Let G be a reductive linear algebraic group, B, a Borel subgroup. and T, a maximal torus contained in B. I am interested in the flag variety(or manifold) F= G/B or more particularly in the cohomology of F, where by cohomology, I mean in a general sense: singular, equivariant, K-theory. equivariant K-theory, and beyond. For each of these theories, there are two descriptions of cohomology: One is in terms of generators and relations(typically involving the Lie algebra of T) and the other in terms of Schubert classes, which are cohomology classes associated to the Schubert subvarieties of F. There is one Schubert class S_{w} for each element w of the Weyl group of G.

The invertible diagonal matrix acts on a matrix g in G by scaling the rows of g. Therefore, the column echelon form of g remains the same and so T.C_{w}=C_{w},where C_{w} is a Schubert cell. In addition, the fixed points of the T-action on G/B are exactly the permutation matrices. One T- fixed point in each C_{w}.

Given a subtorus T_{0} subset T, we may consider its centralizer G'=Z_{G}(T_{0}) in G, which is a connected, reductive subgroup of G. For any subgroup H subset G, let $\psi(H)= H\cap G'$. Set $B'=\psi (B)$ and $T'=\psi(T)$, which are a Borel subgroup and maximal torus of G'. Let F' = G'/B' be the flag variety of G' and set F^{T_{0}} to be the fixed points of F under the T_{0} action. This consists of the Borel subgroups of G that contain T_{0}. By [4, Theorem 6.4.7], the map $\psi: \mathcal{F}^{T_{0}}\longrightarrow \mathcal{F}'$ is G'-invariant(action preserving). Billey and Braden[2] showed that ψ is an isomorphism on each connected component of $\mathcal{F'}$, and they so that, for every Schubert cell C_{w} of $\mathcal{F}$, $\psi $ is an isomorphism between $C_{w}\cap \mathcal{F}^{T_{0}}$ and Schubert a cell $C_{\phi (w)}$ of $\mathcal{F'},$ where $w\longmapsto \phi (w)$ is their pattern map from the Weyl group of G to the Weyl group of G'. When $G =GL_{n}(\C)$, this geometry was studied earlier by Bergeron and Sottile [1] who were interested the effect of these maps on Schubert classes.\\

The map $\psi: \mathcal{F}^{T_{0}}\longrightarrow \mathcal{F}'$ is G'- invariant(action preserving). One can think of $B'=\psi (B)$ and $T'=\psi(T)$, as $B'=\psi (B\cap G')$ and $T'=\psi(T\cap G')$ which are Borel subgroup and maximal torus of G'. The Weyl group of G' is given by $W'=N_{G'}(T')/T'=W\cap(G'/B')$. Where W is the Weyl group of G. The Schubert varieties of $\mathcal{F'}$ are indexed by elements of W'.\\

The groups G' constructed in this manner are Levi subgroup of $G.$ It is also known that W' is a parabolic subgroup of W. Indeed there is a unique map $\phi: W\longrightarrow W'$, the so-called pattern map. A pattern of an element in W is its image under ϕ. I am not sure for now, if the requirement that the parabolic subgroup W' to be standard is important. Billey and Braden[2.] said all the subgroups which are generated by reflections are parabolic in type A but in other types the case is not true. I really want to understand this in full detail. \\

Let $\varphi: \mathcal{F}'\longrightarrow \mathcal{F}^{T_{0}}$ be a section of this map ψ, so that $\psi\circ \varphi$ is the identity. The singular cohomology of $\mathcal{F}$ and $\mathcal{F}'$ has generators corresponding to certain line bundles on $\mathcal{F}$ and $\mathcal{F}'$ and the induced map $\varphi ^{\ast}: H^{\ast}(\mathcal{F})\longrightarrow H^{\ast}(\mathcal{F})$ is a surjection mapping generators to generators. However, for the basis of Schubert classes, there are certain decomposition coefficients d^{u}_{w} for $w\in W$ and $u\in W'$ defined by $$\varphi ^{\ast}(S_{w})=\sum_{u\in W}d_{w}^{u} S_{u}.$$

Bergeron and Sottile[1] identified these decomposition coefficients as certain Littlewood-Richardson coefficients for the singular cohomology of $\mathcal{F}$. Later, Lenart, Robinson, and Sottile[3.] extended this to K-theory of $\mathcal{F}$. \\

{\bf The Goal}\\

The goal of this Research is to formulate results in the largest possible generality, and then explore their consequences for the cohomology theories most relevant for combinatorialists, such as equivariant cohomology, Cobordism. This would be done by computing the effect of the pattern map of Billey and Braden on the cohomology of flag varieties for very general cohomology theories.\\

The decomposition coefficient is little bit technical for me to grasp, I am still studying it. I really want to understand the relevance of flattening functions to pattern maps? Of course, The opposite Borel subgroup exists in G since G is reductive. I think the intersection of any two Borel subgroup contains a maximal torus. The approach of Sottile to visualize pattern map is a bit different from Billey's as I observed. Sottile used the product of flag manifolds. He identified ${\mathcal{F}'}$ with $\mathbb{F}\ell (U)\times \mathbb{F}\ell (V)$ and described the map $\pi:\mathcal{F}^{T_{0}}\longrightarrow \mathbb{F}\ell (U)\times \mathbb{F}\ell (V)$ by $G_{\bullet}\longmapsto (G_{\bullet}\cap U, G_{\bullet}\cap V)$ as not injective. The claim is that there is a bijection between the set of flags in $\mathcal{F}^{T_{0}}$ and the cosets $S_{m}\times S_{n}\simeq W'$ in $S_{m+n}=W$, where $dim(U)=m$ and $dim(V)=n$ and to a flag $G_{\bullet}\in \mathcal{F}$ there exist two subsets P and Q. These two subsets must be disjoint.\\ I think the description resembles flattening function. The most general parabolic subgroup of S_{n} is obtained according to Billey by allowing $P_{1}, P_{2},\dots, P_{k}$ to be disjoint subsets of $[n]$. There is associated parabolic subgroup W'_{i} to each P_{i} and therefore, $$W'=W'_{1}W'_{2}\dot W'_{k}\cong S_{\mid P_{1}\mid}\times\dots \times S_{\mid P_{k}\mid}$$ is a parabolic subgroup. The corresponding flattening function is $$w\longmapsto (fl_{\mid P_{1}\mid}(w),\dots , fl_{\mid P_{k}\mid}(w)).$$

I believe the detailed study of cohomology on manifolds, in cases where there is enough structure to make effective computations, is an area of great interest in algebraic geometry. This interest comes in part from the modern theories of physics which make predictions on certain symmetries that one should find in the cohomology of specific manifolds.\\

I am currently talking with Professor Sottile in this direction of research

[1.] N. Bergeon and F. Sottile, Schubert polynimials, the Bruhat order, and the geometry of flag manifolds. Duke Math. J 95(1998) no.2 373-423S

[2.] S. Billey and T.Braden, Lower bounds for Kazhdan-Lusztig polynomial from patterns, Transform. groups 8 (2003) no 4, 345-374.

[3.]C. Lenart, Shawn Robinson, and Frank Sottile, Grothendieck poynomials via permutation patterns and chains in the Bruhat order, Amer. J. Math. 128(2006) no.4, 805-848.

[4.] T. A Springer, Linear algebraic groups, second ed., Progress in Mathematics, vol. 9 Birkhauser Boston Inc., Boston, MA 1998.