H} - \mu_v S_v \quad (2.6)$$

$$\frac{dI_V}{dt} = \frac{apS_V(I_H + I_{NH})}{N_H} - \mu_V I_V \quad (2.7)$$

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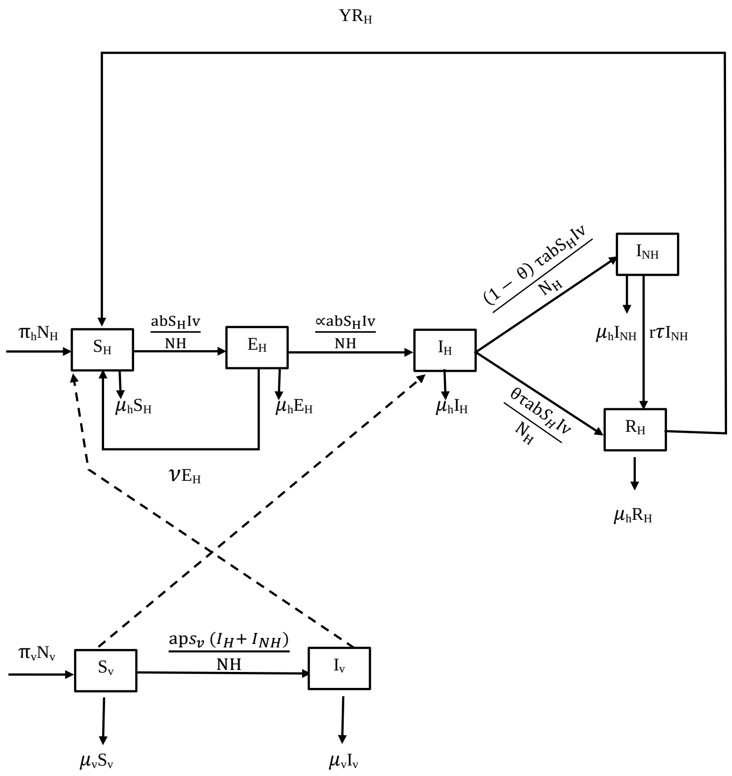

Diagram for Malaria Model Incorporating a New Class of Non-Drug Compliant Human Compartment

The restriction on the initial population arises from the fact that the variables describe the dynamics of human and mosquito populations. Therefore, for the model to be biologically meaningful, all the initial conditions and parameters must be non-negative. Thus $S_H(0) \geq 0$, $E_H(0) \geq 0$, $I_H(0) \geq 0$, $I_{NH}(0) \geq 0$, $R_H(0) \geq 0$, $S_V(0) \geq 0$, $I_V(0) \geq 0$.

The total population sizes N_H and N_V are

$$\frac{dN_H}{dt} = (\pi_h - \mu_h)N_H - \delta I_H - \mu_N I_{NH} \quad (2.8)$$

$$\frac{dN_V}{dt} = (\pi_v - \mu_v)N_V \quad (2.9)$$

which are derived by adding (2.1)-(2.5) for the human population and (2.6)-(2.7) for the mosquito vector population.

3 Transformation of the model

It is convenient to use fraction of population instead of population number. This is done by dividing each population class by the total population and hence, we have:
$s_h = \frac{S_H}{N_H}$; $i_h = \frac{I_H}{N_H}$; $e_h = \frac{E_H}{N_H}$, $i_{nh} = \frac{I_{NH}}{N_H}$, $r_h = \frac{R_H}{N_H}$; $s_v = \frac{S_V}{N_V}$; $i_v = \frac{I_V}{N_V}$; $m = \frac{N_V}{N_H}$.
Differentiating the fraction with respect to time t gives the following:

$$\frac{ds_h}{dt} = \pi_h(1-s_h) - abms_hi_v + \nu e_h + \gamma r_h + \delta s_hi_h + \mu_N s_hi_{nh} \tag{3.1}$$

$$\frac{de_h}{dt} = abms_hi_v - (\nu+\pi_h)e_h - \alpha abms_hi_v + \delta e_hi_h + \mu_N e_hi_{nh} \tag{3.2}$$

$$\frac{di_h}{dt} = \alpha abms_hi_v - \tau abms_hi_v - (\delta+\pi_h)i_h + \delta i_h^2 + \mu_N i_hi_{nh} \tag{3.3}$$

$$\frac{di_{nh}}{dt} = (\tau - \theta\tau)abms_hi_v - (\mu_N + r\tau + \pi_h)i_{nh} + \delta i_hi_{nh} + \mu_N i_{nh}^2 \tag{3.4}$$

$$\frac{dr_h}{dt} = \theta\tau abms_hi_v - (\gamma+\pi_h)r_h + \delta i_hr_h + r\tau i_{nh} + \mu_N i_{nh}r_h \tag{3.5}$$

$$\frac{ds_v}{dt} = \pi_v(1-s_v) - aps_v(i_h + i_{nh}) \tag{3.6}$$

$$\frac{di_v}{dt} = aps_v(i_h+i_{nh}) - \pi_v i_v \tag{3.7}$$

From the relation $s_h+e_h+i_h+i_{nh}+r_h = 1$ and $s_v+i_v = 1$, it implies that $r_h = 1-s_h-e_h-i_h-i_{nh}$ and $s_v = 1-i_v$ which reduces to the following system of differential equations:

$$\frac{ds_h}{dt} = \pi_h(1-s_h) - abms_hi_v + \nu e_h + \gamma(1-s_h-e_h-i_h-i_{nh}) + \delta s_hi_h + \mu_N s_hi_{nh} \tag{3.8}$$

$$\frac{de_h}{dt} = abms_hi_v - (\nu+\pi_h)e_h - \alpha abms_hi_v + \delta e_hi_h + \mu_N e_hi_{nh} \tag{3.9}$$

$$\frac{di_h}{dt} = \alpha abms_hi_v - \tau abms_hi_v - (\delta+\pi_h)i_h + \delta i_h^2 + \mu_N i_hi_{nh} \tag{3.10}$$

$$\frac{di_{nh}}{dt} = (\tau-\theta\tau)abms_hi_v - (\mu_N + r\tau + \pi_h)i_{nh} + \delta i_hi_{nh} + \mu_N i_{nh}^2 \tag{3.11}$$

$$\frac{di_v}{dt} = api_h(1-i_v) + api_{nh}(1-i_v) - \pi_v i_v \tag{3.12}$$

3.1 Existence of Solutions

Here, we provide the following result which guarantees that the malaria model governed by the system (3.1)-(3.7) is epidemiologically well-posed in a feasible region Γ defined by
$\Gamma \in \Re^7_+$ and $\Gamma_h \cup \Gamma_v \subset \Re^5_+ * \Re^2_+$
Lemma 1: The solutions of the system are contained and bounded in the region, $\Gamma \in \Re^7$ and $\Gamma_c \cup \Gamma_t \subset \Re^5_+ * \Re^2_+$.

Proof: We show that the feasible solutions are uniformly bounded in proper subsets $\Gamma \in \Re^7_+$. Let $(s_h, e_h, i_h, i_{nh}, r_h, s_v, i_v) \in \Re^7$ be any solution of the system given by $N_h = s_h+e_h+i_h+i_{nh}+r_h$ and $N_v = s_v + i_v$ with non-negative initial conditions. In differential form, we write

$$\frac{dN_h}{dt} = \frac{ds_h}{dt} + \frac{de_h}{dt} + \frac{di_h}{dt} + \frac{di_{nh}}{dt} + \frac{dr_h}{dt}$$

$$\frac{dN_h}{dt} = \pi_h - (\pi_h - \delta i_h - \mu_N i_{nh})N_h - \delta i_h - \mu_N i_{nh}$$

since

$$s_h + e_h + i_h + i_{nh} + r_h = N_h$$

$$\frac{dN_h}{dt} = \pi_h - (\pi_h - \delta i_h - \mu_N i_{nh})N_h - \delta i_h - \mu_N i_{nh}$$

Hence we have

$$\frac{N_h}{dt} + (\pi_h - \delta i_h - \mu_N i_{nh})N_h = \pi_h - \delta i_h - \mu_N i_{nh}$$

Solving yields

$$N_h = 1 + Be^{-(\pi_h - \delta i_h - \mu_N i_{nh})t}$$

Applying the initial condition $N_h(0) = N_h^o$ leads to

$$N_h = 1 + (N_h^o - 1)e^{-(\pi_h - \delta i_h - \mu_N i_{nh})t}$$

Thus $N_h \to 1$ as $t \to \infty$

And

$$\frac{dN_v}{dt} = \pi_h - \pi_v N_v$$
$$\frac{dN_h}{dt} + \pi_v N_v = \pi_v$$

We solve to obtain

$$N_v = 1 + (N_v^o - 1)e^{-\pi_v t}$$

Thus $N_v \to 1$ as $t \to \infty$

Hence the feasible region for the model is given by

$\Gamma =$

$(s_h, e_h, i_h, i_{nh}, r_h, s_v, i_v) \in \Re_+^7; s_h, e_h, i_h, i_{nh}, r_h, s_v, i_v \geq 0, s_h + e_h + i_h + i_{nh} + r_h = 1; s_v + i_v = 1$ which is positively invariant set for the model system. Hence, the model is well-posed and biologically realistic and meaningful. Thus, all solutions of the human population only are confined in the feasible region Γ_h and all solutions of the mosquito vector population are confined in Γ_v

3.2 Basic Reproduction Number

The computation of the basic reproduction number R_o is needed in order to assess the global stability of disease-free equilibrium. This is obtained by expressing (3.8)-(3.12) as the difference between the rate of new infection in each infected compartment F and the rate of transfer between each infected compartment G. Hence, we have

$$\begin{bmatrix} \frac{de_h}{dt} \\ \frac{di_h}{dt} \\ \frac{i_{nh}}{dt} \\ \frac{i_v}{dt} \end{bmatrix} = F - G = \begin{bmatrix} abms_h i_v - \alpha abms_h i_v \\ \alpha abms_h i_v - \tau abms_h i_v \\ (\tau - \theta\tau) abms_h i_v \\ aps_v i_h - aps_v i_{nh} \end{bmatrix} - \begin{bmatrix} (\nu + \pi_h)e_h + \delta e_h i_h + \mu_N e_h i_{nh} \\ (\delta + \pi_h)i_h + \delta i_h^2 + \mu_N i_h i_{nh} \\ (\mu_N + r\tau + \pi_h)i_{nh} + \delta i_h i_{nh} + \mu_N i_{nh}^2 \\ \pi_v i_v \end{bmatrix}$$

The Jacobian matrices J_F and J_G of F and G are found about E_0.

$$S = J_F J_G^{-1} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{ap}{\pi_v} \\ 0 & 0 & 0 & \frac{ap}{\pi_v} \\ \frac{-abm\alpha + abm}{\nu + \pi_h} & \frac{-abm\tau + abm\tau}{\delta + \pi_h} & \frac{(-\tau\theta + \tau)abm}{r\tau + \mu_N + \pi_h} & 0 \end{bmatrix}$$

R_o is the maximum eigenvalue of S given as

$$R_o = \frac{a^2bmp(r\tau\alpha - r\tau^2 - r\delta\theta - \tau\theta\pi_h + \tau\delta - \tau\mu_N + \alpha\mu_N + \alpha\pi_h)}{\pi_v(r\tau\delta + r\tau\pi_h + \delta\mu_N + \delta\pi_h + \mu_N\pi_h + \pi_h^2)}$$

$$R_o = \frac{a^2bmp(r\tau\alpha - r\tau^2 - r\delta\theta - \tau\theta\pi_h + \tau\delta - \tau\mu_N + \alpha\mu_N + \alpha\pi_h)}{\pi_v A_T B_T}$$

where

$A_T = \delta + \pi_h$ and $B_T = \mu_N + r\tau + \pi_h$

4 Results and Discussions

4.1 Global Stability of the disease-free equilibrium

The disease-free equilibrium solution is obtained by setting the right-hand side of (3.8)-(3.12) to zero to obtain $E_o = (1, 0, 0, 0, 0)$. Hence, we provide the dynamical behaviour of the model (3.8)-(3.12) as its solution trajectories approach the disease-free equilibrium solution in what follows:

Theorem 1: The disease-free equilibrium E_o of (3.8)-(3.12) is globally asymptotically stable in Γ if $R_o \leq 1$ and unstable if $R_o > 1$.

Proof: Consider the Lyapunov function $L = apr\delta\theta e_h + (apr\tau + ap\mu_N + ap\pi_h)i_h + (apr\alpha + ap\pi_h + ap\delta)i_{nh} + A_T B_T i_v$. Its time derivative is

$$\begin{aligned}
L' &= apr\delta\theta\frac{de_h}{dt} + (apr\tau + ap\mu_N + ap\pi_h)\frac{di_h}{dt} + (apr\alpha + ap\pi_h + ap\delta)\frac{di_{nh}}{dt} + A_T B_T\frac{di_v}{dt} \\
&= apr\delta\theta(abms_hi_v - C_Te_h - \alpha abms_hi_v + \delta e_hi_h + \mu_Ne_hi_{nh}) + \\
&\quad (apr\tau + ap\mu_N + ap\pi_h)(\alpha abms_hi_v - \tau abms_hi_v - A_Ti_h + \delta i_h^2 + \mu_Ni_hi_{nh}) + \\
&\quad (apr\alpha + ap\pi_h + ap\delta)(\tau abms_hi_v - \theta\tau abms_hi_v - B_Ti_{nh} + \delta i_hi_{nh} + \mu_Ni_{nh}^2) + \\
&\quad A_TB_T[api_h(1 - i_v) + api_{nh}(1 - i_v) - \pi_vi_v] \\
&= a^2bmp(r\tau\alpha - r\tau^2 - r\delta\theta - \tau\theta\pi_h + \tau\delta - \tau\mu_N + \alpha\mu_N + \alpha\pi_h)s_hi_v - \\
&\quad A_TB_T\pi_vi_v - Dape_h - Eapi_h - Fapi_{nh} - Ga^2bmps_hi_v \\
&= A_TB_T\pi_vi_v\left(\frac{a^2bmp(r\tau\alpha - r\tau^2 - r\delta\theta - \tau\theta\pi_h + \tau\delta - \tau\mu_N + \alpha\mu_N + \alpha\pi_h)s_h}{A_TB_T\pi_v} - 1\right) - \\
&\quad Dape_h - Eapi_h - Fapi_{nh} - Ga^2bmps_hi_v \\
&= A_TB_T\pi_vi_v(R_os_h - 1) - Dape_h - Eapi_h - Fapi_{nh} - Ga^2bmps_hi_v \\
&\quad \leq A_TB_T\pi_vi_v(R_0s_h - 1) \leq 0 \quad \text{if} \quad R_0 \leq 1
\end{aligned}$$

where
$A_T = \delta + \pi_h$
$B_T = \mu_N + r\tau + \pi_h$
$C_T = \nu + \pi_h$
$D = r\delta\theta C_T - \delta^2 i_h r\theta - \mu_N i_{nh} r\delta\theta$
$E = A_T r\tau + A_T\mu_N + A_T B_T i_v + A_T\pi_h - \delta i_h r\tau - \mu_N i_{nh} r\tau - \delta i_h\mu_N - \mu_N i_{nh}^2 - \delta i_h\pi_h - \mu_N i_{nh}\pi_h - \delta i_{nh} r\alpha - \delta i_{nh}\pi_h - \delta^2 i_{nh} - A_T B_T$
$F = B_T r\alpha + B_T\pi_h + B_T\delta + A_T B_T i_v - A_T B_T - i_{nh}\mu_N\delta - i_{nh}\mu_N\pi_h - \mu_N i_{nh} r\alpha$
$G = \alpha r\delta\theta + \tau\pi_h + \theta\tau r\alpha - \theta\tau\delta - \tau r\alpha - \tau\pi_h$

Therefore, $L' \leq 0$ for $R_o \leq 1$. One sees further that $(s_h, e_h, i_h, i_{nh}, i_v) \to (1, 0, 0, 0, 0)$ as $t \to \infty$. Consequently, the largest compact invariant set in $\{(s_h, e_h, i_h, i_{nh}, i_v) \in \Gamma : L' = 0\}$ is the E_0 and by Lyapunov-Lasalle's principle [14, 12], the disease-free equilibrium point is globally asymptotically stable in Γ if $R_o \leq 1$ and this completes the proof of Theorem 1. The epidemiological implication of the result implies that the disease can be eradicated with population that starts with either large or small number of infectious humans whenever $R_o < 1$.

4.2 Local Stability of Endemic Equilibrium

We shall first show the interval where the endemic equilibrium exists using the idea of Tumwiine et al.[23]. Hence, for the existence and uniqueness of endemic equilibrium $E_1 = (s_h^*, e_h^*, i_h^*, i_{nh}^*, i_v^*)$, its coordinates should satisfy the conditions $s_h^* > 0, e_h^* > 0, i_h^* > 0, i_{nh}^* > 0, i_v^* > 0$. Adding (3.8)-(3.12), we have
$\pi_h(1 - s_h^* - e_h^* - i_h^* - i_{nh}^*) + \gamma(1 - s_h^* - e_h^* - i_h^* - i_{nh}^*) - \delta i_h^*(1 - s_h^* - e_h^* - i_h^* - i_{nh}^*) - \mu_N i_{nh}^*(1 - s_h^* - e_h^* - i_h^* - i_{nh}^*) + api_h^*(1 - i_v^*) + api_{nh}^*(1 - i_v^*) - \pi_v i_v^* + r\tau i_{nh}^* - \theta\tau abms_h^* i_v^* = 0$
From (3.12), $api_h^*(1 - i_v^*) + api_{nh}^*(1 - i_v^*) - \pi_v i_v^* = 0$
This yields
$(\pi_h + \gamma - \delta i_h^* - \mu_N i_{nh}^*)(1 - s_h^* - e_h^* - i_h^* - i_{nh}^*) = \theta\tau abms_h^* i_v^* - r\tau i_{nh}^*$.
Since $(1 - s_h^* - e_h^* - i_h^* - i_{nh}^*) > 0$ and $\theta\tau abms_h^* i_v^* - r\tau i_{nh}^* > 0$, then

$$\pi_h + \gamma - \delta i_h^* - \mu_N i_{nh}^* > 0 \tag{4.1}$$

Further simplification gives
$\pi_h + \gamma > \delta i_h^* + \mu_N i_{nh}^*$
since death due to non-drug compliance μ_N implies death due to malaria δ, then
$\mu_N = \delta$
Therefore, $\pi_h + \gamma > \delta i_h^* + \delta i_{nh}^*$
$\delta(i_h^* + i_{nh}^*) < (\lambda_h + \gamma)$
This gives
$(i_h^* + i_{nh}^*) < \dfrac{\lambda_h + \gamma}{\delta}$.
Therefore, an endemic equilibrium point exists, where $(i_h^* + i_{nh}^*)$ lie in the interval $\left(0, \min\left\{1, \frac{\lambda_h+\gamma}{\delta}\right\}\right)$.
If $\delta < \lambda_h + \gamma$, the interval becomes large and this means that malaria persists in the population.

We next analyze the stability of endemic equilibrium E_1 using the Jacobian matrix computed for (3.8)-(3.12) given by

$$J_{E_1} = \begin{bmatrix} J_{11} & \nu - \gamma & -\gamma + \delta s_h^* & -\gamma + \mu_N s_h^* & -abms_h^* \\ abmi_v^* - \alpha abmi_v^* & J_{22} & \delta e_h^* & \mu_N e_h^* & abms_h^* - \tau abms_h^* \\ \alpha abmi_v^* - \tau abmi_v^* & 0 & J_{33} & \mu_N i_h^* & \alpha abms_h^* - \tau abms_h^* \\ \tau abmi_v^* - \theta\tau abmi_v^* & 0 & \delta i_{nh}^* & J_{44} & \tau abms_h^* - \theta\tau abms_h^* \\ 0 & 0 & ap(1-i_v^*) & ap(1-i_v^*) & -\pi_v^* \end{bmatrix} \qquad (4.2)$$

where
$J_{11} = -\pi_h - abmi_v^* - \gamma + \delta i_h^* + \mu_N i_{nh}^*$
$J_{22} = -\nu - \pi_h + \delta i_h^* + \mu_N i_{nh}^*$
$J_{33} = -\delta - \pi_h + 2\delta i_h^* + \mu_N i_{nh}^*$
$J_{44} = -\mu_N - r\tau - \pi_h + \delta i_h^* + 2\mu_N i_{nh}^*$

Using Sarrus diagram, i.e.,
$a_1b_2c_3d_4e_5 + a_2b_3c_4d_5e_1 + a_3b_4c_5d_1e_2 + a_4b_5c_1d_2e_3 + a_5b_1c_2d_3e_4 -$
$a_1b_5c_4d_3e_2 - a_2b_1c_5d_4e_3 - a_3b_2c_1d_5e_4 - a_4b_3c_2d_1e_5 - a_5b_4c_3d_2e_1,$
the characteristic equation of the Jacobian matrix (4.2) at the endemic equilibrium point $E_1 = (s_h^*, e_h^*, i_h^*, i_{nh}^*, i_v^*)$ is a fifth degree polynomial given by

$$\lambda^5 + a_1\lambda^4 + a_2\lambda^3 + a_3\lambda^2 + a_4\lambda + a_5 = 0 \qquad (4.3)$$

where

$$\begin{aligned}
a_0 &= 1 \\
a_1 &= (abmi_v^* + \gamma + \nu + 2\pi_h - 2\delta i_h^* - 2\mu_N i_{nh}^*) + (api_h^* + api_{nh}^* + \pi_v) + (\delta + \pi_h - 2\delta i_h^* - \mu_N i_{nh}^*) + \\
&\quad (\mu_N + r\tau + \pi_h - \delta i_{nh}^* - 2\mu_N i_{nh}^*) \\
a_2 &= (\delta + \pi_h - 2\delta i_h^* - \mu_N i_{nh}^*)(\mu_N + r\tau + \pi_h - \delta i_{nh}^* - 2\mu_N i_{nh}^*) \\
&\quad +(api_h^* + api_{nh}^* + \pi_v)(\delta + \pi_h - 2\delta i_h^* - \mu_N i_{nh}^* + \mu_N + r\tau + \pi_h - \delta i_{nh}^* - 2\mu_N i_{nh}^*) + \\
&\quad (abmi_v^* + \gamma + \nu + 2\pi_h - 2\delta i_h^* - 2\mu_N i_{nh}^*)(\delta + \pi_h - 2\delta i_h^* - \mu_N i_{nh}^* + \mu_N + r\tau + \pi_h - \delta i_{nh}^* - 2\mu_N i_{nh}^*) + \\
&\quad (abmi_v^* + \gamma + \nu + 2\pi_h - 2\delta i_h^* - 2\mu_N i_{nh}^*)(api_h^* + api_{nh}^* + \pi_v) + (abmi_v^* + \pi_h - \gamma - \delta i_h^* - \mu_N i_{nh}^*) \\
&\quad (\nu + \pi_h - \delta i_h^* - \mu_N i_{nh}^*) \\
a_3 &= [(api_h^* + api_{nh}^* + \pi_v)(\delta + \pi_h - 2\delta i_h^* - \mu_N i_{nh}^*)(\mu_N + r\tau + \pi_h - \delta i_{nh}^* - 2\mu_N i_{nh}^*) + \\
&\quad (abmi_v^* + \gamma + \nu + 2\pi_h - 2\delta i_h^* - 2\mu_N i_{nh}^*)(\delta + \pi_h - 2\delta i_h^* - \mu_N i_{nh}^*) \\
&\quad (\mu_N + r\tau + \pi_h - \delta i_{nh}^* - \mu_N i_{nh}^*) + (abmi_v^* + \gamma + \nu + 2\pi_h - 2\delta i_h^* - 2\mu_N i_{nh}^*)(api_h^* + api_{nh}^* + \pi_v) \\
&\quad (\delta + \pi_h - 2\delta i_h^* - \mu_N i_{nh}^* + \mu_N + r\tau + \pi_h - \delta i_{nh}^* - 2\mu_N i_{nh}^*) + \\
&\quad (abmi_v^* + \pi_h - \gamma - \delta i_h^* - \mu_N i_{nh}^*)(\nu + \pi_h - \delta i_h^* - \mu_N i_{nh}^*) \\
&\quad (\delta + \pi_h - 2\delta i_h^* - \mu_N i_{nh}^* + \mu_N + r\tau + \pi_h - \delta i_{nh}^* - 2\mu_N i_{nh}^*) + \\
&\quad (api_h^* + api_{nh}^* + \pi_v)(abmi_v^* + \pi_h - \gamma - \delta i_h^* - \mu_N i_{nh}^*)(\nu + \pi_h - \delta i_h^* - \mu_N i_{nh}^*)] \\
a_4 &= [(abmi_v^* + \gamma + \nu + 2\pi_h - 2\delta i_h^* - 2\mu_N i_{nh}^*)(api_h^* + api_{nh}^* + \pi_v)(\delta + \pi_h - 2\delta i_h^* - \mu_N i_{nh}^*) \\
&\quad (\mu_N + r\tau + \pi_h - \delta i_{nh}^* - 2\mu_N i_{nh}^*) + (api_h^* + api_{nh}^* + \pi_v) \\
&\quad (\delta + \pi_h - 2\delta i_h^* - \mu_N i_{nh}^* + \mu_N + r\tau + \pi_h - \delta i_{nh}^* - 2\mu_N i_{nh}^*)] \\
a_5 &= (abmi_v^* + \pi_h - \gamma - \delta i_h^* - \mu_N i_{nh}^*)(\nu + \pi_h - \delta i_h^* - \mu_N i_{nh}^*) \\
&\quad (api_h^* + api_{nh}^* + \pi_v)(\delta + \pi_h - 2\delta i_h^* - \mu_N i_{nh}^*)(\mu_N + r\tau + \pi_h - \delta i_{nh}^* - 2\mu_N i_{nh}^*) + \\
&\quad abm(\nu - \gamma)ap(1 - i_v^*)(\alpha - a)s_h^*(\mu_N + r\tau + \pi_h + \delta i_{nh}^* - 2\mu_N i_{nh}^*) + \\
&\quad a^2b^2m^2 i_v^* s_h^*(\tau - \alpha)ap(1 - i_v^*)(\nu + \pi_h - \delta i_h^* - \mu_N i_{nh}^*)(\delta s_h^* - \gamma) = 0
\end{aligned}$$

Clearly, $a_0 > 0$. Since $\pi_h + \gamma - \delta i_h^* - \mu_N i_{nh}^* > 0$ from (4.1), then $abmi_v^* + \gamma + \nu + 2\pi_h - 2\delta i_h^* - \mu_N i_{nh}^* > 0$, $\delta + \pi_h - 2\delta i_h^* - \mu_N i_{nh}^* > 0$ and $\mu_N + r\tau + \pi_h - \delta i_{nh}^* - 2\mu_N i_{nh}^* > 0$. Clearly, $api_h^* + api_{nh}^* + \pi_v > 0$. It then follows that $a_1 > 0$, $a_2 > 0$, $a_3 > 0$, $a_4 > 0$ and $a_5 > 0$ for $abmi_v^* + \gamma + \nu + 2\pi_h - 2\delta i_h^* - \mu_N i_{nh}^* > 0$, $\delta + \pi_h - 2\delta i_h^* - \mu_N i_{nh}^* > 0$, $\mu_N + r\tau + \pi_h - \delta i_{nh}^* - 2\mu_N i_{nh}^* > 0$ and $api_h^* + api_{nh}^* + \pi_v > 0$. Therefore, all the coefficients $a_i s$ are positive. The necessary and sufficient conditions for the local stability of the endemic equilibrium E_1 are that the Hurwitz determinants, H_i, are all positive for the Routh-Hurwitz criteria [18]. Hence, since $a_1 > 0$, $a_2 > 0$, $a_3 > 0$, $a_4 > 0$ and $a_5 > 0$, then
$H_1 > 0$,
$H_2 > a_1a_2 - a_3 > 0$,
$H_3 = a_1a_2a_3 + a_1a_5 - a_1^2a_4 - a_3^2 > 0$,
$H_4 = (a_3a_4 - a_2a_5)(a_1a_2 - a_3) - (a_1a_4 - a_5)^2 > 0$,
$H_5 = a_5H_4 > 0$
Therefore by Routh-Hurwitz theorem [15, 18], all the eigenvalues of the polynomial $P(\lambda)$ have negative real parts and the endemic equilibrium is locally asymptotically stable.

The theorem below summarizes the above result:
Theorem: The endemic equilibrium is locally asymptotically stable if all the eigenvalues of the polynomial $P(\lambda)$ have negative real part.

5 Discussion of Results and Conclusion

In this work, a mathematical model is formulated and analysed to study the transmission and spread of malaria parasite in a population. The model incorporates a class of non-drug compliant human compartment into the population. A 7-dimensional system of nonlinear ordinary differential equations is modelled. It is shown that there exist a domain Γ where the model is well-posed and biologically meaningful. The disease-free equilibrium points of the model are obtained and analysed for stability. The condition for disease spread which is the basic reproduction number, R_0, is calculated respectively. It is shown that when $R_0 < 1$, malaria is cleared from the population. Whereas, if $R_0 > 1$, the disease persists in the population. Thus, a new class of non-drug compliant humans can contribute to the spread of malaria as susceptible mosquitoes get infected when they bite this group thereby spreading malaria in the population. Also, public health can educate people on the effect of the incorporated non-drug compliant human compartment on transmission dynamics of malaria model by using this article as a study guide for seminars, workshop or training programs.

References

[1] Alles, H.K., Mendis, K.N., & Carter, R. Malaria mortality rates in South Asia and in Africa:implications for malaria control *Parasitology Today* 14, 369-375 (1998).

[2] Aneke, S.J. Mathematical modeling of drug resistant malaria parasites and vector populations.*Mathematical Methods in the Applied Sciences* 90, 385-396 (2012).

[3] Antino, B., Corbett, Y., Catelli, F. & Taramelli, D. Pathogenesis of malaria in tissues and blood.*Mediterranean Journal of Hematology and Infectious Diseases* 2035-3006 (2012).

[4] Aron, J.L. Mathematical modeling of immunity to malaria. *Mathematical Biosciences* 90, 385-396 (1982).

[5] Bakary, T., Sangare, B. & Traore, S. Mathematical modeling of malaria transmission with structured vector population and seasonality. *Journal of Applied Mathematics* 15, (2017).

[6] Blayneh, K., Cao, Y. & Kwon, H. Optimal control of vector-borne diseases. Treatment and prevention. *Discrete Continuous Dyn Syst Ser B* 11, 587-611 (2009).

[7] Chitnis, N., Cushing, J.M & Hyman, J.M. Bifurcation analysis of a mathematical model for malaria transmission. *SIAM Journal of Applied Mathematics* 67, 24-45 (2018).

[8] Coutinho, F.A.B., Burattini, M.N., Lopez, L.F. & Massad, E. An approximate threshold condition for non-autonomous system:an application to a vector-borne infection. *Mathematics and Computer in Simulation* 70, 149-158 (2005).

[9] Gemperli, A., Vounatsou P., Sogoba, N. & Smith, T. Malaria mapping using transmission models: application to survey data. *American Journal of Epidemiology* 163, 289-297 (2006).

[10] Ishikawa, H., Ishii, A., Nagai, N., Ohmae, H., Mazakazu, H., Shuguri, S. & Leafasia, J. A mathematical model for the transmission of plasmodium vivax malaria. *Parasitology International* 52, 81-93 (2013).

[11] Jongwutiwes, S., Putaporntip, C., Iwasaki, T., Sata T. & Kanbara, H. Naturally acquired plasmodium knowlesi in malaria in human, Thailand. *Emerging Infectious Diseases* 10, 2211-2232 (2010).

[12] Khalil, H. Nonlinear systems. *Prentice Hall* (2002).

[13] Laxminarayan, R. Act now or later? Economics of malaria resistance. *The American Journal of Tropical Medicine and Hyiegene* 71, 187-195 (2004).

[14] Lasalle, J.P. The stability of dynamical systems. *Philadelphia, PA: SIAM* (1976).

[15] Lancaster, P. Theory of matrices. *Academic Press, New York* (1969).

[16] Lieshonta, M.V., Kovats, R.S., Livermore, M.T.J & Martens, P. Climate change and malaria: analysis of the SRES climate and socio-economic scenarios. *Global Environmental Change* 14, 87-99 (2004).

[17] Murray, C.J.L, Rosenfeld, L.C., Lim, S.S., Andrews, K.G., Foreman, K.J., Haring, D., Fullman, N., Naghari, M., Lozano, R. & Lopez, A.D. Global malaria mortality between 1980 and 2010: a systematic analysis. *The Lancet* 379, 413-431 (2012).

[18] Murray, J.D. Mathematical Biology I, An introduction. *Springer-Verlag, Berlin* (2002).

[19] Macdonald, G. The epidemiology and control of malaria. *Oxford: Oxford University Press* (1957).

[20] Ousmane, K., Traore, B. & Sangare, B. Mathematical modeling of malaria transmission global dynamics: taking into account the immatured stages of the vectors. *Advances in Difference Equations* 22, (2018).

[21] Tumwiine, J., Mugisha, J.Y.T & Luboobi, L.S. On oscillatory pattern of malaria dynamics in a population with temporary immunity. *Computational and Mathematical Mathematical Methods in Medicine* 8, 191-203 (2007).

[22] Tumwiine, J., Luboobi, L.S. & Mugisha, J.Y.T. Modeling the effect of treatment and mosquito control on malaria transmission. *International Journal of Management and Systems* 21, 107-124 (2005).

[23] Tumwiine, J., Mugisha, J.Y.T & Luboobi, L.S. A mathematical model for the dynamics of malaria in a human host and mosquito vector with temporary immunity. *Applied Mathematics and Computation* 189, 1953-1965 (2007).

[24] Wedajo, A.J., Bole, B.K, Koya, P.R. Analysis of SIR mathematical model for malaria disease with the inclusion of infected immigrants. *IOSR Journal of Mathematics* 14, 10-21 (2018).

[25] World Health Organization. World malaria report. *WHO Library Cataloguing-in-publication data* (2013).

[26] World Health Organization. Malaria control in complex emergencies, an inter-agency field handbook. *WHO Library Cataloguing-in-publication data* (2005).